A common fixed point theorem for six mappings in bicomplex valued metric spaces

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Abstract

The application of the fixed point theory in complex valued metric spaces has motivated the researchers to explore the immense possibilities in the field of mathematical analysis. The conceptualisation of bicomplex valued metric spaces is one of the recent developments in this area and this paper is inspired by this new phenomenon. It investigates a common fixed point theorem for six self contracting, commuting and weakly compatible mappings in a bicomplex valued metric space and extends some theorems of Azam et al.[1] and Rouzkard & Imdad[12] regarding common fixed point theorems in complex valued metric spaces. Moreover, some important concepts of Choi et al.[7], and Datta et al.[8] are used here.

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1 Introduction.

Segre [13] conceptualized commutative generalization of complex numbers as bicomplex numbers, tricomplex numbers, etc. as elements of an infinite set of algebras. Unfortunately this pioneering work failed to earn the attention of the mathematicians for almost a century. However, recently a renewed interest in this subject contributes a lot in the different fields of mathematical sciences and other branches of science and technology $\{cf.$[6], [10] &[11]$\}$.

The Banach contraction principle $\{cf.$[3]$\}$ is a very popular and effective tool to solve the existence problems in many branches of mathematical analysis and it is an active area of research since 1922. The famous Banach theorem $\{cf.$[3]$\}$ states that "Let (X, d) be a metric space and T be a mapping of X into itself satisfying $d(Tx, Ty) \leq kd(x, y), \forall x, y \in X$, where k is a constant in $(0, 1)$. Then T has a unique fixed point $x^* \in X$". In this connection one can see the attempts in $\{cf.$[2],[4],[5],[8]$\}$.

In recent times, Choi et al.[7], define the bicomplex valued metric space and prove some common fixed point theorems in connection with two weakly compatible mappings in bicomplex valued metric spaces.

2 Definitions and Notations.

In this section we state some relevant definitions.

2.1 Bicomplex number.

Segre [13] defined the bicomplex number as $\xi = a_1+a_2i_1+a_3i_2+a_4i_1i_2 = z_1+i_2z_2,$, where $a_1, a_2, a_3, a_4 \in \mathbb{C}_0$ (the set of reals) and $z_1 = a_1+a_2i_1, z_2 = a_3+a_4i_1 \in \mathbb{C}_1$ (the set of complex numbers), the independent units i_1, i_2 are such that $i_1^2 = i_2^2 = -1$ & $i_1i_2 = i_2i_1$.

We denote the set of bicomplex numbers as $\mathbb{C}_2$.

2.2 Norm of a bicomplex number.

The norm $\|\cdot\|$ of $\mathbb{C}_2$ is a positive real valued function and $\|\cdot\| : \mathbb{C}_2 \to \mathbb{C}_0^+$ for any $\xi \in \mathbb{C}_2$ is defined by

$$\|\xi\| = \|z_1 + i_2z_2\| = \left\{|z_1|^2 + |z_2|^2\right\}^{\frac{1}{2}} = \left(a_1^2 + a_2^2 + a_3^2 + a_4^2\right)^{\frac{1}{2}}.$$

2.3 Partial order relation.

The partial order relation $\precsim_{i_2}$on $\mathbb{C}_2$ defined as:

For any $\xi = z_1 + i_2z_2, \eta = w_1 + i_2w_2 \in \mathbb{C}_2$, $\xi \precsim_{i_2} \eta$ if and only if $z_1 \precsim w_1$ and $z_2 \precsim w_2$, i.e., $\xi \precsim_{i_2} \eta$ if one of the following conditions is satisfied:

(i) $z_1 = w_1,\ z_2 = w_2$, (ii) $z_1 \prec w_1,\ z_2 = w_2$, (iii) $z_1 = w_1,\ z_2 \prec w_2$ and (iv) $z_1 \prec w_1,\ z_2 \prec w_2$.

In particular we can write $\xi \precnsim_{i_2} \eta$ if $\xi \precsim_{i_2} \eta$ and $\xi \neq \eta$ i.e. one of $(ii), (iii)$ and (iv) is satisfied and we will write $\xi \prec_{i_2} \eta$ if only (iv) is satisfied.

For any two bicomplex numbers $\xi, \eta \in \mathbb{C}_2$ we can verify the following:

(i) $\|\xi\eta\| \leq \sqrt{2}\,\|\xi\|\,\|\eta\|$ and the equality holds only when at least one of ξ and η is degenerated, (ii) $\|\xi+\eta\| \leq \|\xi\| + \|\eta\|$, (iii) $\|a\xi\| = a\,\|\xi\|$, where a is a non negative real number and (iv) $\left\|\frac{\xi}{\eta}\right\| = \frac{\|\xi\|}{\|\eta\|}$ if η is a degenerated bicomplex number.

In their Paper, J. Choi et al.[7] defined the bicomplex valued metric space as:

Definition 2.1 *Let X be a nonempty set. Suppose the mapping $d : X \times X \to \mathbb{C}_2$ satisfies the following conditions:*

(i). $0 \precsim_{i_2} d(x,y)$ for all $x, y \in X$, (ii). $d(x,y) = 0$ if and only if $x = y$, (iii). $d(x,y) = d(y,x)$ for all $x, y \in X$ and (iv). $d(x,y) \precsim_{i_2} d(x,z) + d(z,y)$ for all $x, y, z \in X$.

Then (X,d) is called the bicomplex valued metric space.

Example 1 *Consider $X = [0,\infty)$, define a bicomplex valued metric $d : X \times X \to \mathbb{C}_2$ by $d(x,y) = i_1 i_2 |x-y|$, $\forall x, y \in X$.*

From the above definition of d one can easily verify that

1. $0 \precsim_{i_2} d(x,y)$ for all $x, y \in X$
2. $d(x,y) = 0$ if and only if $x = y$
3. $d(x,y) = d(y,x)$ for all $x, y \in X$ and
4. $d(x,y) = i_1 i_2 |x-y| = i_1 i_2 |x-z+z-y| \precsim_{i_2} i_1 i_2 [|x-z| + |z-y|] \precsim_{i_2} d(x,y) + d(x,y)$, therefore (X,d) is a bicomplex valued metric space.

Definition 2.2 *For a bicomplex valued metric space (X,d)*

(i). A sequence $\{x_n\}$ in X is said to be a convergent sequence and converges to a point x if for any $0 \prec_{i_2} r \in \mathbb{C}_2$ there is a natural number $n_0 \in \mathbb{N}$ such that $d(x_n, x) \prec_{i_2} r$, for all $n > n_0$. And we write $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.

(ii). A sequence $\{x_n\}$ in X is said to be a Cauchy sequence in (X,d) if for any $0 \prec_{i_2} r \in \mathbb{C}_2$ there is a natural number $n_0 \in \mathbb{N}$ such that $d(x_n, x_{n+m}) \prec_{i_2} r$, for all $m, n \in \mathbb{N}$ and $n > n_0$.

(iii). If every cauchy sequence in X is convergent in X then (X,d) is said to be a complete bicomplex valued metric space.

Definition 2.3 *Two families of self-mappings $\{T_i\}_1^m$ and $\{S_i\}_1^n$ are said to be pairwise commuting if:*

1. *$T_i T_j = T_j T_i$ for all $i, j \in \{1, 2, ...m\}$,*
2. *$S_i S_j = S_j S_i$ for all $i, j \in \{1, 2, ...n\}$ and*
3. *$T_i S_j = S_j T_i$ for all $i \in \{1, 2, ...m\}$ & $j \in \{1, 2, ...n\}$.*

Definition 2.4 *Let $T, S : X \to X$ be two self-mappings. Then*

(i). A point $x \in X$ is said to be a fixed point of T if $Tx = x$.

(ii). A point $x \in X$ is said to be a common fixed point of T and S if $Tx = Sx = x$.

Definition 2.5 *Let (X,d) be a bicomplex valued metric space and $S,T:X\to X$ be two Self-mappings then S and T are said to be cmpatible if* $\lim_{n\to\infty} d(STx,TSx_n)=0$, *whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n=\lim_{n\to\infty} Tx_n=u$ for some $u\in X$.*

Definition 2.6 *Let $S,T:X\to X$ be two Self-mappings then, S and T are said to be weakly compatible if $STx=TSx$ whenever $Sx=Tx$ for all $x\in X$.*

3 Lemmas.

In this section we present some lemmas which will be needed in the sequel.

Lemma 3.1 *[8] Let (X,d) be a bicomplex valued metric space and a sequence $\{x_n\}$ in X is said to be convergent to a point x if and only if $\lim_{n\to\infty}\|d(x_n,x)\|=0$.*

Lemma 3.2 *[8] Let (X,d) be a bicomplex valued metric space and a sequence $\{x_n\}$ in X is said to be a Cauchy sequence in X if and only if $\lim_{n\to\infty}\|d(x_n,x_{n+m})\|=0$.*

4 Main Theorems.

In this section we are going to prove a fixed point theorem for weakly compatible mappings in bicomplex valued metric spaces.

Theorem 4.1 *Let (X,d) be a complete bicomplex valued metric space and let $F,G,I,J,K,L:X\to X$ be six self-mappings such that*

$$KL(X)\subseteq F(X) \text{ and } IJ(X)\subseteq G(X) \tag{1}$$

and satisfying

$$d(IJx,KLy) \precsim_{i_2} Ad(Fx,Gy)+Bd(Fx,IJx)+Cd(Gy,KLy)+D\left[d(Fx,KLy)+d(Gy,IJx)\right] \tag{2}$$

for all $x,y\in X$ where A,B,C and D are non-negative real numbers with $A+B+C+2D<1$. Suppose that the pairs $\{IJ,F\}$ and $\{KL,G\}$ are weakly compatible and the pairs $\{K,L\}$, $\{K,G\}$, $\{L,G\}$, $\{I,J\}$, $\{I,F\}$, and $\{J,F\}$, are commuting pair of mappings. Then K,L,I,J,F and G have a unique common fixed point.

Proof. Let y_0 be an arbitrary point in X and we construct a sequence $\{y_n\}$ such that

$$y_{2k}=IJx_{2k}=Gx_{2k+1},\quad y_{2k+1}=KLx_{2k+1}=Fx_{2k+2},\quad k=0,1,2,.....$$

where $\{x_n\}$ is another sequence in X.

Then from (2), we get that

$$\begin{aligned} d(y_{2k}, y_{2k+1}) &= d(IJx_{2k}, KLx_{2k+1}) \\ &\precsim_{i_2} Ad(Fx_{2k}, Gx_{2k+1}) + Bd(Fx_{2k}, IJx_{2k}) + Cd(Gx_{2k+1}, KLx_{2k+1}) \\ &+ D[d(Fx_{2k}, KLx_{2k+1}) + d(Gx_{2k+1}, IJx_{2k})] \\ &\precsim_{i_2} Ad(y_{2k-1}, y_{2k}) + Bd(y_{2k-1}, y_{2k}) + Cd(y_{2k}, y_{2k+1}) \\ &+ D[d(y_{2k-1}, y_{2k+1}) + d(y_{2k}, y_{2k})] \\ &\precsim_{i_2} Ad(y_{2k-1}, y_{2k}) + Bd(y_{2k-1}, y_{2k}) + Cd(y_{2k}, y_{2k+1}) \\ &+ D[d(y_{2k-1}, y_{2k}) + d(y_{2k}, y_{2k+1})] \\ &\precsim_{i_2} (A + B + D)\, d(y_{2k-1}, y_{2k}) + (C + D)\, d(y_{2k}, y_{2k+1}) \end{aligned}$$

$$i.e., \quad d(y_{2k}, y_{2k+1}) \precsim_{i_2} \frac{A + B + D}{1 - C - D} d(y_{2k-1}, y_{2k})$$

Similarly,

$$\begin{aligned} & d(y_{2k+1}, y_{2k+2}) \\ &= d(KLx_{2k+1}, IJx_{2k+2}) = d(KLx_{2k+2}, IJx_{2k+1}) \\ &\precsim_{i_2} Ad(Fx_{2k+2}, Gx_{2k+1}) + Bd(Fx_{2k+2}, IJx_{2k+2}) + Cd(Gx_{2k+1}, KLx_{2k+1}) \\ &+ D[d(Fx_{2k+2}, KLx_{2k+1}) + d(Gx_{2k+1}, IJx_{2k+2})] \\ &\precsim_{i_2} Ad(y_{2k+1}, y_{2k}) + Bd(y_{2k+1}, y_{2k+2}) + Cd(y_{2k}, y_{2k+1}) \\ &+ D[d(y_{2k+1}, y_{2k+1}) + d(y_{2k}, y_{2k+2})] \\ &\precsim_{i_2} Ad(y_{2k+1}, y_{2k}) + Bd(y_{2k+1}, y_{2k+2}) + Cd(y_{2k}, y_{2k+1}) \\ &+ D[d(y_{2k}, y_{2k+1}) + d(y_{2k+1}, y_{2k+2})] \\ &\precsim_{i_2} (A + C + D)\, d(y_{2k}, y_{2k+1}) + (B + D)\, d(y_{2k+1}, y_{2k+2}) \end{aligned}$$

$$i.e., d(y_{2k+1}, y_{2k+2}) \precsim_{i_2} \frac{A + C + D}{1 - B - D} d(y_{2k}, y_{2k+1})$$

Let us consider $\lambda = \max\left[\frac{A+B+D}{1-C-D}, \frac{A+C+D}{1-B-D}\right]$ then $\lambda < 1$ as $A + B + C + 2D < 1$ and it follows that

$$d(y_{n+1}, y_{n+2}) \precsim_{i_2} \lambda d(y_n, y_{n+1}) \precsim_{i_2} ...... \precsim_{i_2} \lambda^{n+1} d(y_0, y_1) \text{ for all } n = 0, 1, 2, ...$$

Then for any two positive integers m, n with $m > n$ we get that

$$\begin{aligned} d(y_n, y_m) &\precsim_{i_2} d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + ... + d(y_{m-1}, y_m) \\ &\precsim_{i_2} \left[\lambda^n + \lambda^{n+1} + ... + \lambda^{m-1}\right] d(y_0, y_1) \\ &\precsim_{i_2} \lambda^n \left[1 + \lambda + \lambda^2 ...... + \lambda^{m-n-1}\right] d(y_0, y_1) \end{aligned}$$

Since, $0 \leq \lambda < 1$,Then $1 + \lambda + \lambda^2 ...... + \lambda^{m-n-1} \leq \frac{1}{1-\lambda}$.
Hence

$$d(y_n, y_m) \precsim_{i_2} \frac{\lambda^n}{1 - \lambda} d(y_0, y_1).$$

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Which yields that,

$$\|d(y_n, y_m)\| \leq \frac{\lambda^n}{1-\lambda} \|d(y_0, y_1)\|$$

Again since $\frac{\alpha^n}{1-\alpha} \longrightarrow 0$ as $n \longrightarrow \infty$, therefore for any $\varepsilon > 0$ there exists a positive integer n_0 such that $\|d(y_n, y_m)\| < \varepsilon$, for all $m, n > n_0$. Hence $\{y_n\}$ is a Cauchy sequence in X. Also X is a complete bicomplex valued metric space, then there is some $z \in X$ such that $\lim_{n \longrightarrow \infty} y_n = z$.

Form (1) we have, $KL(X) \subseteq F(X)$. Therefore there exists some $u \in X$ such that $z = Fu$.

Now we show that $IJu = z$, if not, then $0 \prec_{i_2} d(IJu, z) \in \mathbb{C}_2$.

Therefore,

$$\begin{aligned} d(IJu, z) &\precsim_{i_2} d(IJu, KLx_{2n-1}) + d(KLx_{2n-1}, z) \\ &\precsim_{i_2} Ad(Fu, Gx_{2n-1}) + Bd(Fu, IJu) + Cd(Gx_{2n-1}, KLx_{2n-1}) \\ &+ D[d(Fu, KLx_{2n-1}) + d(Gx_{2n-1}, IJu)] + d(KLx_{2n-1}, z) \\ &\precsim_{i_2} Ad(z, y_{2n-2}) + Bd(z, IJu) + Cd(y_{2n-2}, y_{2n-1}) \\ &+ D[d(z, y_{2n-1}) + d(y_{2n-2}, IJu)] + d(y_{2n-1}, z). \end{aligned}$$

Taking limit as $n \to \infty$, we obtain that

$$d(IJu, z) \precsim_{i_2} Ad(z, z) + Bd(z, IJu) + Cd(z, z) + D[d(z, z) + d(z, IJu)] + d(z, z)$$

$$i.e., d(IJu, z) \precsim_{i_2} (B + D)\, d(IJu, z)$$

which is a contradiction. Therefore $IJu = Fu = z$.

Again form (1) we have $IJ(X) \subseteq G(X)$. Therefore there exists some $v \in X$ such that $z = Gv$.

Now we show that $KLv = z$, if not, then $0 \prec_{i_2} d(KLv, z) \in \mathbb{C}_2$.

Therefore,

$$\begin{aligned} &d(KLv, z) \\ &\precsim_{i_2} d(IJu, KLv) \\ &\precsim_{i_2} Ad(Fu, Gv) + Bd(Fu, IJu) + Cd(Gv, KLv) + D[d(Fu, KLv) + d(Gv, IJu)] \\ &\precsim_{i_2} Ad(z, z) + Bd(z, z) + Cd(z, KLv) + D[d(z, KLv) + d(z, z)]. \end{aligned}$$

$$i.e., d(KLv, z) \precsim_{i_2} (C + D)\, d(KLv, z)$$

which is a contradiction. Therefore $KLv = Gv = z$ and so $IJu = Fu = KLv = Gv = z$.

Since F and IJ are weakly compatible mappings, therefore $IJFu = FIJu \Rightarrow IJz = Fz$. Now we show that $IJz = z$, if not, then $0 \prec_{i_2} d(IJz, z) \in \mathbb{C}_2$.

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Therefore,

$$\begin{aligned}
&d(IJz, z)\\
&= d(IJz, KLv)\\
&\precsim_{i_2} Ad(Fz, Gv) + Bd(Fz, IJz) + Cd(Gv, KLv) + D[d(Fz, KLv) + d(Gv, IJz)]\\
&\precsim_{i_2} Ad(IJz, z) + Bd(IJz, IJz) + Cd(z, z) + D[d(IJz, z) + d(z, IJz)].
\end{aligned}$$

$$i.e., d(IJz, z) \precsim_{i_2} (A + 2D)\, d(IJz, z)$$

which is a contradiction. Therefore $IJz = Fz = z$.

Again since G and KL are weakly compatible mappings, therefore $KLGv = GKLv \Rightarrow KLz = Gz$. Now we show that $KLz = z$, if not, then $0 \prec_{i_2} d(KLz, z) \in \mathbb{C}_2$.

Therefore,

$$\begin{aligned}
&d(KLz, z)\\
&= d(KLz, IJz) = d(IJz, KLz)\\
&\precsim_{i_2} Ad(Fz, Gz) + Bd(Fz, IJz) + Cd(Gz, KLz) + D[d(Fz, KLz) + d(Gz, IJz)]\\
&\precsim_{i_2} Ad(z, KLz) + Bd(z, z) + Cd(KLz, KLz) + D[d(z, KLz) + d(KLz, z)].
\end{aligned}$$

$$i.e., d(KLz, z) \precsim_{i_2} (A + 2D)\, d(KLz, z)$$

which is a contradiction. Therefore $KLz = Gz = z$ and so $IJz = KLz = Fz = Gz = z$. This shows that z is a common fixed point of F, G, IJ and KL.

Now we show that F, G, IJ and KL have a unique common fixed point, if possible suppose $z^* \in X$ be another common fixed point of IJ and KL.

Then

$$\begin{aligned}
d(z, z^*) = d(IJz, KLz^*) &\precsim_{i_2} Ad(Fz, Gz^*) + Bd(Fz, IJz) + Cd(Gz^*, KLz^*)\\
&+ D[d(Fz, KLz^*) + d(Gz^*, IJz)]\\
&\precsim_{i_2} Ad(z, z^*) + Bd(z, z) + Cd(z^*, z^*) + D[d(z, z^*) + d(z^*, z)]\\
&\precsim_{i_2} (A + 2D)\, d(z, z^*).
\end{aligned}$$

which is a contradiction. Therefore $z = z^*$ .This shows that z is the unique common fixed point of F, G, IJ and KL.

Since the self-mappings are pairwise commutative, therefore

$$Kz = K(KLz) = K(LKz) = KL(Kz) \text{ and } Kz = K(Fz) = F(Lz)$$

Also

$$Lz = L(KLz) = LK(Lz) = KL(Lz) \text{ and } Lz = L(Fz) = F(Lz).$$

which shows that Kz and Lz are common fixed point of KL and F.

Since the fixed point of KL and F is unique, therefore

$$Kz = Lz = z = Fz = KLz$$

Similarly we can show that

$$Iz = Jz = z = Gz = IJz$$

This shows that K, L, I, J, F and G have a unique common fixed point. This completes the proof. ■

Taking $G = F$ in the Theorem 4.1 we get the following corollary.

Corollary 4.1 *Let (X, d) be a complete bicomplex valued metric space and let $F, I, J, K, L : X \to X$ be five self-mappings such that $IJ(X) \subseteq F(X)$ and $KL(X) \subseteq F(X)$ and satisfying*

$$d(IJx, KLy) \precsim_{i_2} Ad(Gx, Fy) + Bd(Fx, IJx) + Cd(Fy, KLy) + D[d(Fx, KLy) + d(Fy, IJx)]$$

for all $x, y \in X$ where A, B, C and D are non-negative real numbers with $A + B + C + 2D < 1$. Suppose that the pairs $\{IJ, F\}$ and $\{KL, F\}$ are weakly compatible and the pairs $\{K, L\}$, $\{K, F\}$, $\{L, F\}$, $\{I, J\}$, $\{I, F\}$, and $\{J, F\}$, are commuting pair of mappings. Then K, L, I, J and F have a unique common fixed point.

5 Future Prospect.

In the line of the works as carried out in the paper one may think of the deduction of fixed point theorems using fuzzy metric, quasi metric, partial metric, probabilistic metric, p-adic metric (where p is a prime number), cone metric, quasi semi metric and other different types of metrics under the flavour of bicomplex analysis. This may be regarded as an active area of research to the future workers in this branch.

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