STRUCTURAL PROPERTIES OF LENGTH BIASED WEIGHTED POWER FUNCTION DISTRIBUTION

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Abstract- In this paper, we proposed a new probability model called as the length biased weighted power function distribution (LBWPFD) and discussed its various statistical properties. The probability density function, moments, hazard rate function, reverse hazard rate function and survival function have been discussed. The maximum likelihood method has been used to estimate the parameter and their asymptotics and Likelihood ratio test have been discussed.

Keywords – Weighted distribution, Moments, Reliability Analysis, Entropy, Order Statistics, Maximum Likelihood Estimation.

1. INTRODUCTION

The concept of weight distribution is discussed by Fisher (1934). After being modified by C R Rao (1965) in a different way, in which with a weighty distribution many situations can be resolved. Weighted distribution is used in a variety of research fields related to reliability, environment, engineering and biomedicine. If the weight function looks at the length of one, the weight distribution reduces the length of the average distribution. If the weight function looks at the size of one, the weight distribution reduces the size or length biased distribution. The concept of length-biased distribution can be work in development of proper models for lifetime data. Length-biased distribution is a special case of the more general form known as weighted distribution. The concept of length biased sampling was introduced first by Cox (1969) and Zelen (1974). Patil and Ord (1976) examined the use of weight-bearing statistics related to population and the environment can be found in Patil and Rao (1978). Van Deusen's (1986) relevant data related to the extent to chest height from a sample of the above point in color distribution. Lappi and Bailey (1987), used a randomized distribution of research in the analysis of the ascending data rate of sample length. Abdul et.al.(2018) have discussed Length-Biased weighted Exponentiated Lomax Distribution.

Rogers (1963) Consider the probability density function (pdf) of power function distribution is given by

$$f(x;\alpha,\beta)=\frac{x^{\beta-1}\beta}{\alpha^{\beta}};\quad 0<x\le\alpha\quad \alpha,\beta>0 \tag{1.1}$$

and its mean is

$$E(X)=\frac{\alpha\beta}{(\beta+1)} \tag{1.2}$$

II. LENGTH BIASED WEIGHTED POWER FUNCTION (LBWPF) DISTRIBUTION

Suppose X is a non-negative random variable with probability density function $f(x)$. Let $w(x)$ be the non negative weight function, and then the probability density function of the weighted random variable X_w is given by:

$$f_w(x) = \frac{w(x)f(x)}{E(w(x))};\ x > 0$$

Where $w(x)$ be a non-negative weight function and $E\big(w(x)\big) = \int w(x)f(x)dx < \infty$.

For different weighted models, we have different choice of the weight function $w(x)$. When $w(x) = x^c$, the resulting distribution is termed as weighted distribution. In this paper, the length biased version of weighted distribution, take c = 1 in weights x^1 , in order to get the length biased distribution and its pdf is given by:

$$f_1(x) = \frac{xf(x)}{E(x)};\, x > 0 \qquad (2.1)$$

Using the values of (1.1) and (1.2) in equation (2.1), we will get the pdf of length biased power function distribution

$$f_1(x) = \frac{x^{\beta}\,(\beta+1)}{\alpha^{\beta+1}};\quad 0 < x \leq \alpha \quad \alpha,\beta > 0 \qquad (2.2)$$

And the cumulative distribution function of length biased weighted power function (*LBWPF)* distribution is obtained as

$$F_1(x) = \int_0^x f_1(x)dx$$

$$F_1(x) = \int_0^x \frac{x^{\beta}\,(\beta+1)}{\alpha^{\beta+1}}\,dx$$

After simplification, we will get the cumulative distribution function (cdf) of *LBWPF* distribution is

$$F_1(x) = \left(\frac{x}{\alpha}\right)^{\beta+1} \qquad (2.3)$$

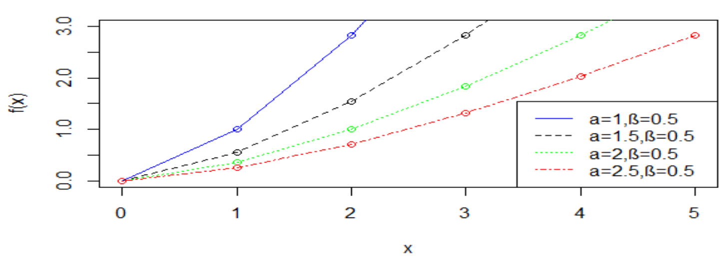

Fig 1. The Probability Density Function of *LBWPFD* for the indicated values of α and β.

III. RELIABILITY ANALYSIS

In this session, we will introduce the reliability function or survival function, hazard rate function or failure rate and reverse hazard rate function, Mills ratio for the length biased weighted power function distribution

3.1 Survival Function

The survival function of LBWPFD is defined as

$$S(x) = 1 - F_1(x)$$

$$S(x) = 1 - \left(\frac{x}{\alpha}\right)^{\beta+1}$$

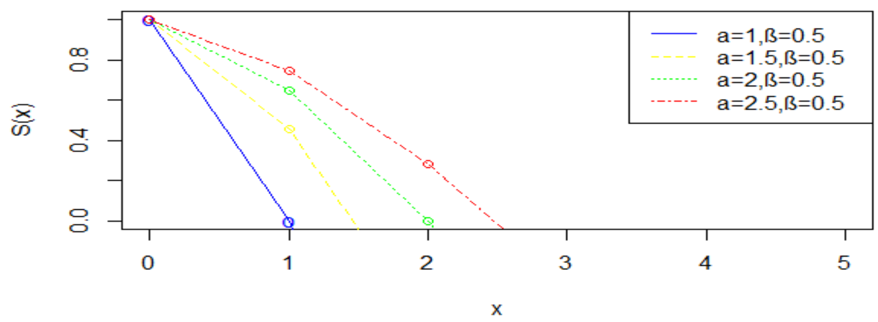

Fig.2 Survival function curve of LBWPFD

3.2 Hazard Rate Function of LBWPFD

The hazard function is also known as the hazard rate is defined as

$$h(x) = \frac{f_1(x)}{S(x)}$$

$$h(x) = \frac{(\beta + 1)x^{\beta}}{\alpha^{\beta+1} - x^{\beta+1}}$$

3.3 Reverse Hazard Rate Function of LBWPFD

The reverse hazard rate is given by

$$h_r(x) = \frac{f_1(x)}{F_1(x)}$$

$$h_r(x) = \frac{(\beta+1)}{x}$$

3.4 Mills Ratio

The Mills ratio of the LBWPF distribution is given by,

$$Mills\ ratio = \frac{1}{h_r(x)}$$

$$= \frac{x}{\beta+1}$$

IV. MOMENTS AND ASSOCIATED MEASURES

Let X denotes the random variable of LBWPF distribution with parameters α and β then the r^{th} order moment of LBWPF distribution can be defined as

$$E(X^r) = \mu_r' = \int_0^\infty x^r f_1(x)dx$$

$$= \int_0^\alpha x^r \frac{x^\beta (\beta+1)}{\alpha^{\beta+1}} \quad dx$$

$$= \frac{(\beta+1)}{\alpha^{\beta+1}} \int_0^\alpha x^{r+\beta}\, dx$$

$$= \frac{(\beta+1)\alpha^{\beta+r+1}}{(\beta+r+1)\alpha^{\beta+1}}$$

$$= \frac{(\beta+1)\alpha^r}{(\beta+r+1)}$$

$$\mu_r' = \frac{\alpha^r (\beta+1)}{(\beta+r+1)} \tag{4.1}$$

Putting $r = 1$ in equation (4.1), we will get the mean of LBWPF distribution which is given by

$$E(X) = \frac{\alpha\,(\beta+1)}{(\beta+2)}$$

And putting $r = 2$ we obtain the second moment is

$$E(X^2) = \frac{\alpha^2\,(\beta+1)}{(\beta+3)}$$

$$E(X^3) = \frac{\alpha^3\,(\beta+1)}{(\beta+4)}$$

$$E(X^4) = \frac{\alpha^4\,(\beta+1)}{(\beta+5)}$$

Therefore,

$$Variance = \sigma^2 = E(X^2) - (E(X))^2$$

$$\sigma^2 = \frac{\alpha^2\,(\beta+1)}{(\beta+2)^2(\beta+3)}$$

$$Standard\ Deviation = \sigma = \frac{\alpha}{(\beta+2)}\sqrt{\frac{(\beta+1)}{(\beta+3)}}$$

$$Coefficient\ of\ Variation = \frac{\sigma}{\mu} = \frac{1}{(\beta+1)}\sqrt{\frac{(\beta+1)}{(\beta+3)}}$$

$$Coefficient\ of\ Dispersion(\gamma) = \frac{\sigma^2}{\mu} = \frac{\alpha}{(\beta+2)(\beta+3)}$$

4.1 Moment generation function and characteristic function

Let X be LBWPF distribution, then the MGF of X is obtained as

$$M_X(t) = E(e^{tx}) = \int_0^\alpha e^{tx}\ f_1(x)\, dx$$

$$= \int_0^{\alpha} \left(1 + (tx) + \frac{(tx)^2}{2!} + \frac{(tx)^3}{3!} + \cdots + \cdots\right) f_1(x) dx$$

$$= \int_0^{\alpha} \sum_{r=0}^{\infty} \frac{(tx)^r}{r!} f_1(x)\, dx$$

$$= \sum_{r=0}^{\infty} \frac{(t)^r}{r!} \int_0^{\alpha} \mathrm{x}^{\mathrm{r}} f_1(x)\, dx$$

$$= \sum_{r=0}^{\infty} \frac{(t)^r}{r!} E(x^r) \quad (4.1.1)$$

Using equation (4.1) in equation (4.1.1), we have

$$M_X(t) = \sum_{r=0}^{\infty} \frac{(t)^r}{r!} \frac{\alpha^r\, (\beta+1)}{(\beta+r+1)}$$

Similarly, the characteristic function of LBWPF distribution can be obtained as

$$\varphi_X(t) = M_X(it)$$

$$\Rightarrow M_X(it) = \sum_{r=0}^{\infty} \frac{(it)^r}{r!} \frac{\alpha^r (\beta+1)}{(\beta+r+1)}$$

4.2 Harmonic Mean

Let X is a LBWPFD, and then the harmonic mean is obtained as

$$\frac{1}{H} = \int_0^{\infty} \frac{1}{x} f_1(x)\, dx$$

$$= \int_0^{\alpha} \frac{1}{x} \frac{x^{\beta}\, (\beta+1)}{\alpha^{\beta+1}}\ dx$$

$$= \frac{(\beta+1)}{\alpha^{\beta+1}} \int_0^{\alpha} x^{\beta-1} dx$$

$$= \frac{(\beta+1)}{\alpha^{\beta+1}} \frac{\alpha^{\beta}}{\beta}$$

$$= \frac{(\beta+1)}{\alpha\beta}$$

Thus, the harmonic mean of LBWPFD

$$H = \frac{\alpha\beta}{(\beta+1)}$$

V. ENTROPIES

5.1 Renyi Entropy

Renyi entropy is important in nature and mathematics as an indicator of diversity. Renyi entropy is also important in quantum data, where it can be used as a catch measure. Renyi entropy is given by

$$Re(\delta) = \frac{1}{1-\delta} \log\left(\int_0^{\infty} {f_1}^{\delta}(x) dx\right)$$

Where $\delta > 0$ and $\delta \neq 1$

$$Re(\delta) = \frac{1}{1-\delta} log\left(\int_0^{\alpha} \left(\frac{x^{\beta}\, (\beta+1)}{\alpha^{\beta+1}}\right)^{\delta} dx\right)$$

$$Re(\delta) = \frac{1}{1-\delta} log\left(\left(\frac{(\beta+1)}{\alpha^{\beta+1}}\right)^{\delta} \int_0^{\alpha} x^{\delta\beta} dx\right)$$

$$Re(\delta) = \frac{1}{1-\delta} log\left(\left(\frac{(\beta+1)}{\alpha^{\beta+1}}\right)^{\delta} \frac{\alpha^{\delta\beta+1}}{\delta\beta+1}\right)$$

$$Re(\delta) = \frac{1}{1-\delta} log\left(\frac{(\beta+1)^{\delta}\alpha^{1-\delta}}{\delta\beta+1}\right)$$

5.2 Tsallis Entropy

The Boltzmann-Gibbs (B-G) measurable mechanics started by Tsallis has centered a lot to consideration. The mathematical expression of Tsallis entropy (Tsallis, 1988) for a continuous random variable x is given by

$$S_{\lambda} = \frac{1}{1-\lambda}\left(1 - \int_0^{\infty} f_1{}^{\lambda}(x)dx\right)$$

$$S_{\lambda} = \frac{1}{1-\lambda}\left(1 - \int_0^{\alpha}\left(\frac{x^{\beta}(\beta+1)}{\alpha^{\beta+1}}\right)^{\lambda} dx\right)$$

$$S_{\lambda} = \frac{1}{1-\lambda}\left(1 - \left(\frac{(\beta+1)}{\alpha^{\beta+1}}\right)^{\lambda}\int_0^{\alpha} x^{\lambda\beta} dx\right)$$

$$S_{\lambda} = \frac{1}{1-\lambda}\left(1 - \left(\frac{(\beta+1)}{\alpha^{\beta+1}}\right)^{\lambda}\frac{\alpha^{\lambda\beta+1}}{\lambda\beta+1}\right)$$

$$S_{\lambda} = \frac{1}{1-\lambda}\left(1 - \frac{(\beta+1)^{\lambda}\alpha^{1-\lambda}}{\lambda\beta+1}\right)$$

6. ORDER STATISTICS OF LBWPF DISTRIBUTION

The probability density function of the j^{th} order statistics $X_{(j)}$ for $1 \le j \le n$ is

$$f_{X_{(j)}}(x) = \frac{n!}{(j-1)!(n-j)!}[F(x)]^{j-1}[1-F(x)]^{n-j} f_1(x) \tag{6.1}$$

Substitute the value of pdf and cdf of length biased weighted power function distribution in equation (6.1), we get

$$f_{X_{(j)}}(x) = \frac{n!}{(j-1)!(n-j)!}\left[\left(\frac{x}{\alpha}\right)^{\beta+1}\right]^{j-1} \times \left[1-\left(\frac{x}{\alpha}\right)^{\beta+1}\right]^{n-j} \frac{x^{\beta}(\beta+1)}{\alpha^{\beta+1}} \tag{6.2}$$

Put $j = 1$ in equation (6.2), we will get the probability density function of first order statistics of LBWPFD.

$$f_{X_{(1)}}(x) = n\left[1-\left(\frac{x}{\alpha}\right)^{\beta+1}\right]^{n-1} \frac{x^{\beta}(\beta+1)}{\alpha^{\beta+1}}$$

Put $j = n$ in equation (6.2), we will get the probability density function of n^{th} order statistics of LBWPFD.

$$f_{X_{(n)}}(x) = n\left[\left(\frac{x}{\alpha}\right)^{\beta+1}\right]^{n-1} \frac{x^{\beta}(\beta+1)}{\alpha^{\beta+1}}$$

VII. BONFERRONI AND LORENZ CURVES

The Bonferroni and Lorenz curves is obtained by

$$B(p) = \frac{1}{p\mu}\int_0^q x f_1(x)dx$$

& $L(p) = pB(p) = \frac{1}{\mu}\int_0^q x\, f_1(x)dx$

where $\mu = E(x) = \frac{\alpha\,(\beta+1)}{(\beta+2)}$ $and\ q = F^{-1}(p)$

$$\therefore\ B(p) = \frac{(\beta+2)}{p\alpha(\beta+1)}\int_0^q \frac{x^{\beta+1}\,(\beta+1)}{\alpha^{\beta+1}}dx$$

$$B(p) = \frac{(\beta+2)}{p\alpha(\beta+1)}\frac{(\beta+1)}{\alpha^{\beta+1}}\int_0^q x^{\beta+1}\,dx$$

after simplification

$$B(p) = \frac{1}{p}\left(\frac{q}{\alpha}\right)^{\beta+2}$$

and

$$L(p) = pB(p) = \left(\frac{q}{\alpha}\right)^{\beta+2}$$

VIII. MAXIMUM LIKELIHOOD ESTIMATION OF LBWPF

In this section, we will discuss the maximum likelihood estimators (MLE) of the parameters of length biased weighted power function (LBWPF) distribution. Consider be the random sample of length *n* from the LBWPF distribution, then the likelihood function is given by

$$L(x;\alpha,\beta) = \frac{(\beta+1)^{\mathrm{n}}}{\alpha^{n(\beta+1)}}\prod_{i=1}^{n} {x_i}^{\beta}$$

The log likelihood function is given by

$$logl = nlog(\beta+1) - n(\beta+1)log(\alpha) + \beta\sum_{i=1}^{n} logx_i \qquad (8.1)$$

The MLE of α and β can be obtained by differentiating equation (8.1) with respect to α and β, we get

$$\frac{\partial log\ l}{\partial\alpha} = -\frac{n(\beta+1)}{\alpha} \qquad (8.2)$$

$$\frac{\partial log\ l}{\partial\beta} = \frac{n}{(\beta+1)} - nlog(\alpha) + \sum_{i=1}^{n} logx_i \qquad (8.3)$$

$$\frac{\partial log\ l}{\partial\alpha} = 0 \ \text{ and } \ \frac{\partial log\ l}{\partial\beta} = 0$$

From 8.2 we get

$$-\frac{n(\beta+1)}{\alpha} = 0$$

$\hat{\alpha} = \infty$, which is an absurd result.

Here we can apply inspection method. Let us consider the n[th] ordered samples,

$0 \le X_{(1)} \le X_{(2)} \le X_{(3)} \ \le \cdots \le X_{(n)} \le \alpha$

$\Longrightarrow \alpha \ge X_{(n)}$

Therefore MLE of $\hat{\alpha} = X_{(n)} =$ there highest sample observation.

From (8.3) we get

$$\frac{n}{(\beta+1)} - nlog(\alpha) + \sum_{i=1}^{n} logx_i = 0$$

After solving

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$$\hat{\beta} = \frac{n - nlog(\alpha) + \sum_{i=1}^{n} logx_i}{nlog(\alpha) - \sum_{i=1}^{n} logx_i}$$

To obtain confidence interval we use the asymptotic normality tests. If as $\hat{\lambda} = (\hat{\alpha}, \hat{\beta})$ denote the MLE of $\lambda = (\alpha, \beta)$, state the results as follows:

$$\sqrt{n}(\hat{\lambda} - \lambda) \to N(0, I^{-1}(\lambda))$$

Where $I(\lambda)$ is Fisher's Information Matrix, i.e.

$$I(\lambda) = -\frac{1}{n}\begin{bmatrix} E\left(\frac{\partial^2 log\, l}{\partial^2 \alpha}\right) & E\left(\frac{\partial^2 log\, l}{\partial\alpha\partial\beta}\right) \\ E\left(\frac{\partial^2 log\, l}{\partial\beta\partial\alpha}\right) & E\left(\frac{\partial^2 log\, l}{\partial^2 \beta}\right) \end{bmatrix}$$

Where

$$E\left(\frac{\partial^2 log\, l}{\partial^2 \alpha}\right) = \frac{n(\beta+1)}{\alpha^2},$$

$$E\left(\frac{\partial^2 log\, l}{\partial^2 \beta}\right) = \frac{-n}{(\beta+1)^2}$$

$$E\left(\frac{\partial^2 log\, l}{\partial\alpha\partial\beta}\right) = E\left(\frac{\partial^2 log\, l}{\partial\beta\partial\alpha}\right) = \frac{-n}{\alpha}$$

Since λ being unknown, we estimate $I^{-1}(\lambda)$ by $I^{-1}(\hat{\lambda})$ and this can be used to obtain asymptotic confidence intervals for α and β .

IX. LIKELIHOOD RATIO TEST:

Let $x_1, x_2, x_3, \ldots\ldots, x_n$ be a random sample from the LBWPF distribution. We use the hypothesis
$H_0: f(x) = f(x, \alpha, \beta)$ Against $H_1: f(x) = f_1(x, \alpha, \beta)$
In order test the random sample of length n comes from power function distribution and length biased weighted power function (LBWPF) distribution then following test statistics is used

$$\Delta = \frac{L_1}{L_0} = \prod_{i=1}^{n} \frac{f_1(x, \alpha, \beta)}{f(x, \alpha, \beta)}$$

$$\Delta = \left(\frac{(\beta+1)}{\alpha\beta}\right)^n \prod_{i=1}^{n} x_i$$

We reject the null hypothesis, if

$$\Delta = \left(\frac{(\alpha+3)}{\beta^2(\alpha+1)}\right)^n \prod_{i=1}^{n} x_i > k$$

$$\Delta = \prod_{i=1}^{n} x_i > k\left(\frac{\beta^2(\alpha+1)}{(\alpha+3)}\right)^n$$

or

$$\Delta^* = \prod_{i=1}^{n} x_i > k^* \quad \text{Where} \quad k^* = k\left(\frac{\beta^2(\alpha+1)}{(\alpha+3)}\right)^n > 0$$

For large sample length n, $2log\Delta$ distributed as chi-square distribution with one degree of freedom and also p-value is obtained from the chi-square distribution. Thus, we reject the null hypothesis, when the probability value is given by

$$p(\Delta^* > \theta^*)$$

Where $\theta^* = \prod_{i=1}^{n} x_i$ is less than specified level of significance and $\prod_{i=1}^{n} x_i$ is observed value of the statistics Δ^*.

X. CONCLUSION:

In this paper, we have studied a new distribution called as the length biased weighted power function distribution. By using certain special functions, its statistical properties, moments, failure rate, survival function, entropy has been obtained. The parameters have been estimated by using maximum likelihood method and also order statistics and fisher's information matrix have been obtained.

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