Solutions of one dimensional heat equation and heat-like equation with variable coefficients using Variational Iteration Method

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Abstract: J.H. He has initiated the Variational Iteration Method. Usually modeling problems yield either differential or partial differential equations. Their solutions are to be obtained for their physical interpretation. The methods given in the theory of differential equations may fail to give solutions of certain differential and partial differential equations. There are several methods to get the approximate solutions of such equations. Variational Iteration method is among such approximate methods. Numerous researchers are using variation iteration method to find the solutions of such problems. Present work is devoted to the application of Variational Iterative method i.e., VIM to obtain the solution of the heat equation and heat-like equation with variable coefficients. Few illustrative examples of heat equation and heat-like equation with variable coefficients are presented to show the efficiency of this method. It is observed that the successive approximations obtained by VIM are converging to their exact solutions. The obtained solutions show that VIM is simple, effective and easy compared to other methods.

Key words: Successive Approximation Solution, Variational Iterative method, Variable Coefficients, Boundary conditions, Lagrange's multiplier.

1. Introduction

Some problems in physics and engineering yield partial differential equations with some initial and boundary conditions for example gas equation, heat equation and wave equation. For their physical interpretation we need their solutions. It is not possible to obtain the exact solutions of some problems. Using numerical methods, Adomain's decomposition method, Homotopy perturbation method etc. we can find the approximate solutions. He J.H.[6-8] developed the variational iteration method for solving linear, non-linear and boundary value problems. The successive approximations of a partial differential equation with initial and boundary conditions obtained by this method are converging to the exact solution. H.Ahmad [1,2,3] has used VIM to obtain the solutions of fifth order differential equations, wave-like vibration equations and telegraph equations. Muhammad MunibKhan[11] has applied VIM to obtain the Solution of Differential Equation of Motion of the Mathematical Pendulum and Duffing-Harmonic Oscillator.Approximate solution for Burger's Fisher equation by variational iteration transform method obtained by ALI. AL-Fayadh[4].E.Rama, K.Somaiah and K.Sambaiah [12] have used this method for obtaining solution of various types of problems like differential equations of first and second order, isoperimetric problem and Volterra integral equation of second kind.

Many researchers have developed algorithms using VIM.BehzadKafash et al[5] developed a computational method for determining solution for the optimal control problems using VIM. Hijaz Ahmad and Tufail A Khan[10] developed an algorithm for solution of differential equations of motion of simple and damped mass-spring systems using VIM. J.H.He and H.Latifizadeh[9] developed a general numerical algorithm for non-linear differential equations using VIM.

In the present work we obtained exact solutions of heat-equation and heat-like equation with variable coefficients. We have shown graphically the error in u(x,t) and sixth approximations against t values for fixed values of x for the all problems. It is observed that the error is reducing for decreasing values of t.

2. Description of the method

In this section we describe the basic idea of the variational iteration method. Consider the following partial differential equation

$$L_t u(x,t) + L_x u(x,t) + Nu(x,t) = g(x,t) \qquad (1)$$

where u is a function of two independent variables x and t, L_t is a linear operator involving partial derivatives with respect to t only, L_x is linear operator involving partial derivatives with respect to x only, N is non-linear operator and g(x,t) is a continuous function. According to variational iteration method the correctional functional of equation (1) in t and x directions respectively are given by

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda_1(x,s)\left[L_s u_n + \{L_x + N\}\tilde{u}_n(x,s) - g(x,s)\right]ds \qquad (2)$$

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda_2(x,s)\left[L_s u_n + \{L_t + N\}\tilde{u}_n(x,s) - g(x,s)\right]ds \qquad (3)$$

where λ_1 and λ_2 are Lagrangian multipliers which can be identified optimally using the variational theory and $\tilde{u}_n$ is the restricted variation. i.e., $\delta\tilde{u}_n$ =0. After determination of Lagrange multipliers λ_1 and λ_2 the successive approximations $u_{n+1}(x, t)$ of u(x, t) can be obtained by suitable choice of initial approximation $u_0(x, t)$. The solution of equation (1) is obtained as

u(x,t)= $\underset{n\to\infty}{Lt}\; u_n(x,t)$.

3. Illustrative examples

3.1 Example: Consider $\frac{\partial u}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2}$ where 0<x<π ,t>0 and the initial, boundary conditions are i) u(0,t)=0 ii) u(π ,t)=0 iii) u(x,0)=3sin(2x).

The correction formula in t-direction to obtain the solution of $\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial u}{\partial t}$

by the VIM is

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(x,s)\left[\frac{\partial u_n}{\partial s} - \frac{\partial^2 \tilde{u}_n}{\partial x^2}\right]ds \qquad (4)$$

where $u_n(x,t)$ is the n^{th} approximation of u(x,t), $\lambda(x,s)$ is Lagrangian multiplier and $\frac{\partial^2 \tilde{u}}{\partial x^2}$ is restricted variation

i.e., $\delta\left(\frac{\partial^2 \tilde{u}_n}{\partial x^2}\right) = 0$ (5)

In view of (5) the equation (4) becomes

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(x,s)\left[\frac{\partial u_n}{\partial s}\right]ds$$

Using integration by parts the above equation becomes

$$u_{n+1}(x,t) = u_n(x,t) + [\lambda(x,s)u_n(x,s)]_{s=0}^{s=t} - \int_0^t \{\frac{\partial}{\partial s}\lambda(x,s)\}u_n(x,s)ds$$

$$u_{n+1}(x,t) = u_n(x,t) + [\lambda(x,s)u_n(x,s)]_{s=t} - [\lambda(x,s)u_n(x,s)]_{s=0} - \int_0^t \{\frac{\partial}{\partial s}\lambda(x,s)\}u_n(x,s)ds$$

Taking the variation with respect to u_n we obtain

$$\delta u_{n+1}(x,t) = \delta u_n(x,t) + [\lambda(x,s)]_{s=t}\,\delta u_n(x,t) - \delta\int_0^t \{\frac{\partial}{\partial s}\lambda(x,s)\}u_n(x,s)ds$$

$$\delta u_{n+1}(x,t) = [1+\lambda(x,s)]_{s=t}\,\delta u_n(x,t) - \int_0^t \{\frac{\partial}{\partial s}\lambda(x,s)\}\delta u_n(x,s)ds$$

$\delta u_{n+1}(x,t)$ will become zero if $[1+\lambda(x,s)]_{s=t} = 0$ and $\frac{\partial}{\partial s}\lambda(x,s) = 0$. From these two equations we get the Lagrangian multiplier

$\lambda(x,s) = -1$ (6)

Hence the variational iteration formula for the problem considered is

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t \{\frac{\partial}{\partial s}u_n(x,s) - \frac{\partial^2 u_n(x,s)}{\partial x^2}\}ds \quad (7)$$

Choose $u_0(x,t)=u(x,0)=3\sin(2x)$. Using this initial approximation and (7) with n=0 we get

$$u_1(x,t) = 3\sin(2x) - \int_0^t \left[\frac{\partial}{\partial s}u_0(x,s) - \frac{\partial^2}{\partial x^2}u_0(x,s)\right]ds$$

$$u_1(x,t) = 3\sin(2x) - \int_0^t \left[\frac{\partial}{\partial s}\{3\sin(2x)\} - \frac{\partial^2}{\partial x^2}\{3\sin(2x)\}\right]ds$$

$$u_1(x,t) = 3\sin(2x)[1-4t] \quad (8)$$

Continuing this process we get

$$u_2(x,t) = 3\sin(2x)[1-4t+\frac{4^2t^2}{2!}] \quad (9)$$

$$u_3(x,t) = 3\sin(2x)[1-4t+\frac{4^2t^2}{2!}-\frac{4^3t^3}{3!}] \quad (10)$$

$$u_4(x,t) = 3\sin(2x)[1-4t+\frac{4^2t^2}{2!}-\frac{4^3t^3}{3!}+\frac{4^4t^4}{4!}] \quad (11)$$

$$u_5(x,t) = 3\sin(2x)[1-4t+\frac{4^2t^2}{2!}-\frac{4^3t^3}{3!}+\frac{4^4t^4}{4!}-\frac{4^5t^5}{5!}] \quad (12)$$

$$u_6(x,t) = 3\sin(2x)[1-4t+\frac{4^2t^2}{2!}-\frac{4^3t^3}{3!}+\frac{4^4t^4}{4!}-\frac{4^5t^5}{5!}+\frac{4^6t^6}{6!}] \quad (13)$$

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u(x,t)= $\underset{n\leftarrow\infty}{Lt}\, u_n(x,t) = 3\sin(2x)e^{-4t}$

It is observed that the successive approximations obtained by VIM are converging to the exact solution of the problem considered.

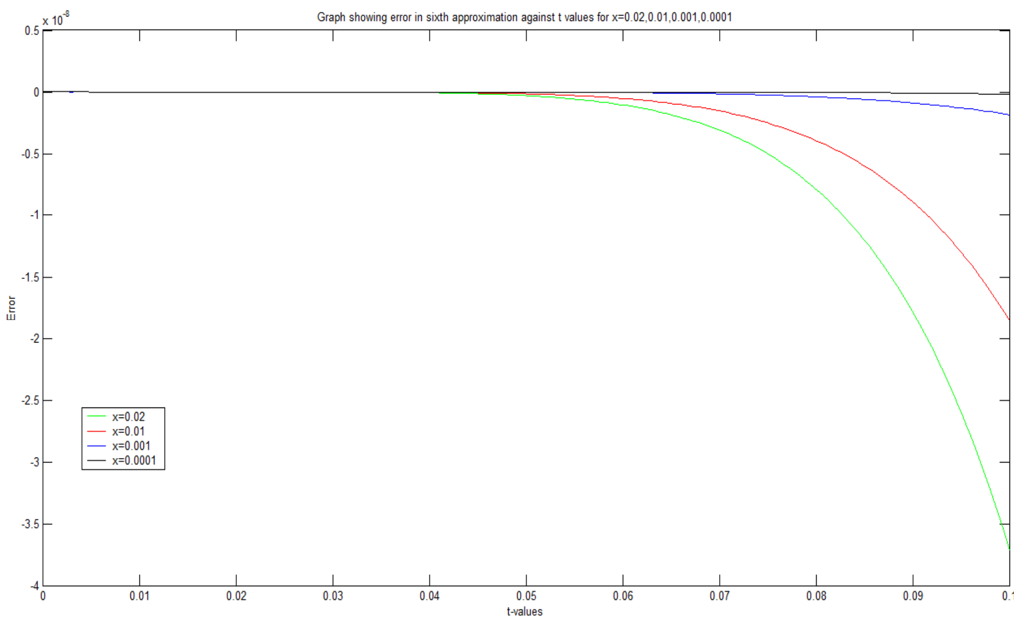

Fig.1

3.2 Example: Consider $\frac{\partial u}{\partial t}=\frac{\partial^2 u(x,t)}{\partial x^2}$ where 0<x<1, t>0 and the initial and boundary conditions are i) u(0,t)=0 ii) u(1,t)=0 iii) u(x,0)=$-\frac{1}{2}\sin(3\pi x)+\frac{3}{2}\sin(\pi x)$.

The variational iteration formula for this problem proceeding as in the case of earlier problem we get

$$u_{n+1}(x,t)=u_n(x,t)-\int_0^t\{\frac{\partial}{\partial s}u_n(x,s)-\frac{\partial^2 u_n(x,s)}{\partial x^2}\}ds \quad (14)$$

Choose $u_0(x,t)=u(x,0)=$ $-\frac{1}{2}\sin(3\pi x)+\frac{3}{2}\sin(\pi x)$

Substituting n=0 and substituting $u_0(x,t)$ in equation (14) we get

$$u_1(x,t)=u_0(x,t)-\int_0^t\{\frac{\partial}{\partial s}u_0(x,s)-\frac{\partial^2 u_0(x,s)}{\partial x^2}\}ds$$

$$=-\frac{1}{2}\sin(3\pi x)+\frac{3}{2}\sin(\pi x)-\int_0^t\left[-\frac{3^2}{2}\pi^2\sin(3\pi x)-\frac{3\pi^2}{2}\right]ds$$

$$u_1(x,t)=-\frac{1}{2}\sin(3\pi x)\left\{1-3^2\pi^2 t\right\}+\frac{3}{2}\sin(\pi x)\left\{1-\pi^2 t\right\} \quad (15)$$

Similarly it can be shown that

$$u_2(x,t)=-\frac{1}{2}\sin(3\pi x)\left\{1-3^2\pi^2 t+3^4\pi^4\frac{t^2}{2}\right\}+\frac{3}{2}\sin(\pi x)\left\{1-\pi^2 t+\pi^4\frac{t^2}{2}\right\} \quad (16)$$

$$u_3(x,t)=-\frac{1}{2}\sin(3\pi x)\left\{1-3^2\pi^2 t+3^4\pi^4\frac{t^2}{2}-\frac{3^6\pi^6 t^3}{3!}\right\}$$

$$+\frac{3}{2}\sin(\pi x)\left\{1-\pi^2 t+\pi^4\frac{t^2}{2}-\frac{\pi^6 t^3}{3!}\right\} \quad (17)$$

$$u_4(x,t)=-\frac{1}{2}\sin(3\pi x)\left\{1-3^2\pi^2 t+3^4\pi^4\frac{t^2}{2}-\frac{3^6\pi^6 t^3}{3!}+\frac{3^8\pi^8 t^4}{4!}\right\}$$

$$+\frac{3}{2}\sin(\pi x)\left\{1-\pi^2 t+\pi^4\frac{t^2}{2}-\frac{\pi^6 t^3}{3!}+\frac{\pi^8 t^4}{4!}\right\} \quad (18)$$

$$u_5(x,t)=-\frac{1}{2}\sin(3\pi x)\left\{1-3^2\pi^2 t+3^4\pi^4\frac{t^2}{2}-\frac{3^6\pi^6 t^3}{3!}+\frac{3^8\pi^8 t^4}{4!}-\frac{3^{10}\pi^{10} t^5}{5!}\right\}$$

$$+\frac{3}{2}\sin(\pi x)\left\{1-\pi^2 t+\pi^4\frac{t^2}{2}-\frac{\pi^6t^3}{3!}+\frac{\pi^8t^4}{4!}-\frac{\pi^{10}t^5}{5!}\right\} \quad (19)$$

$$u_6(x,t)=-\frac{1}{2}\sin(3\pi x)\left\{1-3^2\pi^2t+3^4\pi^4\frac{t^2}{2}-\frac{3^6\pi^6t^3}{3!}+\frac{3^8\pi^8t^4}{4!}-\frac{3^{10}\pi^{10}t^5}{5!}+\frac{3^{12}\pi^{12}t^6}{6!}\right\}$$

$$+\frac{3}{2}\sin(\pi x)\left\{1-\pi^2 t+\pi^4\frac{t^2}{2}-\frac{\pi^6t^3}{3!}+\frac{\pi^8t^4}{4!}-\frac{\pi^{10}t^5}{5!}+\frac{\pi^{12}t^6}{6!}\right\} \quad (20)$$

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u(x,t)= $\underset{n\leftarrow\infty}{Lt}\, u_n(x,t)=-\frac{1}{2}\sin(3\pi x)e^{-3^2\pi^2t}+\frac{3}{2}\sin(\pi x)e^{-\pi^2t}$

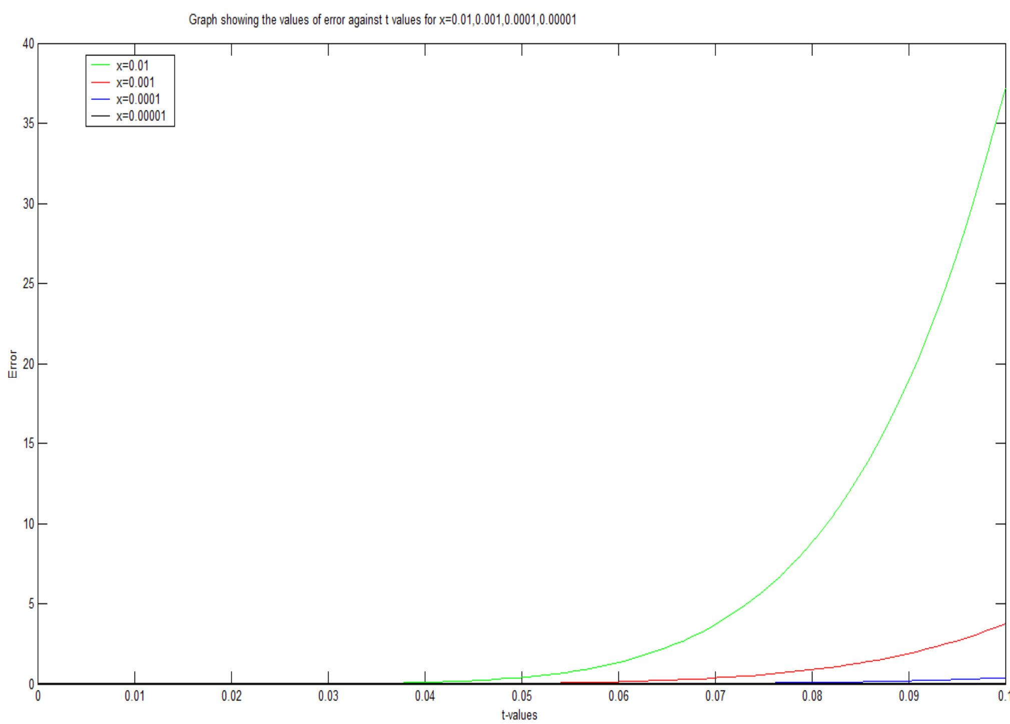

Fig.2

The successive approximations of this problem obtained by VIM are also converging to the exact solution of it.

3.3 Example : Consider heat-like equation with variable coefficients

$$2\frac{\partial u}{\partial t}-(x^2+1)\frac{\partial^2u(x,t)}{\partial x^2}=0 \quad (21)$$

with initial condition $u(x,0)=x^2+1$.

The equation (21) can be written as

$$\frac{\partial u}{\partial t}-\frac{x^2+1}{2}\frac{\partial^2 u(x,t)}{\partial x^2}=0 \qquad (22)$$

So that Lagrangian multiplier is λ= -1.

The variational iteration formula for (22) can be shown that

$$u_{n+1}(x,t)=u_n(x,t)-\int_0^t[\frac{\partial}{\partial s}u_n(x,s)-(\frac{x^2+1}{2})\frac{\partial^2 u_n(x,s)}{\partial x^2}]ds \quad (23)$$

Choose $u_0(x,t)= u(x,0)=x^2+1$.

Substituting n=0 in equation (23) and using $u_0(x,t)= x^2+1$ we get

$$u_1(x,t)=u_0(x,t)-\int_0^t[\frac{\partial}{\partial s}u_0(x,s)-(\frac{x^2+1}{2})\frac{\partial^2 u_0(x,s)}{\partial x^2}]ds$$

$$=(x^2+1)+\int_0^t[\frac{\partial}{\partial s}(x^2+1)-(\frac{x^2+1}{2})\frac{\partial^2 (x^2+1)}{\partial x^2}]ds$$

$$=(x^2+1)+\int_0^t[\frac{\partial}{\partial s}(x^2+1)-(\frac{x^2+1}{2})\frac{\partial^2 (x^2+1)}{\partial x^2}]ds$$

$$=(x^2+1)+(x^2+1)t$$

Thus,

$$u_1(x,t)=(x^2+1)(1+t) \qquad (24)$$

Continuing this process we get

$$u_2(x,t)=(x^2+1)(1+t+\frac{t^2}{2!}) \qquad (25)$$

$$u_3(x,t)=(x^2+1)(1+t+\frac{t^2}{2!}+\frac{t^3}{3!}) \qquad (26)$$

$$u_4(x,t)=(x^2+1)(1+t+\frac{t^2}{2!}+\frac{t^3}{3!}+\frac{t^4}{4!}) \qquad (27)$$

$$u_5(x,t)=(x^2+1)(1+t+\frac{t^2}{2!}+\frac{t^3}{3!}+\frac{t^4}{4!}+\frac{t^5}{5!}) \qquad (28)$$

$$u_6(x,t)=(x^2+1)(1+t+\frac{t^2}{2!}+\frac{t^3}{3!}+\frac{t^4}{4!}+\frac{t^5}{5!}+\frac{t^6}{6!}) \qquad (29)$$

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u(x,t)= $\underset{n\leftarrow\infty}{Lt}\, u_n(x,t)=(x^2+1)e^t$

The solution obtained by the variational method is same as the exact solution of the equation (21).

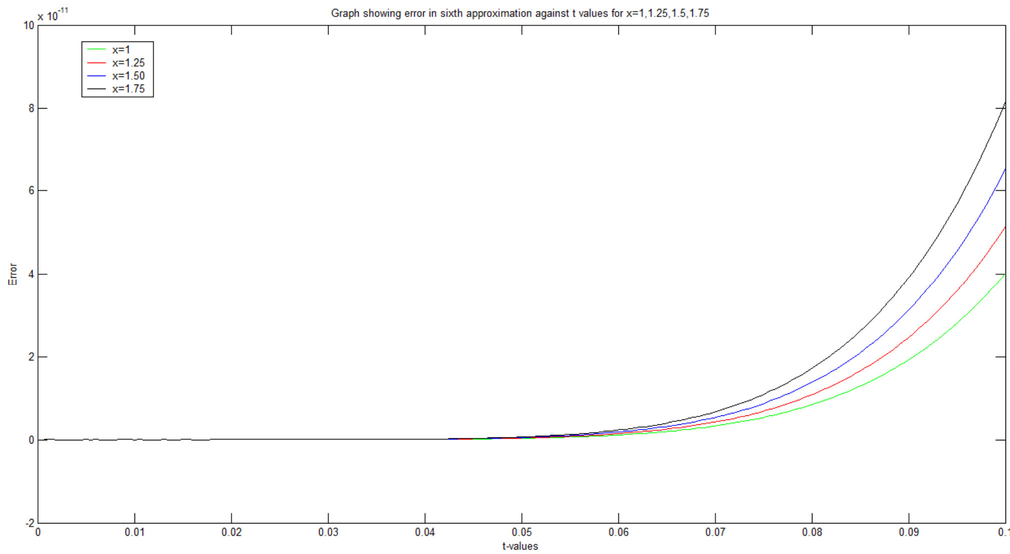

Fig.3

Conclusion: This paper is aimed to application of Variational Iterative method to obtain the solution of one dimensional heat equation and wave like equation with variable coefficients. We observed that VIM is simple and solutions obtained by it are valid for the whole domain. It is also observed that the successive approximations are converging to the exact solution of each problem. Further this mehod is easy to adopt for obtaining the solution of boundary value problems.

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