Quantitative Method of Singularly Perturbed Delay Differential Equations

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Abstract – **In this paper, singularly perturbed delay differential equation is solved by converting to first order ordinary differential equation and using two- point Gaussian quadrature rule to approximate the solution. Numerically the values of the derivatives and the integrals are approximated to get tridiagonal system. Thomas algorithm is used to solve the tridiagonal system of the method. Convergence of the proposed method is analyzed**

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Keywords: **singularly perturbed delay differential equation, Taylor series, Boundary Layer, Tridiagonal system**

I. Introduction

Singularly perturbed differential-difference equations (SPDDEs) arise very frequently in the mathematical modeling of real life situations in science and engineering variational problems in control theory[2],[6],[14]. In general, any ordinary differential equation in which the highest derivative is multiplied by a small positive parameter and containing at least one delay/advance parameter is known as singularly perturbed differential difference equation. Lange and Miura [14]-[18] published a series of papers extending the method of matched asymptotic expansions initially developed for ordinary differential equations to obtain approximate solution of singularly perturbed differential-difference equations. Numerical analysis of singularly perturbed differential - difference turning point problems was initiated by Kadalbajoo and Sharma. In a series of papers, see [8]-[10], they gave many robust numerical techniques for the solution of such type of problems. Kadalbajoo and Sharma [8] elucidate a numerical method to solve boundary value problems for singularly perturbed differential difference equation of the form: , under the interval and boundary conditions on and on . Kadalbajoo and Sharma [9] proposed a numerical method to solve boundary-value problems for a singularly perturbed differential-difference equation of a mixed type, i.e. which contains both types of terms having negative shifts as well as positive shifts, and considered the case in which the solution of the problem exhibits rapid oscillations. Kadalbajoo and Sharma [10] described a numerical approach based on finite difference method to solve a mathematical model arising from a model of neuronal variability. Patidar and Sharma [20] combined fitted-operator methods with Micken's non-standard finite difference techniques for the numerical approximations of singularly perturbed linear delay differential equations. Kadalbajoo et al. [11] derived ϵ-uniformly convergent fitted methods for the solution of singularly perturbed differential difference equation (SPDDE). Kumar and Sharma [13] presented a numerical scheme based on B-spline collocation to approximate the solution of boundary value problems for singularly perturbed differential-difference equations with delay as well as advance. Amiraliyev and Cimen [5] derived a numerical method for singularly perturbed boundary value problem for a linear second order delay differential equation with a large delay in the reaction term. The authors constructed an exponentially fitted differential scheme on a uniform mesh accomplished by the method of integral identities with the use of exponential basis functions and interpolating quadrature rules with weight and the remainder term in the integral form.

II. Description of the problem

Consider a linear singularly perturbed two -point boundary value problem of the form:

$$\varepsilon y''(x)+a(x)y(x-\delta)+b(x)y(x)=f(x), 0<x<1, \quad (1)$$

on (0, 1), under the boundary conditions

$$y(x)=\varphi(x) \quad \text{on } -\delta \le x \le 0, \qquad y(1)=\beta \quad , \qquad (2)$$

where ε is small parameter, $0<\varepsilon<<1$ and δ is also delay parameter $0<\delta<1$ $a(x), b(x), f(x),$ and $\varphi(x)$ are bounded continuous functions in (0,1) and β is a finite constant. For $\delta = 0$ the solution of the boundary value problem (1) and (2) exhibits layer or oscillatory behavior depending on the sign of $\left(a(x)+b(x)\right)$. If $\left(a(x)+b(x)\right) \le -M < 0$, where M is a positive constant, the solution of the problem (1) and (2) exhibits layer behavior and if $\left(a(x)+b(x)\right) \le M > 0$, it exhibits oscillatory behavior. The boundary value Problem considered here is of the reaction-diffusion type, therefore if the solution exhibits layer behavior, there will be two boundary layers which will be at both end point i.e., at $x=0$ and $x=1$.

III. Numerical Method

We divide the interval $[0,1]$ into N equal parts of mesh size h, with $0 = x_0, x_1, ..., x_{N-1}, x_N = 1$ being the mesh points. Hence, we have $x_i = ih,$ for $i = 0$ *to* N. We choose n such that $x_n = \frac{1}{2}$.In the Interval $\left[0,\frac{1}{2}\right]$,the boundary layer will be at the left hand side i.e., at $x=0$ and in $\left[\frac{1}{2},1\right]$ the boundary layer will be at the right hand side i.e., at $x=1$. Therefore we calculate the solution for three Different intervals $\left[0,\frac{1}{2}\right], \frac{1}{2}$ and $\left[\frac{1}{2},1\right]$.

III.1 Left – End Layer

For left hand layer i.e., $0 \le x \le \frac{1}{2}$, by using Taylor's expansion for $y'(x-\varepsilon)$,we can say that,

$$y'(\text{x-}\varepsilon) \approx y'(x) - \varepsilon y''(x)$$

Hence,

$$\varepsilon y''(\text{x}) \approx y'(x) - y'(x-\varepsilon)$$

By using this approximation in (1) we get,

$$y'(x) \approx y'(x-\varepsilon) - a(x)y(x-\delta) - b(x)y(x) + f(x) \quad (3)$$

Inregrating this equation from x_i *to* x_{i+1} we get,

$$y_{i+1} - y_i = \int_{x_i}^{x_{i+1}} y'(x-\varepsilon)dx + \int_{x_i}^{x_{i+1}} a(x)y(x-\delta)dx - \int_{x_i}^{x_{i+1}} b(x)y(x)dx + \int_{x_i}^{x_{i+1}} f(x)dx \quad (4)$$

We use Gause-Legendre quadrature to approximate the value of this integration, for which we need to change the limits to -1 to 1. For this purpose, we introduce a new variable u

Where,

$$u = \frac{2x - (x_i + x_{i+1})}{x_{i+1} - x_i} \Rightarrow x = \frac{uh + (x_i + x_{i+1})}{2}$$

Using this approximation in (4) we get

$$y_{i+1} - y_i = y(\text{x}_{i+1} - \varepsilon) - y(\text{x}_i - \varepsilon) \quad (5)$$

Using two- point Gaussian quadrature we get

$$\int_{-1}^{1} f(x)dx \approx f\left(\frac{1}{\sqrt{3}}\right) + f\left(\frac{-1}{\sqrt{3}}\right),$$

For any function f, continuous and differentiable in interval [-1,1].

when $u = \frac{1}{\sqrt{3}}$,

$$\frac{uh + x_i + x_{i+1}}{2} = x_{i+1} - \frac{h\left(1 - \frac{1}{\sqrt{3}}\right)}{2} = x_{i+1} - k,$$

where $\text{k} = \frac{h\left(1 - \frac{1}{\sqrt{3}}\right)}{2}$,

$$u = \frac{-1}{\sqrt{3}}, \ \frac{uh + x_i + x_{i+1}}{2} = x_i + \frac{h\left(1 - \frac{1}{\sqrt{3}}\right)}{2} = x_i + k,$$

Where $\text{k} = \frac{h\left(1 - \frac{1}{\sqrt{3}}\right)}{2}$

Substituting this in (5) we get

$$y_{i+1} - y_i = y(x_{i+1} - \varepsilon) - y(x_i - \varepsilon) - \frac{h}{2}\left[\begin{aligned} & a(x_{i+1} - k)y(x_{i+1} - k - \delta) + a(x_i + k)y(x_i + k - \delta) \\ & + b(x_{i+1} - k)y(x_{i+1} - k) + b(x_i + k)y(x_i + k) - f(x_{i+1} - k) - f(x_i + k) \end{aligned}\right]$$

(6)

Now we use following approximations for the y values,

$$y(x_{i+1} - k - \delta) = y_{i+1} - (k+\delta)y'_{i+1},$$

$$= y_{i+1} - \frac{(k+\delta)(y_{i+1} - y_i)}{h},$$

$$= \frac{k+\delta}{h} y_i + \left(1 - \frac{k+\delta}{h}\right) y_{i+1}.$$

Similarly,

$$y(x_i + k - \delta) = \left[1 + \frac{k-\delta}{h}\right] y_i - \frac{k-\delta}{h} y_{i-1},$$

$$y(x_{i+1} - k) = \frac{k}{h} y_i + \left(1 - \frac{k}{h}\right) y_{i+1},$$

$$y(x_i + k) = \left(1 + \frac{k}{h}\right) y_i - \frac{k}{h} y_{i-1}$$

By substituting this in (6) we get,

$$\frac{h\left[f(x_{i+1}-k)+f(x_i+k)\right]}{2}=$$
$$\left[\frac{\varepsilon}{h}-a(x_i+k)\frac{k-\delta}{2}-b(x_i+k)\frac{k}{2}\right]y_{i-1}+$$
$$\left[\frac{-2\varepsilon}{h}+a(x_{i+1}-k)\frac{k+\delta}{2}+a(x_i+k)\frac{h+k-\delta}{2}+b(x_{i+1}-k)\frac{k}{2}+b(x_i+k)\frac{h+k}{2}\right]y_i$$
$$+\left[\frac{\varepsilon}{h}+a(x_{i+1}-k)\frac{h-k-\delta}{2}+b(x_{i+1}-k)\frac{h-k}{2}\right]y_{i+1}. \quad (7)$$

We can say that,

$$A_i y_{i-1}+B_i y_i+C_i y_{i+1}=b_i,$$

For $1\le i\le n-1,$ where

$$A_i=\frac{\varepsilon}{h}-a(x_i+k)\frac{k-\delta}{2}-b(x_i+k)\frac{k}{2},$$
$$B_i=\frac{-2\varepsilon}{h}+a(x_{i+1}-k)\frac{k+\delta}{2}+a(x_i+k)\frac{h+k-\delta}{2}+b(x_{i+1}-k)\frac{k}{2}+b(x_i+k)\frac{h+k}{2},$$
$$C_i=\frac{\varepsilon}{h}+a(x_{i+1}-k)\frac{h-k-\delta}{2}+b(x_{i+1}-k)\frac{h-k}{2},$$
$$b_i=\frac{h\left[f(x_{i+1}-k)+f(x_i+k)\right]}{2}.$$

III.2 Right – End layer

For right hand layer i.e., $\frac{1}{2}\le x\le 1,$ by using Taylor's expansion for $y'(x+\varepsilon)$ we get,

$$y'(x+\varepsilon)\approx y'(x)+\varepsilon y''(x)$$

Hence, $\varepsilon y''(x)\approx y'(x+\varepsilon)-y'(x)$

By using this approximation in (1) we get,

$$y'(x)=y'(x+\varepsilon)+a(x)y(x-\delta)+b(x)y(x)-f(x) \quad (8)$$

Integrating (8) from x_{i-1} to x_i gives us,

$$y_i-y_{i-1}=\int_{x_{i-1}}^{x_i}y'(x+\varepsilon)dx+\int_{x_{i-1}}^{x_i}a(x)y(x-\delta)dx+\int_{x_{i-1}}^{x_i}b(x)y(x)dx-\int_{x_{i-1}}^{x_i}f(x)dx \quad (9)$$

Here the value of u for using Gauss-Legendre Quadrature will be,

$$u=\frac{2x-(x_{i-1}+x_i)}{x_i-x_{i-1}}\Rightarrow x=\frac{uh+x_{i-1}+x_i}{2}$$

$$u=\frac{1}{\sqrt{3}},$$

$$\frac{uh+x_{i-1}+x_i}{2}=x_i-\frac{h\left(1-\frac{1}{\sqrt{3}}\right)}{2}=x_i-k,$$

Where $k=\frac{h\left(1-\frac{1}{\sqrt{3}}\right)}{2},$

At $u=\frac{-1}{\sqrt{3}},$

$$\frac{uh+x_{i-1}+x_i}{2}=x_{i-1}+\frac{h\left(1-\frac{1}{\sqrt{3}}\right)}{2}=x_{i-1}+k,$$

Where $k=\frac{h\left(1-\frac{1}{\sqrt{3}}\right)}{2},$

Substituting this is (8) giyes us,

$$y_i-y_{i-1}=y(x_i+\varepsilon)-y(x_{i-1}+\varepsilon)+\frac{h}{2}\left[\begin{array}{l}a(x_i-k)y(x_i-k-\delta)+a(x_{i-1}+k)y(x_{i-1}+k-\delta)\\+b(x_i-k)y(x_i-k)+b(x_{i-1}+k)y(x_{i-1}+k)-f(x_i-k)-f(x_{i-1}+k)\end{array}\right] \quad (10)$$

Now we use following approximations for the y values

$$y(x_i-k-\delta)=y_i-(k+\delta)y_i',$$
$$=y_i-\frac{(k+\delta)(y_{i+1}-y_i)}{h},$$
$$=\left[1+\frac{k+\delta}{h}\right]y_i-\frac{k+\delta}{h}y_{i+1}.$$

Similarly,

$$y(x_{i-1}+k-\delta)=\left[1-\frac{k-\delta}{h}\right]y_{i-1}+\frac{k-\delta}{h}y_i,$$
$$y(x_i-k)=\left(1+\frac{k}{h}\right)y_i-\frac{k}{h}y_{i+1},$$
$$y(x_{i-1}+k)=\left(1-\frac{k}{h}\right)y_{i-1}+\frac{k}{h}y_i.$$

By substituting this in (10) we get,

$$\frac{h\left[f(x_i-k)+f(x_{i-1}+k)\right]}{2}=\left[\frac{\varepsilon}{h}+a(x_{i-1}+k)\frac{h-k+\delta}{2}+b(x_{i-1}+k)\frac{h-k}{2}\right]y_{i-1}$$
$$+\left[\frac{-2\varepsilon}{h}-a(x_i-k)\frac{h+k+\delta}{2}-a(x_{i-1}+k)\frac{k-\delta}{2}-b(x_i-k)\frac{h+k}{2}-b(x_{i-1}+k)\frac{k}{2}\right]y_i$$
$$+\left[\frac{\varepsilon}{h}-a(x_i-k)\frac{k+\delta}{2}-b(x_i-k)\frac{k}{2}\right]y_{i+1} \quad (11)$$

We can say that,

$$G_i y_{i-1}+H_i y_i+I_i y_{i+1}=b_i,$$

for $n\le i\le N-1$, where ,

$$G_i=\frac{\varepsilon}{h}+a(x_{i-1}+k)\frac{h-k+\delta}{2}+b(x_{i-1}+k)\frac{h-k}{2}$$

$$H_i = \frac{-2\varepsilon}{h} - a(x_i - k)\frac{h+k+\delta}{2} - a(x_{i-1}+k)\frac{k-\delta}{2} - b(x_i - k)\frac{h+k}{2} - b(x_{i-1}+k)\frac{k}{2}$$

$$I_i = \frac{\varepsilon}{h} + a(x_{i+1}-k)\frac{h-k-\delta}{2} + b(x_{i+1}-k)\frac{h-k}{2}$$

$$b_i = \frac{h\left[f(x_i - k) + f(x_{i-1}+k)\right]}{2}$$

For $x_n = \frac{1}{2}$,

$$\varepsilon y''(x_n) + a(x_n)\, y(x_n - \delta) + b(x_n)\, y(x_n) = f(x_n)$$

(12)

We use following approximations in this equation.

$$y(x_n - \delta) = y(x_n) - \delta y'(x_n) = y_n - \delta \frac{y_{n+1} - y_{n-1}}{2h},$$

$$y''(x_n) = \frac{y_{n+1} - 2y_n + y_{n-1}}{h^2}$$

To get,

$$f_n = \varepsilon\left[\frac{y_{n+1} - 2y_n + y_{n-1}}{h^2}\right] + a_n\left[y_n - \delta\frac{y_{n+1} - y_{n-1}}{2h}\right] + b_n y_n$$

(13)

Rearranging this equation gives us,

$$f_n = \left[\frac{\varepsilon}{h^2} - a_n\frac{\delta}{2h}\right] y_{n+1} + \left[2\frac{\varepsilon}{h^2} - a_n - b_n\right] y_n + \left[\frac{\varepsilon}{h^2} + a_n\frac{\delta}{2h}\right] y_{n-1}$$

(14)

We can say that,

$$D_n y_{n-1} + E_n y_n + F_n y_{n+1} = b_n,$$

Where,

$$D_n = \frac{\varepsilon}{h^2} + a_n\frac{\delta}{2h},$$

$$E_n = 2\frac{\varepsilon}{h^2} - a_n - b_n,$$

$$F_n = \frac{\varepsilon}{h^2} - a_n\frac{\delta}{2h},$$

$$b_n = f_n.$$

Finally we can write the formulation in the matrix form as, $Ay = b$, where

$$A = \begin{bmatrix} B_1 & C_1 & 0 & \cdots & \cdots & \cdots & 0 \\ A_2 & B_2 & C_2 & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & A_{n-1} & B_{n-1} & C_{n-1} & \ddots & \ddots & \vdots \\ \vdots & \ddots & D_n & E_n & F_n & \ddots & \vdots \\ \vdots & \ddots & \ddots & G_{n+1} & H_{n+1} & I_{n+1} & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & G_{N-2} & H_{N-2} & I_{N-2} \\ 0 & \cdots & \cdots & \cdots & 0 & G_{N-1} & H_{N-1} \end{bmatrix},$$

$$y = [y_1 \cdots y_{N-1}]',$$

$$b = [b_1 - A_1 y_0 \;\; b_2 \cdots b_{N-2} \;\; b_{N-1} - C_{N-1} y_N]'$$

IV. Numerical Example

An example of this type of equation is,

$$\varepsilon y''(x) - 2y(x-\delta) - y(x) = 1,\ y(0) = 1,$$

$$-\delta \le x \le 0 \text{ and } y(1) = 1$$

For this equation, solution was calculated for N =20,40,80,160,and 320,for different values of ε and δ.

TABLE I

THE MAXIMUM ERRORS IN SOLUTION OF EXAMPLE 1 WITH $\varepsilon = 0.1$

$\delta \downarrow$	N=100	N = 200	N=300	N=400
0.01	3.157 e-003	1.605e-003	1.075e-003	8.083e-010
0.04	3.143 e-003	1.599e-003	1.075e-003	8.033e-004
0.06	3.078 e-003	1.566e-003	1.050e-003	7.9001e-014

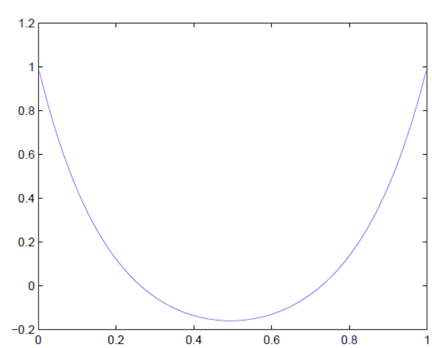

Fig.1.Numerical solution for $\varepsilon = 0.1$ and $\delta = 0.01$

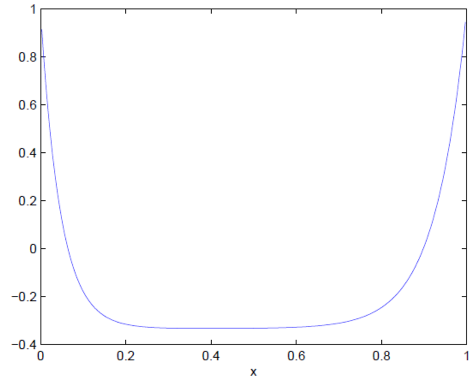

Fig.2.Numerical solution for $\varepsilon = 0.01$ and $\delta = 0.04$

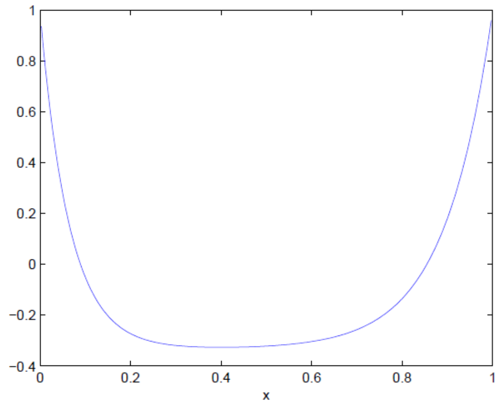

Fig.3..Numerical solution for $\varepsilon = 0.02$ and $\delta = 0.06$

Error term is given by,

$$|u - u_h| = ch^p,$$

Where u is the exact solution, u_h is the numerical solution is a constant, h is the mesh size and p is the order of convergence.

Here, as the exact solution is not knows we assume the numerical solution at N =320 to be the exact solution and use other numerical solutions to plot $\log|u - u_h|$ versus $\log h$, for $\varepsilon = 0.01$ and $\delta = 0.04$,to obtain the convergence plot. Slope of convergence plot is equal to the order of convergence.

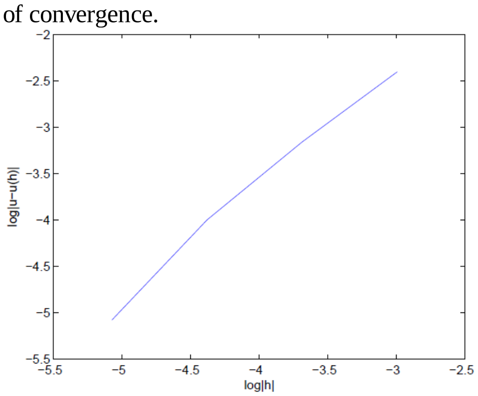

Fig.4. Convergence Plot

From this plot, it can be said that order of convergence for this method is 1.

V.Conclusion

The numerical analysis of singularly perturbed delay differential equation is discussed. In general numerical solution of second order differential equation will be more difficult than numerical solution of first order differential equation. Even though the numerical results are computed at all points of the mesh size h, only few results have been reported. It is observed that the present method approximates the exact solution very well for $h \geq \varepsilon$ for which other classical finite difference methods fail to give good results. The effect of the delay parameter on the solution has also investigated and presented by using graphs. Convergence of the proposed method is analyzed. The present method is simple, very easy to implement and efficient technique for solving singularly perturbed delay differential equations.

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