Graphs associated with a commutative group (Z_n, o)

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Abstract- Due to the interplay between group/ring theory and graph theory many of researches attempted to develop the extensive studies on various graphs arising, by defining adjacency using various algebraic conditions. In this paper the author has taken a similar approach and studied such type of graphs by using algebraic properties like square of the values, idem potency etc.

Key words: Idempotent, bipartite graph ,regular graph .

I.INTRODUCTION

Basic concepts of group theory:

2.1 Introduction: Algebraic structures whose binary operations satisfy particularly important properties are semi groups group, rings ,fields ,modules, and so on. The simplest algebraic structure to recognize is a semi group ,which is defined as a non empty set S with an associative binary operation. Any algebraic structure S with a binary operation $'+' or$ $'.'$ is normally written $(S,+) or (S, .)$.

Preliminaries:

Group: A non empty set S with a binary operation '.' on S is called a group if the following axioms hold:
(i) $a(bc) = (ab)c$ for all $a,b,c \in S$.
(ii) There exists $e \in S$ such that $ea = a$ $\forall a \in S$.
(iii)For every $a \in S$ $\exists$ $a' \in S$ such that $aa' = e$.
Note: $'o'$ is with respect to ' + ' and ' .'

Semi group: A semi group is a pair $(S,.)$ in which S is a non empty set and ' .' is a Binary associative operation on S, i.e. the equation $a(bc) = (ab)c$ holds $\forall a,b,c \in S$ [1].

Ring: By a ring we mean a non empty set R with two binary operations $'+'$ and ' .' called addition and multiplication(also called product), respectively, such that
(i) $(R, +)$ is an addition abelian group .

(ii) $(R, +)$ is a multiplicative semi group .
(iii) Multiplication is distributive(on both sides) over addition ; i.e. $\forall a,b,c \in R$
$a.(b+c) = a.b + a.c$, $(a+b).c = a.c + b.c$.
Commutative group: The group $(S,+)$ is called a commutative group if it satisfies Commutative i.e. $a.b = b.a \ \forall a,b \in S$
Idempotent group: In a group (S, o) the identity element e (0&1) is the only idempotent element. Indeed , x is an element of S such that $x^2 = x, \forall x \in S$.

Addition modulo(m):

Definition: Let $a,b \in Z$ and $'m'$ be a fixed positive integer. If $'r'$ is the remainder $(0 \le r < m)$ when $a+b$ (ordinary sum of a,b) is divided by $'m'$,we define $a +_m b = r$ and we read $'a +_m b'$ as $'a'$ addition modulo $'m'$ b .

Example: (1) $20 +_6 5 = 1$. Since $20 + 5 = 4(6) + 1$ is the remainder when $20 + 5$ is divided by 6.
(2) $24 +_5 4 = 3$ (3) $2 +_7 3 = 5$

Example **:** $S = Z_3 = \{0,1,2\}$ is abelian group of order 3 with respect to modulo '3'

$+_3$	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

The composition table for S w.r.t $+_3$ is given
Now we can prove all the axioms of an abelian group
$\therefore (S,+_3)$ is a finite abelian group of order 3.

Multiplication modulo(m):

Definition: If $a \& b$ are integers and $'p'$ is a fixed positive integer and if ab (ordinary product of a,b) is divided by $'p'$ such that $'r'$ is the remainder $(0 \le r < p)$,we define $a \times_p b = r$ we read $a \times_p b$ as $'a'$ multiplication modulo $'p'$ b .

Example: (1) $20 \times_6 5 = 4$ .Since $20 \times 5 = 100 = 16(6) + 4$ i.e. is the remainder when 20×5 is divided by 6.
(2) $20 \times_5 4 = 1$ (3) $2 \times_7 3 = 6$.

Example : $S = Z_3 = \{0,1,2\}$ is abelian group of order 3 with respect to modulo '3'

$\times_3$	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

The composition table for S w.r.t $\times_3$ is given
Now we can prove all the axioms of an abelian group

$\therefore\ (S, \times_3)$ is a finite abelian group of order 3.

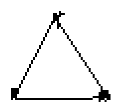
Basic concepts and properties of graph theory:

Graph theory Introduction: In mathematics, graphs are useful on geometry and certain parts of topology, for example knot theory. Algebraic graph theory has close links with group theory. In this paper we discuss the graph theoretic properties on the group z_n .That is throughout this paper, we consider the group $(z_n, +_n)$,where $z_n = \{0,1,2 \ldots\ldots\ldots\ldots\}$ with the binary operation '$+_n$ '(addition modulo n). z_n is called the group of integers modulo n. The graph on the group z_n is denoted G=(V,E) and is called the graph of z_n with vertex set V(G) and edge set E(G).The elements of z_n are considered as the vertices of G and edge set of G is defined by assuming condition on the elements of z_n .It is observed that .The resultant graphs obtained on group z_n are regular graphs[2-7].

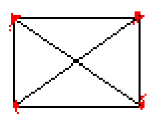
Preliminaries:

Graph: $G = (V, E)$ consists of a set of objects $V = \{v_1, v_2, ...\}$ called vertices, and another set $E = \{e_1, e_2, ...\}$ whose elements are called edges , such that each edge e_k is identified with an unordered pair (v_i, v_j) of vertices.

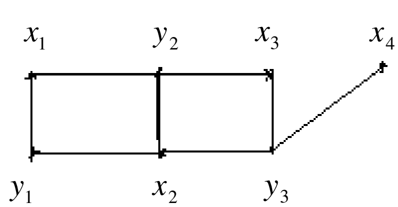
Regular graph: If each vertex of G has degree k then G is said to be k-regular graph



Complete graph : A complete graph is a simple graph in which every pair of vertices can be joined by an edge complete graph of n-vertices is denoted by K_n .



Bipartite Graph : Let G be a graph. If the vertex set V of G can be portioned in to two non-empty subsets X and Y $(X \cup Y = V,\ X \cap Y = \phi)$ in such way that each edge of G has one end in X and one end in Y , then G is called bipartite. The partition $X \cup Y = V$ is called a bipartition of G.



Walk: A walk is defined as a finite alternative sequence of vertices and edges.

Example: $v_i e_j v_{i+1} e_{j+1} \ldots\ldots\ldots\ldots\ldots e_k v_m$.

Path: In a walk , a vertex can not appear more than once.

2. Graphs and their illustration on group Z_n :

This section deals with the graphs obtained on the elements of z_n, which are considered as the vertices of the vertices of the graph G of $Z_n (= S)$. The resultant graph obtained is regular graphs or complete graphs or bipartite graphs.

Theorem 2.2.1: Let $S = Z_n$ be a commutative group of integers multiplication mod(n) and let $G = (V, E)$ denote the graph of S with vertex set $V(S)$ where the elements of S are taken on the vertices in V and edge set $E(S)$ is defined as

.(i) If $E(S) = \{x, y \in S, x \neq y / x \text{ and } y \text{ are adjacent iff their square values are same } \}$

Then the resultant graph is bipartite graph.

.(ii) If $E(S) = \{x, y \in S, x \neq y / x \text{ and } y \text{ are adjacent iff they are idempotent } \}$

Then the resultant graph is complete regular graph.

<u>Proof:</u>

Given that S be a commutative group of integers multiplication mod(n) and let $G = (V, E)$ denote the graph of G with vertex set $V(S)$ and edge set $E(S)$.

where $E(S)$ is defined by

.Case(i):

$E(S) = \{x, y \in S, x \neq y / x \text{ and } y \text{ are adjacent iff their square values are same } \}$

For n is odd there does not exists elements in S whose square values are same.

If n is even then it is observed that for $n = 2$ no elements have the same square values.

If $n > 2$ then $n = 6$ we know that for any two elements a, b $(a \neq b)$.

If $n = a + b$ or $n / a + b$ then a and b are have the same square values and in that case no elements other than $a, b \in Z_n$ satisfies the above condition with either a or b .

Then we say that 'a' and 'b' are adjacent i.e. disjoint pair of elements in Z_n satisfies the condition $n / a + b$ or $n = a + b$.

i.e. no other elements of Z_n are adjacent to a or b .

In the graph $G(S)$, distinct pair of elements of S are adjacent.

Hence the resultant graph $G(S)$ is a bipartite graph[2].

Case(ii):

If the edge set $E(S)$ of the graph G is defined as

$E(S) = \{x, y \in S, x \neq y / x \text{ and } y \text{ are adjacent iff they are idempotent } \}$

Then we observed that in general for any n , there are two trivial idempotents.

i.e. '0' and '1' ($0^2 = 0, 1^2 = 1$)

There is a non trivial idempotent in any group Z_n if and only if for $1 < a < n$ such that $n / a(a-1)$. This is because $a^2 \equiv a (\mathrm{mod}\, n) \Leftrightarrow a^2 - a \equiv 0 (\mathrm{mod}\, n)$.

Since a and $a - 1$ are relatively prime , we get that such 'a' exists if and only if n divides the product of two relatively prime consecutive non trivial elements a , $a - 1$.

Then the resultant graph is complete regular graph[2].

Illustration 2.2.2 : When we consider the number of elements $n \; in \; Z_n$ **has** $n = 4t + 2, t = 123,.........$ **then the resultant bipartite** (m, n) **graph /regular graphs are in A.P series**

If $Z_n = \{0,1,2,3,4,........n-1.\}$.Then find square of each element.

.(i) If $E(S) = \{x, y \in S, x \neq y / x \; and \; y \; are \; adjacent \; iff \; their \, square \; values \; are \; same\}$

Then the resultant graph is bipartite graph.

.(ii) If $E(S) = \{x, y \in S, x \neq y / x \; and \; y \; are \; adjacent \; iff \; they \; are \; idempotent\}$

Then the resultant graph is complete regular graph.

(1) If $Z_6 = \{0,1,2,3,4,5.\}$. Then find square of each element

To find squares:

$0^2 = 0(\bmod 6)$

$1^2 = 1(\bmod 6)$

$2^2 = 4(\bmod 6)$

$3^2 = 3(\bmod 6)$

$4^2 = 4(\bmod 6)$

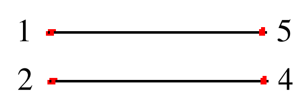
$5^2 = 1(\bmod 6)$

X_6	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	0	2	4
3	0	3	0	3	0	3
4	0	4	2	0	4	2
5	0	5	4	3	2	1

Case(i): If two non zero elements $x, \; y \in Z_6$ are adjacent iff their square values are same.

$1^2 = 1(\bmod 6)$, $5^2 = 1(\bmod 6)$

$2^2 = 4(\bmod 6)$, $4^2 = 4(\bmod 6)$



$(2,2) - Bipartite \; graph$

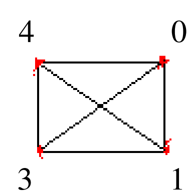
Case(ii): If two non zero elements $x, \; y \in Z_6$ are adjacent iff x and y are idempotent[3,4].

$0^2 = 0(\bmod 6)$

$1^2 = 1(\bmod 6)$

$3^2 = 3(\bmod 6)$

$4^2 = 4(\bmod 6)$



$3 - Complete \; regular \; graph$

(2) If $Z_{10} = \{0,1,2,3,4,5,6,7,8,9.\}$. Then find square of each element.

<u>To find squares</u>:

$0^2 = 0(\bmod 10)$
$1^2 = 1(\bmod 10)$
$2^2 = 4(\bmod 10)$
$3^2 = 9(\bmod 10)$
$4^2 = 6(\bmod 10)$
$5^2 = 5(\bmod 10)$
$6^2 = 6(\bmod 10)$
$7^2 = 9(\bmod 10)$
$8^2 = 4(\bmod 10)$
$9^2 = 1(\bmod 10)$

X_{10}	0	1	2	3	4	5	6	7	8	9
0	0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8	9
2	0	2	4	6	8	0	2	4	6	8
3	0	3	6	9	2	5	8	1	4	7
4	0	4	8	2	6	0	4	8	2	6
5	0	5	0	5	0	5	0	5	0	5
6	0	6	2	8	4	0	6	2	8	4
7	0	7	4	1	8	5	2	9	6	3
8	0	8	6	4	2	0	8	6	4	2
9	0	9	8	7	6	5	4	3	2	1

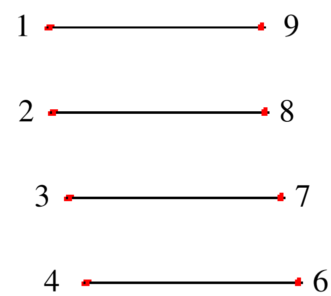
Case(i) If two non zero elements $x, y \in Z_{10}$ are adjacent iff their square values are same.

$1^2 = 1(\bmod 10)$, $9^2 = 1(\bmod 10)$

$2^2 = 4(\bmod 10)$, $8^2 = 4(\bmod 10)$

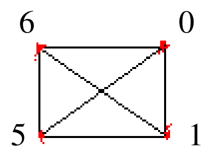
$3^2 = 9(\bmod 10)$, $7^2 = 9(\bmod 10)$

$4^2 = 6(\bmod 10)$, $6^2 = 6(\bmod 10)$



$(4,4) -$ *Bipartite graph*

Case(ii): If two non zero elements $x, y \in Z_{10}$ are adjacent iff x and y are idempotent



$0^2 = 0(\bmod 10)$
$1^2 = 1(\bmod 10)$
$5^2 = 5(\bmod 10)$, $6^2 = 6(\bmod 10)$

$3 -$ *Complete regular graph*

(3) Square elements with respect to $\times_{14}$

$0^2 = 0(\bmod 14)$ $\quad 7^2 = 7(\bmod 14)$
$1^2 = 1(\bmod 14)$ $\quad 8^2 = 8(\bmod 14)$
$2^2 = 4(\bmod 14)$ $\quad 9^2 = 11(\bmod 14)$

$3^2 = 9(\bmod 14)$ $10^2 = 2(\bmod 14)$

$4^2 = 2(\bmod 14)$ $11^2 = 9(\bmod 14)$

$5^2 = 11(\bmod 14)$ $12^2 = 4(\bmod 14)$

$6^2 = 8(\bmod 14)$ $13^2 = 1(\bmod 14)$

Case(i)): If two non zero elements $x,\ y \in Z_{14}$ are adjacent iff their square values are same

$1^2 = 1(\bmod 14)$, $13^2 = 1(\bmod 14)$

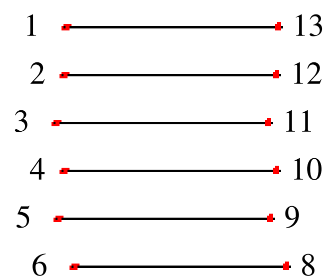
$2^2 = 4(\bmod 14)$, $12^2 = 4(\bmod 14)$

$3^2 = 9(\bmod 14)$, $11^2 = 9(\bmod 14)$

$4^2 = 2(\bmod 14)$, $10^2 = 2(\bmod 14)$

$5^2 = 11(\bmod 14)$, $9^2 = 11(\bmod 14)$

$6^2 = 8(\bmod 14)$, $8^2 = 8(\bmod 14)$

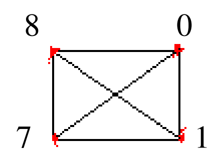


$(6,6) -$ *Bipartite graph*

Case(ii): If two non zero elements $x,\ y \in Z_{14}$ are adjacent iff x and y are idempotent

$0^2 = 0(\bmod 14)$, $1^2 = 1(\bmod 14)$

$7^2 = 7(\bmod 14)$, $8^2 = 8(\bmod 14)$



$3 -$ *Complete regular graph*

(4) Square elements with respect to $\times_{18}$

$0^2 = 0(\bmod 18)$ $9^2 = 9(\bmod 18)$

$1^2 = 1(\bmod 18)$ $10^2 = 10(\bmod 18)$

$2^2 = 4(\bmod 18)$ $11^2 = 13(\bmod 18)$

$3^2 = 9(\bmod 18)$ $12^2 = 0(\bmod 18)$

$4^2 = 16(\bmod 18)$ $13^2 = 7(\bmod 18)$

$5^2 = 7(\bmod 18)$ $14^2 = 16(\bmod 18)$

$6^2 = 0 (\mod 18)$ $15^2 = 9 (\mod 18)$

$7^2 = 13 (\mod 18)$ $16^2 = 4 (\mod 18)$

$8^2 = 10 (\mod 18)$ $17^2 = 1 (\mod 18)$

Case(i) If two non zero elements $x, y \in Z_{18}$ are adjacent iff their square values are same.

$1^2 = 1 (\mod 18)$, $17^2 = 1 (\mod 18)$

$2^2 = 4 (\mod 18)$, $16^2 = 4 (\mod 18)$

$3^2 = 9 (\mod 18)$, $15^2 = 9 (\mod 18)$

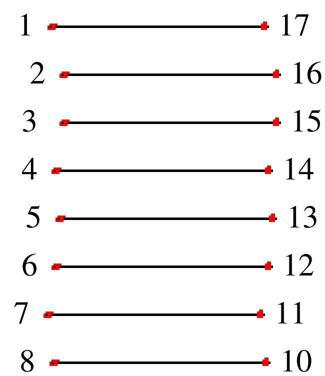
$4^2 = 16 (\mod 18)$, $14^2 = 16 (\mod 18)$

$5^2 = 7 (\mod 18)$, $13^2 = 7 (\mod 18)$

$6^2 = 0 (\mod 18)$, $12^2 = 0 (\mod 18)$

$7^2 = 13 (\mod 18)$, $11^2 = 13 (\mod 18)$

$8^2 = 10 (\mod 18)$, $10^2 = 10 (\mod 18)$

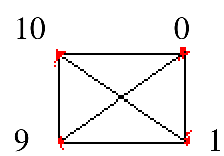


$(8,8) - Bipartite\ graph$

Case(ii): If two non zero elements $x, y \in Z_{18}$ are adjacent iff x and y are idempotent

$0^2 = 0 (\mod 18)$, $1^2 = 1 (\mod 18)$

$9^2 = 9 (\mod 18)$, $10^2 = 10 (\mod 18)$



$3 - Complete\ \ regular\ graph$

Conclusion: From the above illustration it is observed that for any two non zero elements in Z_n are adjacent iff their square values are same

Case(i) table The following table shows the relation between $n\ and\ G(S)$

Number of vertices, $n \geq 6$ $n = 4t+2, t = 1,2,3,.......$	$G(S) = (2t,2t) - Bipartite\ graph$ $t = 1,2,3,.........$
$t=1, n=6$ $t=2, n=10$ $t=3, n=14$ $t=4, n=18$	$t=1,\ (2,2) - Bipartite\ graph$ $t=2,\ (4,4) - Bipartite\ graph$ $t=3,\ (6,6) - Bipartite\ graph$ $t=4,\ (8,8) - Bipartite\ graph$ ... .

Conclusion: From the above illustration it is observed that for any two elements in Z_n are adjacent iff they can be written as idempotent element.

Case(ii) table The following table shows the relation between $n\ and\ G(S)$

Number of vertices, $n \geq 6$ $n = 4t+2, t = 1,2,3,.......$	$G(S) = 3 - Complete\ \ regular\ graph$
$t=1, n=6$ $t=2, n=10$ $t=3, n=14$ $t=4, n=18$	$3 - Complete\ \ regular\ graph$ $3 - Complete\ \ regular\ graph$ $3 - Complete\ \ regular\ graph$ $3 - Complete\ \ regular\ graph$ ... .

Corollary 2.2.3: Let $S = Z_n$ be a commutative group of integers addition $\bmod(n)$ and let $G = (V, E)$ denote the graph of S with vertex set $V(S)$ where the elements of S are taken on the vertices in V and edge set $E(S)$ is defined as

. $E(S) = \{x, y \in S, x \neq y / x\ and\ \ y\ are\ adjacent\ iff\ \ their\ square\ values\ are\ same\ \}$

Then the resultant graph is bipartite graph.[4]

.**Proof:**

Given that S be a commutative group of integers addition mod(n) and let $G=(V,E)$ denote the graph of G with vertex set $V(S)$ and edge set $E(S)$.[2]

where $E(S)$ is defined by

.Case(i):

$E(S)=\{x, y\in S, x\neq y / x \ and \ y \ are \ adjacent \ iff \ their \ square \ values \ are \ same\}$

For n is odd there does not exists elements in $Z_n(=S)$ whose square values are same.

If n is even then it is observed that for $n=2$ no elements have the same square values.

If $n>2$ then $n=6$ we know that for any two elements $a,b\ (a\neq b)$,

if n does not divide by $a+b$ then a and b are have the same square values.

Therefore a and b are adjacent.

By continuing this it is observed that for any two elements $a,b\ (a\neq b)\in Z_n(n \ is \ even)$

a, b have the same square values.

Then n does not divide by $a+b$ and

Hence the resultant graph is a $(2t+1,2t+1)$-bipartite graph.

Illustration 2.2.4 : **When we consider the number of elements** $n \ in \ Z_n$ **has** $n=4t+2, t=123,.........$ **then the resultant bipartite** (m,n) **graph /regular graphs are in A.P series**

If $Z_n=\{0,1,2,3,4,........n-1.\}$.Then find square of each element.

If two elements in Z_n are adjacent iff their square values are same

(i) If $Z_6=\{0,1,2,3,4,5.\}$. Then find square of each element[8]

To find squares:

$0^2=2(0)=0+0=0(\bmod 6)$

$1^2=2(1)=1+1=2(\bmod 6)$

$2^2=2(2)=2+2=4(\bmod 6)$

$3^2=2(3)=3+3=0(\bmod 6)$

$4^2=2(4)=4+4=2(\bmod 6)$

$5^2=2(5)=5+5=4(\bmod 6)$

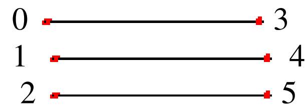
$+_6$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

(1)If two elements x, $y\in Z_6$ are adjacent iff their square values are same[4,

$0^2=2(0)=0+0=0(\bmod 6)$, $3^2=2(3)=3+3=0(\bmod 6)$

$1^2=2(1)=1+1=2(\bmod 6)$, $4^2=2(4)=4+4=2(\bmod 6)$

$2^2 = 2(2) = 2+2 = 4(\bmod 6)$, $5^2 = 2(5) = 5+5 = 4(\bmod 6)$



(3,3) – *Bipartite graph*

(2) If $Z_{10} = \{0,1,2,3,4,5,6,7,8,9.\}$. Then find square of each element.

If two elements x, $y \in Z_{10}$ are adjacent iff their square values are same.

To find squares:

$0^2 = 2(0) = 0+0 = 0(\bmod 10)$

$1^2 = 2(1) = 1+1 = 2(\bmod 10)$

$2^2 = 2(2) = 2+2 = 4(\bmod 10)$

$3^2 = 2(3) = 3+3 = 6(\bmod 10)$

$4^2 = 2(4) = 4+4 = 8(\bmod 10)$

$5^2 = 2(5) = 5+5 = 0(\bmod 10)$

$6^2 = 2(6) = 6+6 = 2(\bmod 10)$

$7^2 = 2(7) = 7+7 = 4(\bmod 10)$

$8^2 = 2(8) = 8+8 = 6(\bmod 10)$

$9^2 = 2(9) = 9+9 = 8(\bmod 10)$

$+_{10}$	0	1	2	3	4	5	6	7	8	9
0	0	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9	0
2	2	3	4	5	6	7	8	9	0	1
3	3	4	5	6	7	8	9	0	1	2
4	4	5	6	7	8	9	0	1	2	3
5	5	6	7	8	9	0	1	2	3	4
6	6	7	8	9	0	1	2	3	4	5
7	7	8	9	0	1	2	3	4	5	6
8	8	9	0	1	2	3	4	5	6	7
9	9	0	1	2	3	4	5	6	7	8

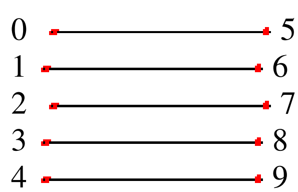
$0^2 = 2(0) = 0+0 = 0(\bmod 10)$, $5^2 = 2(5) = 5+5 = 0(\bmod 10)$

$1^2 = 2(1) = 1+1 = 2(\bmod 10)$, $6^2 = 2(6) = 6+6 = 2(\bmod 10)$

$2^2 = 2(2) = 2+2 = 4(\bmod 10)$, $7^2 = 2(7) = 7+7 = 4(\bmod 10)$

$3^2 = 2(3) = 3+3 = 6(\bmod 10)$, $8^2 = 2(8) = 8+8 = 6(\bmod 10)$

$4^2 = 2(4) = 4+4 = 8(\bmod 10)$, $9^2 = 2(9) = 9+9 = 8(\bmod 10)$.



(5,5) – *Bipartite graph*

(3) Square elements with respect to $+_{14}$

$0^2 = 2(0) = 0+0 = 0(\bmod 14)$ $8^2 = 2(8) = 8+8 = 2(\bmod 14)$

$1^2 = 2(1) = 1+1 = 2(\bmod 14)$ $9^2 = 2(9) = 9+9 = 4(\bmod 14)$

$2^2 = 2(2) = 2+2 = 4(\bmod 14)$ $10^2 = 2(10) = 10+10 = 6(\bmod 14)$

$3^2 = 2(3) = 3 + 3 = 6(\bmod 14)$ $11^2 = 2(11) = 11 + 11 = 8(\bmod 14)$

$4^2 = 2(4) = 4 + 4 = 8(\bmod 14)$ $12^2 = 2(12) = 12 + 12 = 10(\bmod 14)$

$5^2 = 2(5) = 5 + 5 = 10(\bmod 14)$ $13^2 = 2(13) = 13 + 13 = 12(\bmod 14)$

$6^2 = 2(6) = 6 + 6 = 12(\bmod 14)$

$7^2 = 2(7) = 7 + 7 = 0(\bmod 14)$

If two elements in are adjacent $x, y \in Z_{14}$ iff their square values are same

$0^2 = 2(0) = 0 + 0 = 0(\bmod 14)$, $7^2 = 2(7) = 7 + 7 = 0(\bmod 14)$

$1^2 = 2(1) = 1 + 1 = 2(\bmod 14)$, $8^2 = 2(8) = 8 + 8 = 2(\bmod 14)$

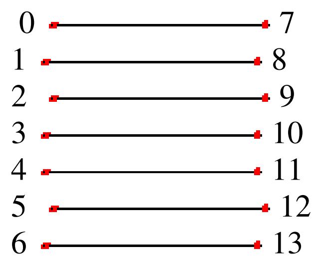
$2^2 = 2(2) = 2 + 2 = 4(\bmod 14)$, $9^2 = 2(9) = 9 + 9 = 4(\bmod 14)$

$3^2 = 2(3) = 3 + 3 = 6(\bmod 14)$, $10^2 = 2(10) = 10 + 10 = 6(\bmod 14)$

$4^2 = 2(4) = 4 + 4 = 8(\bmod 14)$, $11^2 = 2(11) = 11 + 11 = 8(\bmod 14)$

$5^2 = 2(5) = 5 + 5 = 10(\bmod 14)$, $12^2 = 2(12) = 12 + 12 = 10(\bmod 14)$

$6^2 = 2(6) = 6 + 6 = 12(\bmod 14)$, $13^2 = 2(13) = 13 + 13 = 12(\bmod 14)$.



$(7,7) - Bipartite\ graph$

(4) Square elements with respect to $+_{18}$

$0^2 = 2(0) = 0 + 0 = 0(\bmod 18)$ $9^2 = 2(9) = 9 + 9 = 0(\bmod 18)$

$1^2 = 2(1) = 1 + 1 = 2(\bmod 18)$ $10^2 = 2(10) = 10 + 10 = 2(\bmod 18)$

$2^2 = 2(2) = 2 + 2 = 4(\bmod 18)$ $11^2 = 2(11) = 11 + 11 = 4(\bmod 18)$

$3^2 = 2(3) = 3 + 3 = 6(\bmod 18)$ $12^2 = 2(12) = 12 + 12 = 6(\bmod 18)$

$4^2 = 2(4) = 4 + 4 = 8(\bmod 18)$ $13^2 = 2(13) = 13 + 13 = 8(\bmod 18)$

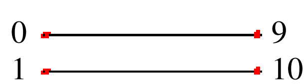
$5^2 = 2(5) = 5 + 5 = 10(\bmod 18)$ $14^2 = 2(14) = 14 + 14 = 10(\bmod 18)$

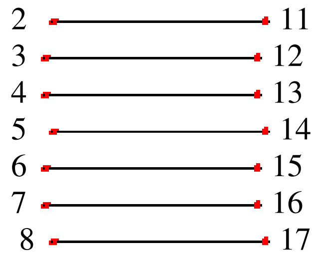
$6^2 = 2(6) = 6 + 6 = 12(\bmod 18)$ $15^2 = 2(15) = 15 + 15 = 12(\bmod 18)$

$7^2 = 2(7) = 7 + 7 = 14(\bmod 18)$ $16^2 = 2(16) = 16 + 16 = 14(\bmod 18)$

$8^2 = 2(8) = 8 + 8 = 16(\bmod 18)$ $17^2 = 2(17) = 17 + 17 = 16(\bmod 18)$

If two elements in $x, y \in Z_{18}$ are adjacent iff their square values are same





$(9,9) - \textit{Bipartite graph}$

Conclusion: From the above illustration it is observed that for any two elements in Z_n are adjacent iff their square values are same.

The following table shows the relation between n and $G(S)$

Number of vertices, $n \geq 6$ $n = 4t + 2, t = 1,2,3,.......$	$G(S) = (2t+1, 2t+1) - \textit{Bipartite graph}$ $t = 1,2,3,...........$
$t = 1, n = 6$ $t = 2, n = 10$ $t = 3, n = 14$ $t = 4, n = 18$	$t = 1,\ (3,3) - \textit{Bipartite graph}$ $t = 2,\ (5,5) - \textit{Bipartite graph}$ $t = 3,\ (7,7) - \textit{Bipartite graph}$ $t = 4,\ (9,9) - \textit{Bipartite graph}$ ... .

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