Bayesian Analysis Of Inverse Topp-Leone Distribution Under Different Loss Functions

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Abstract:-This paper deals with the Bayesian estimation of the shape parameter of Inverse Topp-Leone distribution. The extended Jeffery's prior and gamma prior is used to obtain posterior distributions. The estimators are obtained under squared error loss function, Entropy loss function, precautionary loss function and Linex loss function. In addition maximum likelihood estimation is also discussed. These approaches are compared using mean square error in a simulation analysis of different sample sizes. Eventually a real data set is considered to compare the efficiency of these estimators under different loss function.

Key words:-Bayesian analysis, Jeffery's prior, gamma prior, maximum likelihood estimation, loss functions.

1: Introduction

Topp-Leone distribution belongs to the distribution family which has support $[0,1]$. It indicates the j-shape form of density function along with bathtub shape of its hazard function. This distribution is used for the analysis of failure data. The probability density function of Topp-Leone is given by

$$g(x,\theta)=2\theta x^{\theta-1}(1-x)(2-x)^{\theta-1};0\le x\le 1,\theta>0$$

Since the Topp-Leone distribution is newly formulated distribution proposed by Topp and Leone [12]. This distribution has been studied by several authors such as Nadarajah [10], Ghitney et al [4,5], Genc [6], Zahrani [14], Mostafaee [9], Vicari et al [13].RecentyAmal S. Hassan et al [2] explored the inverse of Topp-Leone distribution.

Let X follows the probability distribution function of Topp-Leone distribution, then the transformation $Y=\frac{1}{X}-1$ is said to follow inverse of Topp-Leone distribution having probability density function (p.d.f) as

$$f(y,\theta)=2\theta y\frac{(1+2y)^{\theta-1}}{(1+y)^{2\theta+1}};y>0,\theta>0\ (1.1)$$

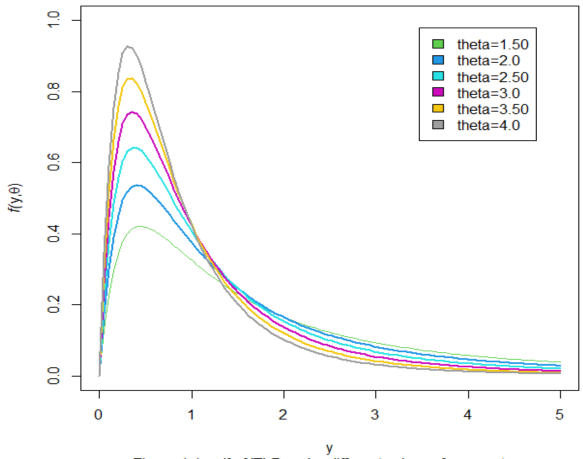

Figure 1.1: pdf of ITLD under different values of parameters

Figure (1.1) illustrates some possible shapes of p.d.f for varying parameters

The corresponding cumulative distribution function (c.d.f) of (1.1) is given by

$$F(y,\theta)=1-\left[\frac{(1+2y)^{\theta}}{(1+y)^{2\theta}}\right];y\geq 0,\theta>0\ (1.2)$$

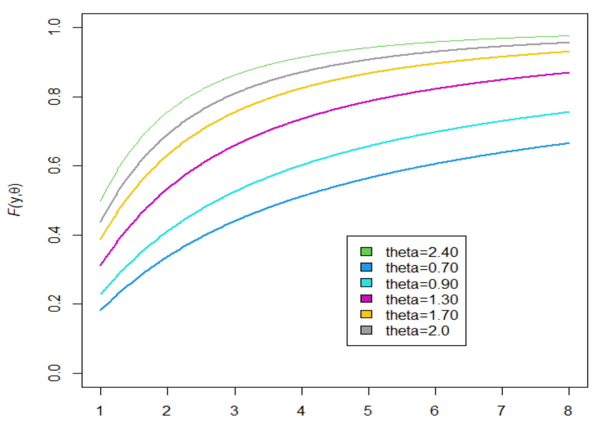

Figure 1.2: cdf of ITLD under different values of parameters

Figure (1.2) illustrates some possible shapes of c.d.f for varying parameters

2: Maximum Likelihood Estimation

Let $y_1, y_2,..., y_n$ be random samples from inverse Topp-Leone distribution given by (1.1), then the likelihood function becomes

$$l=\prod_{i=1}^{n} f\left(y_i,\theta\right)$$

$$=\prod_{i=1}^{n}(2\theta)y_i\left(1+y_i\right)^{-(2\theta+1)}\left(1+2y_i\right)^{\theta-1}$$

The log-likelihood function is

$$\log l=n\log 2+n\log\theta+\sum_{i=1}^{n}\log y_i-(2\theta+1)\sum_{i=1}^{n}\log\left(y_i+1\right)+(\theta-1)\sum_{i=1}^{n}\log\left(2y_i+1\right)$$

Differentiate w.r.t θ we get

$$\frac{\partial\log l}{\partial\theta}=\frac{n}{\theta}-2\sum_{i=1}^{n}\log\left(y_i+1\right)+\sum_{i=1}^{n}\log\left(2y_i+1\right)$$

Solving $\frac{\partial\log l}{\partial\theta}=0$, we get

$$\hat{\theta}=\frac{n}{2\sum_{i=1}^{n}\log\left(y_i+1\right)-\sum_{i=1}^{n}\log\left(2y_i+1\right)}$$

3: Bayesian Estimation Of Inverse Topp-Leone Distribution

The Bayesian estimation technology is a remarkable way of estimating the distribution model parameters. This calculation accounts for the subsequent (posterior) distribution of the life time distribution by considering prior knowledge.From Bayesian point of view there can’t be put the lid on selecting prior(s) by considering one’s prior(s) is more suitable than others. In case of meagre interpretative information about the unknown parameter it is preferable to select non informative prior. However, if one has sufficient information about the parameter(s) it is better to select informative prior. The aim of present study is to obtain a Bayesian estimation of parameter θ of inverseTopp-Leone distribution by using extended Jeffrey’s and gamma prior. In recent past years several research papers have been published in this direction.Afaq et al [1],studied estimation of parameters of two parameter exponentiated gamma distribution. Mudasir et al [8] studied the Bayesian estimation of weighted Erlang distribution. Raqab and Madi [11], studied Bayesian estimation for exponentiated Rayleigh distribution. Recently Ahmad et al [3], studied Bayesian estimation of inverse Ailamujia distribution used different loss functions.

3.1: Posterior Distribution Of Inverse Topp-Leone Distribution under The Assumption Of Extended Jeffrey’s Prior

Suppose $y=\left(y_1,y_2,..,y_n\right)$ denotes the n recorded values of (1.1). Then its likelihood function is given by

$$L(y|\theta)=\prod_{i=1}^{n}2\theta\frac{y_i}{(y_{i+1})^2}\left[\frac{2}{y_i+1}-\frac{1}{(y_i+1)^2}\right]^{\theta-1}$$

$$=2^n\theta^n\prod_{i=1}^{n}\frac{y_i}{(y_i+1)^2}\;e^{-(\theta+1)\sum_{i=1}^{n}-\log\left[\frac{2}{y_1+1}-\frac{1}{(y_i+1)^2}\right]}$$

We assume the prior distribution of θ to be extended Jeffery's prior proposed by Al-Kutubi [7], is given by

$$g(\theta)\propto[I(\theta)]^c,\ c\in R^+$$

Where $[I(\theta)]=-nE\left[\frac{\partial^2\log f(y,\theta)}{\partial^2\theta}\right]$ is the Fisher information matrix, for the distribution (1.1),

$$g(\theta)=K\left[\frac{n}{\theta^2}\right]^c$$

The posterior distribution of θ under the assumption of extended Jeffrey's prior i.e $g(\theta)\propto\frac{1}{\theta^{2c}}$ is given by

$$h(\theta|y)\propto L(y|\theta)g(\theta)$$

$$\Rightarrow h(\theta|y)\propto 2^n\theta^n\prod_{i=1}^{n}\frac{y_i}{(y_i+1)^2}\;e^{-(\theta+1)\sum_{i=1}^{n}-\log\left[\frac{2}{y_1+1}-\frac{1}{(y_i+1)^2}\right]}\frac{1}{\theta^{2c}}$$

$$\Rightarrow h(\theta|y)\propto\prod_{i=1}^{n}2^n\frac{y_i}{(y_i+1)^2}e^{\sum_{i=1}^{n}-\log\left[\frac{2}{y_i+1}-\frac{1}{(y_i+1)^2}\right]}\theta^{n-2c}e^{-\theta\sum_{i=1}^{n}-\log\left[\frac{2}{y_i+1}-\frac{1}{(y_i+1)^2}\right]}$$

$$\Rightarrow h(\theta|y)\propto K\theta^{n-2c}e^{-\theta\sum_{i=1}^{n}-\log\left[\frac{2}{y_i+1}-\frac{1}{(y_i+1)^2}\right]}$$

Where K is independent of θ

$$K^{-1}=\int_0^\infty\theta^{n-2c}e^{-\theta\sum_{i=1}^{n}-\log\left[\frac{2}{y_i+1}-\frac{1}{(y_i+1)^2}\right]}d\theta$$

$$K^{-1}=\frac{\Gamma(n-2c+1)}{\left\{\sum_{i=1}^{n}-\log\left[\frac{2}{y_i+1}-\frac{1}{(y_i+1)^2}\right]\right\}^{n-2c+1}}$$

Therefore

$$K = \frac{\left\{\sum_{i=1}^{n} -\log\left[\frac{2}{y_i+1} - \frac{1}{(y_i+1)^2}\right]\right\}^{n-2c+1}}{\Gamma(n-2c+1)}$$

Hence the posterior distribution of θ is given by

$$h(\theta|y) = \frac{T^{n-2c+1}}{\Gamma(n-2c+1)}\theta^{n-2c}e^{-\theta T}$$

Where $T = \sum_{i=1}^{n} -\log\left[\frac{2}{y_i+1} - \frac{1}{(y_i+1)^2}\right]$

3.1.1: Estimation under square error loss function

The squared error loss function is defined as $l(\hat{\theta},\theta) = c_1(\hat{\theta}-\theta)^2$ for some constant c_1. The risk function is given by

$$R(\hat{\theta},\theta) = E\left[l(\hat{\theta},\theta)\right]$$

$$= \int_0^\infty l(\hat{\theta},\theta)h(\theta|y)d\theta$$

$$= \int_0^\infty c_1(\hat{\theta}-\theta)^2 \frac{T^{n-2c+1}}{\Gamma(n-2c+1)}\theta^{n-2c}e^{-\theta T}d\theta$$

$$= \frac{T^{n-2c+1}}{\Gamma(n-2c+1)}c_1\int_0^\infty (\hat{\theta}-\theta)^2\theta^{n-2c}e^{-\theta T}d\theta$$

$$= \frac{T^{n-2c+1}}{\Gamma(n-2c+1)}c_1\left[\hat{\theta}^2\int_0^\infty \theta^{n-2c}e^{-\theta T}d\theta + \int_0^\infty \theta^{n-2c+2}e^{-\theta T}d\theta - 2\hat{\theta}\int_0^\infty \theta^{n-2c+1}e^{-\theta T}d\theta\right]$$

After solving the integral, we get

$$= \frac{T^{n-2c+1}}{\Gamma(n-2c+1)}c_1\left[\begin{array}{l}\hat{\theta}\frac{\Gamma(n-2c+1)}{T^{n-2c+1}} + \frac{(n-2c+1)(n-2c+1)\Gamma(n-2c+1)}{T^{n-2c+2}} \\ -2\hat{\theta}\frac{(n-2c+1)\Gamma(n-2c+1)}{T^{n-2c+1}}\end{array}\right]$$

$$= c_1\left[\hat{\theta}^2 + \frac{(n-2c+1)(n-2c+2)}{T^2} - 2\hat{\theta}\frac{(n-2c+1)}{T}\right]$$

Now solving $\frac{\partial R(\hat{\theta},\theta)}{\partial\hat{\theta}} = 0$, we get

$$\hat{\theta}_s = \frac{(n-2c+1)}{T}$$

Where $T = \sum_{i=1}^{n} -\log\left[\frac{2}{y_i+1} - \frac{1}{(y_i+1)^2}\right]$

3.1.2: Estimation under entropy loss function

The entropy loss function is defined as $l(\delta) = b[\delta - \log(\delta) - 1]; b > 0, \delta = \frac{\hat{\theta}}{\theta}$. The risk function is given by

$$R(\hat{\theta},\theta) = E[l(\delta)]$$

$$= \int_0^\infty l(\delta) h(\theta|y) d\theta$$

$$= \int_0^\infty b[\delta - \log(\delta) - 1] h(\theta|y) d\theta$$

$$= b\frac{T^{n-2c+1}}{\Gamma(n-2c+1)} \int_0^\infty \left[\frac{\hat{\theta}}{\theta} - \log\left(\frac{\hat{\theta}}{\theta}\right) - 1\right] \theta^{n-2c} e^{-\theta T} d\theta$$

$$= \frac{T^{n-2c+1}}{\Gamma(n-2c+1)} b\left[\hat{\theta}\int_0^\infty \theta^{n-2c-1} e^{-\theta T} d\theta - \log(\hat{\theta})\int_0^\infty \theta^{n-2c} e^{-\theta T} d\theta + \int_0^\infty \log(\theta)\theta^{n-2c} e^{-\theta T} d\theta - \int_0^\infty \theta^{n-2c} e^{-\theta T} d\theta\right]$$

After solving the integral, we get

$$= b\frac{T^{n-2c+1}}{\Gamma(n-2c+1)}\left[\hat{\theta}\frac{\Gamma(n-2c)}{T^{n-2c}} - \log(\hat{\theta})\frac{\Gamma(n-2c+1)}{T^{n-2c+1}} + \frac{\Gamma'(n-2c+1)}{T^{n-2c+1}} - \frac{\Gamma(n-2c+1)}{T^{n-2c+1}}\right]$$

$$= b\left[\hat{\theta}\frac{T}{n-2c} - \log(\hat{\theta}) + \psi'(n-2c+1) - 1\right]$$

Where $\psi'(.)$ denotes the digamma function

Now solving $\frac{\partial R(\hat{\theta},\theta)}{\partial \hat{\theta}} = 0$, we get

$$\hat{\theta}_e = \frac{(n-2c)}{T}$$

Where $T=\sum_{i=1}^{n}-\log\left[\frac{2}{y_i+1}-\frac{1}{(y_i+1)^2}\right]$

3.1.3: Estimation under precautionary loss function

The entropy loss function is defined as $l(\hat{\theta},\theta)=\frac{(\hat{\theta}-\theta)^2}{\hat{\theta}}$. The risk function is given by

$$R(\hat{\theta},\theta)=E[l(\hat{\theta},\theta)]$$

$$=\int_0^\infty l(\hat{\theta},\theta)h(\theta|y)d\theta$$

$$=\frac{T^{n-2c+1}}{\Gamma(n-2c+1)}\int_0^\infty \frac{(\hat{\theta}-\theta)^2}{\hat{\theta}}\theta^{n-2c}e^{-\theta T}d\theta$$

$$=\frac{T^{n-2c+1}}{\Gamma(n-2c+1)}\left[\hat{\theta}\int_0^\infty \theta^{n-2c}e^{-\theta T}d\theta+\frac{1}{\hat{\theta}}\int_0^\infty \theta^{n-2c+2}e^{-\theta T}d\theta-2\int_0^\infty \theta^{n-2c+1}e^{-\theta T}d\theta\right]$$

After solving the integral, we get

$$=\left[\hat{\theta}+\frac{1}{\hat{\theta}}\frac{(n-2c+1)(n-2c+2)}{T^2}-\frac{2(n-2c+1)}{T}\right]$$

Now solving $\frac{\partial R(\hat{\theta},\theta)}{\partial\hat{\theta}}=0$, we get

$$\hat{\theta}_p=\frac{[(n-2c+1)(n-2c+2)]^{\frac{1}{2}}}{T}$$

Where $T=\sum_{i=1}^{n}-\log\left[\frac{2}{y_i+1}-\frac{1}{(y_i+1)^2}\right]$

3.1.4: Estimation under linex loss function

The linex loss function is defined as $l(\hat{\theta},\theta)=\exp\{b_1(\hat{\theta}-\theta)\}-b_1(\hat{\theta}-\theta)-1$. The risk function is given by

$$R(\hat{\theta},\theta)=E[l(\hat{\theta},\theta)]$$

$$=\int_0^\infty l(\hat{\theta},\theta)h(\theta|y)d\theta$$

$$=\frac{T^{n-2c+1}}{\Gamma(n-2c+1)}\int_0^\infty\left[\exp\left\{b_1\left(\hat{\theta}-\theta\right)\right\}-b_1\left(\hat{\theta}-\theta\right)-1\right]\theta^{n-2c}e^{-\theta T}d\theta$$

$$=\frac{T^{n-2c+1}}{\Gamma(n-2c+1)}\left[e^{b_1\hat{\theta}}\int_0^\infty\theta^{n-2c}e^{-\theta(b_1+T)}d\theta-b_1\hat{\theta}\int_0^\infty\theta^{n-2c}e^{-\theta T}d\theta+b_1\int_0^\infty\theta^{n-2c+1}e^{-\theta T}d\theta-\int_0^\infty\theta^{n-2c}e^{-\theta T}d\theta\right]$$

After solving the integral, we get

$$=\left[e^{b_1\theta}\left(\frac{T}{b_1+T}\right)-b_1\hat{\theta}+\frac{b_1(n-2c+1)}{T}-1\right]$$

Now solving $\frac{\partial R\left(\hat{\theta},\theta\right)}{\partial\hat{\theta}}=0$, we get

$$\hat{\theta}_l=\frac{(n-2c+1)}{b_1}\log\left(\frac{b_1+T}{T}\right)$$

Where $T=\sum_{i=1}^{n}-\log\left[\frac{2}{y_i+1}-\frac{1}{(y_i+1)^2}\right]$

4.1: Posterior Distribution Of Inverse Topp-Leone Distribution under The Assumption Of Gamma Prior

Suppose $y=(y_1,y_2,..,y_n)$ denotes the n recorded values of (1.1). Then its likelihood function is given by

$$L(y|\theta)=\prod_{i=1}^{n}2\theta\frac{y_i}{(y_{i+1})^2}\left[\frac{2}{y_i+1}-\frac{1}{(y_i+1)^2}\right]^{\theta-1}$$

$$=2^n\theta^n\prod_{i=1}^{n}\frac{y_i}{(y_i+1)^2}\;e^{-(\theta+1)\sum_{i=1}^{n}-\log\left[\frac{2}{y_1+1}-\frac{1}{(y_i+1)^2}\right]}$$

We assume the prior distribution of θ to be gamma prior

The posterior distribution of θ under the assumption of gamma prior i.e $g(\theta)\propto\frac{a^b}{\Gamma(b)}\theta^{b-1}e^{-a\theta}$ is given by

$$h(\theta|y)\propto L(y|\theta)g(\theta)$$

$$\Rightarrow h(\theta|y)\propto 2^n\theta^n\prod_{i=1}^{n}\frac{y_i}{(y_i+1)^2}\;e^{-(\theta+1)\sum_{i=1}^{n}-\log\left[\frac{2}{y_1+1}-\frac{1}{(y_i+1)^2}\right]}\frac{a^b}{\Gamma(b)}\theta^{b-1}e^{-a\theta}$$

$$\Rightarrow h(\theta|y) \propto \prod_{i=1}^{n} 2^n \frac{y_i}{(y_i+1)^2} e^{\sum_{i=1}^{n} -\log\left[\frac{2}{y_i+1}-\frac{1}{(y_i+1)^2}\right]} \frac{a^b}{\Gamma(b)} \theta^{n+b-1} e^{-\theta\left(a+\sum_{i=1}^{n} -\log\left[\frac{2}{y_i+1}-\frac{1}{(y_i+1)^2}\right]\right)}$$

$$\Rightarrow h(\theta|y) \propto K\theta^{n+b-1} e^{-\theta\left(a+\sum_{i=1}^{n} -\log\left[\frac{2}{y_i+1}-\frac{1}{(y_i+1)^2}\right]\right)}$$

Where K is independent of θ

$$K^{-1} = \int_0^\infty \theta^{n+b-1} e^{-\theta\left(a+\sum_{i=1}^{n} -\log\left[\frac{2}{y_i+1}-\frac{1}{(y_i+1)^2}\right]\right)} d\theta$$

$$K^{-1} = \frac{\Gamma(n+b)}{\left\{a+\sum_{i=1}^{n} -\log\left[\frac{2}{y_i+1}-\frac{1}{(y_i+1)^2}\right]\right\}^{n+b}}$$

Therefore

$$K = \frac{\left\{a+\sum_{i=1}^{n} -\log\left[\frac{2}{y_i+1}-\frac{1}{(y_i+1)^2}\right]\right\}^{n+b}}{\Gamma(n+b)}$$

Hence the posterior distribution of θ is given by

$$h(\theta|y) = \frac{(a+T)^{n+b}}{\Gamma(n+b)} \theta^{n+b-1} e^{-(a+T)\theta}$$

Where $T = \sum_{i=1}^{n} -\log\left[\frac{2}{y_i+1}-\frac{1}{(y_i+1)^2}\right]$

4.1.1: Estimation under square error loss function

The squared error loss function is defined as $l(\hat{\theta},\theta) = c_1(\hat{\theta}-\theta)^2$ for some constant c_1. The risk function is given by

$$R(\hat{\theta},\theta) = E\left[l(\hat{\theta},\theta)\right]$$

$$= \int_0^\infty l(\hat{\theta},\theta) h(\theta|y) d\theta$$

$$= c_1 \frac{(a+T)^{n+b}}{\Gamma(n+b)} \int_0^\infty (\hat{\theta}-\theta)^2 \theta^{n+b-1} e^{-(a+T)\theta} d\theta$$

$$=c_1\frac{(a+T)^{n+b}}{\Gamma(n+b)}\left\{\hat{\theta}^2\int_0^\infty \theta^{n+b-1}e^{-(a+T)\theta}d\theta+\int_0^\infty \theta^{n+b+1}e^{-(a+T)\theta}d\theta-2\hat{\theta}\int_0^\infty \theta^{n+b}e^{-(a+T)\theta}d\theta\right\}$$

After solving the integral, we get

$$=c_1\left\{\hat{\theta}^2+\frac{(n+b)(n+b+1)}{(a+T)^2}-2\hat{\theta}\frac{(n+b)}{a+T}\right\}$$

Now solving $\frac{\partial R(\hat{\theta},\theta)}{\partial\hat{\theta}}=0$, we get

$$\hat{\theta}_s=\frac{n+b}{a+T}$$

Where $T=\sum_{i=1}^{n}-\log\left[\frac{2}{y_i+1}-\frac{1}{(y_i+1)^2}\right]$

4.1.2: Estimation under entropy loss function

The entropy loss function is defined as $l(\delta)=b[\delta-\log(\delta)-1]; b>0$, $\delta=\frac{\hat{\theta}}{\theta}$. The risk function is given by

$$R(\hat{\theta},\theta)=E[l(\delta)]$$

$$=\int_0^\infty l(\delta)h(\theta|y)d\theta$$

$$=\int_0^\infty b[\delta-\log(\delta)-1]h(\theta|y)d\theta$$

$$=b\frac{(a+T)^{n+b}}{\Gamma(n+b)}\int_0^\infty\left[\frac{\hat{\theta}}{\theta}-\log\left(\frac{\hat{\theta}}{\theta}\right)-1\right]\theta^{n+b-1}e^{-(a+T)\theta}d\theta$$

$$=b\frac{(a+T)^{n+b}}{\Gamma(n+b)}\left\{\hat{\theta}\int_0^\infty\theta^{n+b-2}e^{-(a+T)\theta}d\theta-\log\hat{\theta}\int_0^\infty\theta^{n+b-1}e^{-(a+T)\theta}d\theta+\int_0^\infty\log(\theta)\theta^{n+b-1}e^{-(a+T)\theta}d\theta-\int_0^\infty\theta^{n+b-1}e^{-(a+T)\theta}d\theta\right\}$$

After solving the integral, we get

$$=b\left\{\hat{\theta}\frac{(a+T)}{(n+b-1)}-\log\hat{\theta}+\psi'(n+b)-1\right\}$$

Where $\psi'(.)$ denotes the digamma function

Now solving $\frac{\partial R(\hat{\theta},\theta)}{\partial \hat{\theta}}=0$, we get

$$\hat{\theta}_e=\frac{n+b-1}{a+T}$$

Where $T=\sum_{i=1}^{n}-\log\left[\frac{2}{y_i+1}-\frac{1}{(y_i+1)^2}\right]$

4.1.3: Estimation under precautionary loss function

The entropy loss function is defined as $l(\hat{\theta},\theta)=\frac{(\hat{\theta}-\theta)^2}{\hat{\theta}}$ .The risk function is given by

$$R(\hat{\theta},\theta)=E[l(\hat{\theta},\theta)]$$

$$=\int_0^\infty l(\hat{\theta},\theta)h(\theta|y)d\theta$$

$$=\frac{(a+T)^{n+b}}{\Gamma(n+b)}\int_0^\infty \frac{(\hat{\theta}-\theta)^2}{\hat{\theta}}\theta^{n+b-1}e^{-(a+T)\theta}d\theta$$

$$=\frac{(a+T)^{n+b}}{\Gamma(n+b)}\left\{\hat{\theta}\int_0^\infty \theta^{n+b-1}e^{-(a+T)\theta}d\theta+\frac{1}{\hat{\theta}}\int_0^\infty \theta^{n+b+1}e^{-(a+T)\theta}d\theta-2\int_0^\infty \theta^{n+b}e^{-(a+T)\theta}d\theta\right\}$$

After solving the integral, we get

$$=\left\{\hat{\theta}+\frac{1}{\hat{\theta}}\frac{(n+b)(n+b+1)}{(a+T)^2}-2\frac{(n+b)}{(a+T)}\right\}$$

Now solving $\frac{\partial R(\hat{\theta},\theta)}{\partial \hat{\theta}}=0$, we get

$$\hat{\theta}_p=\frac{[(n+b)(n+b+1)]^{\frac{1}{2}}}{a+T}$$

Where $T=\sum_{i=1}^{n}-\log\left[\frac{2}{y_i+1}-\frac{1}{(y_i+1)^2}\right]$

4.1.4: Estimation under linex loss function

The linex loss function is defined as $l(\hat{\theta},\theta)=\exp\{b_1(\hat{\theta}-\theta)\}-b_1(\hat{\theta}-\theta)-1$.The risk function is given by

$$R(\hat{\theta},\theta)=E[l(\hat{\theta},\theta)]$$

$$=\int_0^\infty l(\hat{\theta},\theta)h(\theta|y)d\theta$$

$$=\frac{(a+T)^{n+b}}{\Gamma(n+b)}\int_0^\infty[\exp\{b_1(\hat{\theta}-\theta)\}-b_1(\hat{\theta}-\theta)-1]\theta^{n+b-1}e^{-(a+T)\theta}d\theta$$

$$=\frac{(a+T)^{n+b}}{\Gamma(n+b)}\left[e^{b_1\hat{\theta}}\int_0^\infty\theta^{n+b-1}e^{-(a+b_1+T)\theta}d\theta-b_1\hat{\theta}\int_0^\infty\theta^{n+b-1}e^{-(a+T)\theta}d\theta+b_1\int_0^\infty\theta^{n+b}e^{-(a+T)\theta}d\theta-\int_0^\infty\theta^{n+b-1}e^{-(a+T)\theta}d\theta\right]$$

After solving the integral, we get

$$=\left\{e^{b_1\hat{\theta}}\left(\frac{a+T}{a+b_1+T}\right)^{n+b}-b_1\hat{\theta}+\frac{(n+b)b_1}{(a+T)}-1\right\}$$

Now solving $\frac{\partial R(\hat{\theta},\theta)}{\partial\hat{\theta}}=0$, we get

$$\hat{\theta}_l=\frac{1}{b_1}\log\left(\frac{a+b_1+T}{a+T}\right)^{n+b}$$

Where $T=\sum_{i=1}^{n}-\log\left[\frac{2}{y_i+1}-\frac{1}{(y_i+1)^2}\right]$

5:- Simulation Analysis

This section is dedicated to the simulation analysis, we generate $N=1500$ random samples of size $n=50,100$ and 150 to represent a small, medium and large data set from inverse Topp-Leone distribution for specific values of $\theta=0.5$ and 1. The shape parameter is estimated with maximum likelihood estimation and Bayesian using extended Jeffery's prior and gamma prior. For extended Jeffrey's prior we chose $c=0.5$ and 1 and the value of loss function $b_1=0.06$ and 0.09. In case of gamma prior we chose $a=0.5,1.0$ and $b=0.5,1.0$ with loss function $b_1=0.06$ and 0.09.R software is used for simulation analysis in order to examine and compare the efficiency of the estimates for different sample sizes with different values of loss functions. The results are presented in table 5.1 and 5.2.

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Table 5.1

Mean Square Error for $\hat{\theta}$ Using Jeffery's Prior

n	$\hat{\theta}$	C	$\hat{\theta}_s$	$\hat{\theta}_e$	$\hat{\theta}_p$	$\hat{\theta}_l$	
						$b_1=0.06$	$b_1=0.09$
50	0.5	0.5	0.01019658	0.01019497	0.01019739	0.01019656	0.01019654
		1.0	0.01006666	0.01006506	0.01006746	0.01006662	0.0100666
	1	0.5	0.1605382	0.1605446	0.160535	0.1605383	0.1605383
		1.0	0.1591694	0.1591759	0.1591662	0.1591696	0.1591697
100	0.5	0.5	0.01003337	0.01003257	0.01003377	0.01003335	0.01003335
		1.0	0.01002572	0.01002492	0.01002612	0.0100257	0.0100257
	1	0.5	0.1603843	0.1603875	0.1603827	0.1603844	0.1603844
		1.0	0.1602739	0.1602771	0.1602723	0.160274	0.160274
150	0.5	0.5	0.01001125	0.01001071	0.01001151	0.01001123	0.01001123
		1.0	0.01004535	0.01004483	0.01004563	0.01004535	0.01004533
	1	0.5	0.1600195	0.1600216	0.1600184	0.1600195	0.1600196
		1.0	0.1600716	0.1600738	0.1600706	0.1600717	0.1600717

$\hat{\theta}_s =$ Square error loss function, $\hat{\theta}_e =$ Estimation under Entropy,

$\hat{\theta}_p =$ Estimation under Precautionary, $\hat{\theta}_l =$ Estimation under LINEX

In table5.1, Bayes estimation with squared error loss function under extended Jeffery's prior the lesser values in most cases. Moreover, when sample size increase from 50 to 150, the mean square error decreases quite significantly.

Table 5.2

Mean Square Error for $\hat{\theta}$ Using gammaPrior

n	$\hat{\theta}$	a	b	$\hat{\theta}_s$	$\hat{\theta}_e$	$\hat{\theta}_p$	$\hat{\theta}_l$	
							$b_1=0.06$	$b_1=0.09$
50	0.5	0.5	0.5	0.02337317	0.02337461	0.02337391	0.0233732	0.0233732
		0.5	1.0	0.02335441	0.02335582	0.02335512	0.02335444	0.02335444
	1	0.5	0.5	0.2494094	0.2494161	0.2494127	0.2494095	0.2494096
		0.5	1.0	0.4267061	0.4267121	0.4267091	0.4267061	0.4267062
100	0.5	0.5	0.5	6.735e-06	6.7374e-06	6.7365e-06	6.7356e-06	6.7356e-06
		0.5	1.0	7.1562e-06	7.1517e-06	7.1540e-06	7.1561e-06	7.1561e-06
	1	0.5	0.5	0.2500997	0.250103	0.2501014	0.2500998	0.2500998
		0.5	1.0	0.2490105	0.2490138	0.2490121	0.2490105	0.2490104
150	0.5	0.5	0.5	4.5992e-06	4.6011e-06	4.6002e-06	4.59938e-06	4.59938e-06

		0.5	1.0	4.6406e-06	4.6426e-06	4.6416e-06	4.64060e-06	4.64060e-06
	1	0.5	0.5	0.2502639	0.2502661	0.250265	0.2502639	0.2502639
		0.5	1.0	0.2497457	0.2497479	0.2497468	0.2497457	0.2497457

In table5.2, Bayes estimation with squared error loss function under gamma prior the lesser values in most cases. Moreover, when sample size increase from 50 to 150, the mean square error decreases quite significantly.

6: Application

In this section we provide a real life data sets through which the efficiency of the estimators and posterior risks of different loss function has been obtained.

Real Life Data Set

The following data represent 40 patients suffering from blood cancer(leukaemia) from one ministry of health hospitals in Saudi Arabia (see Abouammah et al.). The ordered lifetimes (in years) are given.

0.315, 0.496, 0.616, 1.145, 1.208, 1.263, 1.414, 2.025, 2.036, 2.162, 2.211, 2.37, 2.532, 2.693, 2.805, 2.91, 2.912, 3.192, 3.263, 3.348, 3.348, 3.427, 3.499, 3.534, 3.767, 3.751, 3.858, 3.986, 4.049, 4.244, 4.323, 4.381, 4.392, 4.397, 4.647, 4.753, 4.929, 4.973, 5.074, 5.381

By using different loss functions the Bayesian estimates and posterior risks of the posterior distribution through both prior's are as follows where posterior risks are in parenthesis.

Table 6.1
Bayes Estimation and Posterior Risks Using Jeffery's Prior

$\hat{\theta}$	C	$\hat{\theta}_s$	$\hat{\theta}_e$	$\hat{\theta}_p$	$\hat{\theta}_l$	
					$b_1 = 0.06$	$b_1 = 0.09$
1.0	0.5	0.5836 (0.0085)	0.5690 (3.622)	0.5908 (8.616)	0.5833 (0.0350)	0.5832 (0.0525)
	1.0	0.5690 (0.0083)	0.5544 (3.597)	0.5763 (8.403)	0.5688 (0.0341)	0.5686 (0.0512)
	1.5	0.5544 (0.0080)	0.5398 (3.571)	0.5617 (8.191)	0.5542 (0.0332)	0.5541 (0.0499)
2.0	0.5	0.5836 (0.0085)	0.5690 (3.622)	0.5908 (8.616)	0.5833 (0.0350)	0.5832 (0.0525)
	1.0	0.5690 (0.0083)	0.5544 (3.597)	0.5763 (8.403)	0.5688 (0.0341)	0.5686 (0.0512)
	1.5	0.5544 (0.0080)	0.5398 (3.571)	0.5617 (8.191)	0.5542 (0.0332)	0.5541 (0.0499)

$\hat{\theta}_s =$ Square error loss function, $\hat{\theta}_e =$ Estimation under Entropy,

$\hat{\theta}_p =$ Estimation under Precautionary, $\hat{\theta}_l =$ Estimation under LINEX

Table 6.2

Bayes Estimation and Posterior Risks Using GammaPrior

$\hat{\theta}$	a	b	$\hat{\theta}_s$	$\hat{\theta}_e$	$\hat{\theta}_p$	$\hat{\theta}_l$	
						$b_1 = 0.06$	$b_1 = 0.09$
1.0	0.5	0.5	0.5866 (0.0084)	0.5721 (4.247)	0.5793 (0.5699)	0.5864 (0.0351)	0.5862 (0.0527)
	0.5	1.0	0.5939 (0.0086)	0.5794 (4.247)	0.5866 (0.5771)	0.5936 (0.0356)	0.5935 (0.0534)
	1.0	0.5	0.5824 (0.0083)	0.5680 (4.254)	0.5752 (0.5637)	0.5821 (0.0349)	0.5820 (0.0524)
2.0	0.5	0.5	0.5866 (0.0084)	0.5721 (4.247)	0.5793 (0.5699)	0.5864 (0.0351)	0.5862 (0.0527)
	0.5	1.0	0.5939 (0.0086)	0.5794 (4.247)	0.5866 (0.5771)	0.5936 (0.0356)	0.5935 (0.0534)
	1.0	0.5	0.5824 (0.0083)	0.5680 (4.254)	0.5752 (0.5637)	0.5821 (0.0349)	0.5820 (0.0524)

Among other loss functions, it is evident from Table 6.1 and Table 6.2. That the square error loss function shows smaller Bayes posterior risk under the both assumptions (extended Jeffery's prior and gamma prior). According to decision rule of less Bayes posterior risk, we accomplish that square error loss function is more useful than others.

Conclusion

In this paper, we have initially obtained the Bayes posterior distribution and estimation of parameter of the inverse Topp-Leone distribution under both informative and non-informative priors. We have discussed different loss functions among them square error loss function provide less Bayes posterior risk. Eventually through simulation analysis and application, the performance of the estimators has been achieved.

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