A Common Fixed Point Result Under Class Θ In b-Metric Spaces

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Abstract

In this paper, we showed that there is a common fixed point of self - mappings T and G on a complete b- metric space under the class Θ which is a generalization of $\alpha : \mathbb{R}_+^3 \to \mathbb{R}_+$ defined by Akram et al.

Keywords: Fixed point, Common fixed point, b-metric space, contraction mapping.

2010 MSC: 47H10, 54H25.

1. Introduction and Preliminaries.

The last century, the theory of fixed points has been revealed as a very powerful and important tool in the study of nonlinear phenomena. In particular, fixed point techniques have been applied in such diverse fields as physics, engineering, and economics.

In 1922, Banach [1] proved the contraction principle theorem which is one of the fundamental theorems in fixed point theory.

Akram et al.[2] in 2008, introduced a new class of contraction maps, called A-contraction, which is includes the classes of contractions studied by Kannan [3], Khan[4], Bianchini [5] and Reich [6].

Akram el at., study common fixed point of a family of self- mappings $\{T_i\}$ on complete metric space, used the following mapping $\alpha : \mathbb{R}_+^3 \to \mathbb{R}_+$ satisties

1. α is continuous on the set $\mathbb{R}_+^3$ of all triplets of nonnegative reals.

2. $a \leq kb$ for some $k \in [0,1)$ whenever $a \leq \alpha(a,b,b)$ or $a \leq \alpha(b,a,b)$ or $a \leq \alpha(b,b,a)$ for all a,b with the following kinds of contraction mapping

$$\varphi(T_i x, T_j y) \leq \alpha\left(\rho(x,y), \rho(x,T_i x), \rho(y,T_j y)\right)$$

for all $x,y \in X$.

Bakhtin [7] and Czerwik [8] introduce a b −metric space, many researcher presented generalization argued the Banach's contraction principle theorem in a b −metric space. Kir [9], Boriceanu [10], and. Krik [11]

Definition 1.1 [7],[8]:

Let X be a nonempty set and $s \geq 1$ be a given real number. A function $d : X \times X \to \mathbb{R}_+ = [0,\infty)$ is a b −metric on X, if for all $x,y,z \in X$, the following conditions hold:

1. $d(x,y) = 0$ if and only if $x = y$,
2. $d(x,y) = d(y,x)$,
3. $d(x,z) \leq s[d(x,y) + d(y,z)]$, ($b$ −triangular inequality).

In this case, the triple (X,d,s) is called a b −metric space

When $s = 1$, b-metric space is metric space while the converse is false e.g. [8],

The space $l_p (0 < p < 1)$,

$$l_p = \left\{\{x_n\}: x_n \in \mathbb{R}, \sum_{n=1}^{\infty} |x_n|^p < \infty\right\},$$

together with the function $d: l_p \times l_p \to \mathbb{R}_+$

$$d(x,y) = \left[\sum_{n=1}^{\infty} |x_n - y_n|^p\right]^{\frac{1}{p}},$$

where $x = \{x_n\}$, $y = \{y_n\}$ in l_p and $s = 2^{\frac{1}{p}}$ is a b −matric space.

Thus the concept of a b −matric space is wider than the concept of a matric space.

Boriceanu et al [12] presented the concept of the complete b −matric space.

Definition 1.2 [12]: Let (X, d, s) be a b-metric space and $\{x_n\}$ is a sequence in X . Then

1) $\{x_n\}$ is called b-convergent (for simplicity we call it convergent) if there exists $x \in X$ such that $d(x_n, x) \to 0$ as $n \to \infty$, we write $\lim_{n\to\infty} x_n = x$:
2) $\{x_n\}$ is called b-Cauchy (for simplicity we call it Cauchy) if $d(x_n, x_m) \to 0$ as $n, m \to \infty$.
3) A b-metric space (X, d, s) is said to be a b- complete b-metric space (for simplicity we call it complete b-metric space) if every Cauchy sequence in X is convergent.

2 The Main Result.

In this section we show that there is a common fixed point of self - mappings T and G on a complete b- metric space under iteration define as: let $x_0 \in X$,

$$x_1 = Gx_0, x_{2n+1} = Gx_{2n}, x_{2n} = Tx_{2n-1} \qquad (1)$$

For all $n \in \mathbb{N}$, where T and G self -mapping onto a set X, and the class Θ which is a generalization of $\alpha : \mathbb{R}_+^3 \to \mathbb{R}_+$ defined as

Definition (2.1):

Let Θ be a class of all mappings define as $\theta: \mathbb{R}_+^6 \to \mathbb{R}_+$ satisfying the conditions:

1. θ is a continuous for each coordinate.

2. For all $a, b \in \mathbb{R}_+$, if $a \le \max\left\{\begin{matrix}\theta(a,b,b,b,b,b), \theta(b,a,b,b,b,b), \\ \theta(b,b,a,b,b,b)\end{matrix}\right\}$,

   then $a \le cb$ for some $c \in [0,1)$.

Proposition (2.2):

Let (X, d, s) be a complete b −metric space and $G, T: X \to X$ be mappings satisfying the condition:

$$d(Gx, TGy) \leq \theta\left(\begin{array}{c} d(x, Gx), d(Gy, TGy), d(x, Gy), d(y, Gy) \\ \frac{d(Gy, TGy) - sd(y, TGy)}{s}, \frac{d(y, TGy) - sd(Gy, TGy)}{s} \end{array}\right) \qquad (2)$$

for all $x, y \in X$ and $\theta \in \Theta$ with $c \in [0,1)$ where $sc \neq 1$.Then a sequence $\{x_n\}$ in X is convergent.

Moreover, if $\lim_{n\to\infty} x_n = x$, is a common fixed point of G and T, then it is unique.

Proof:

Let $x_0 \in X$ define a sequence $\{x_n\}$ in X as (1) and by condition (2) we get:

$$d(x_{2n+1}, x_{2n+2}) = d(Gx_{2n}, TGx_{2n})$$

$$\leq \theta\left(\begin{array}{c} d(x_{2n}, Gx_{2n}), d(Gx_{2n}, TGx_{2n}), \\ d(x_{2n}, Gx_{2n}), d(x_{2n}, Gx_{2n}), \\ \frac{d(Gx_{2n}, TGx_{2n}) - sd(x_{2n}, TGx_{2n})}{s}, \\ \frac{d(x_{2n}, TGx_{2n}) - sd(Gx_{2n}, TGx_{2n})}{s} \end{array}\right)$$

$$= \theta \begin{pmatrix} d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}), \\ d(x_{2n}, x_{2n+1}), d(x_{2n}, x_{2n+1}), \\ \dfrac{d(x_{2n+1}, x_{2n+2}) - sd(x_{2n}, x_{2n+2})}{s}, \\ \dfrac{d(x_{2n}, x_{2n+2}) - sd(x_{2n+1}, x_{2n+2})}{s} \end{pmatrix},$$

$$\leq \theta \begin{pmatrix} d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}), d(x_{2n}, x_{2n+1}), \\ d(x_{2n}, x_{2n+1}), d(x_{2n}, x_{2n+1}), d(x_{2n}, x_{2n+1}) \end{pmatrix}$$

Thus, by the hypothesis, there is $c \in [0,1)$ which $sc \neq 1$ such that

$$d(x_{2n+1}, x_{2n+2}) \leq cd(x_{2n}, x_{2n+1}) \leq c^2 d(x_{2n-1}, x_{2n}).$$

In general, we get $d(x_n, x_{n+1}) \leq c^n\, d(x_0, x_1)$ for all $n \in \mathbb{N}_0$

Take $n \to \infty$, we have $d(x_n, x_{n+1}) \to 0$.

Now, to show that, $\{x_n\}$ is a Cauchy sequence in X, let $m, n > 0$ with $m > n$. By b-triangle inequality

$$d(x_n, x_m) \leq \sum_{k=1}^{\infty} s^k d(x_{n+k-1}, x_{n+k})$$

Since $sc \neq 1$, then, if $sc < 1$ and by geometric infinite series

$$d(x_n, x_m) \leq \sum_{k=1}^{\infty} s^k d(x_{n+k-1}, x_{n+k}) \leq sc^n d(x_0, x_1) \sum_{k=0}^{\infty} (sc)^k$$

$$= \frac{sc^n}{1 - sc} d(x_0, x_1)$$

As $n, m \to \infty$, $d(x_n, x_m) \to 0$.

While if $sc > 1$ thus by geometric finite series,

$$d(x_n, x_{n+m}) \leq \sum_{k=1}^{m} s^k d(x_{n+k-1}, x_{n+k})$$

$$\leq sc^n d(x_0, x_1) \sum_{i=0}^{m-1} (sc)^i$$

$$= sc^n d(x_0, x_1) \left[\frac{1-(sc)^m}{1-sc}\right]$$

$$\leq sc^n d(x_0, x_1)$$

Then by taking $n, m \to \infty$, $d(x_n, x_m) \to 0$.

Hence $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence in X. Therefor $\{x_n\}$ is convergent to a limit point, say x in X, thanks to the complete of X .

Moreover, if x is a common fixed point of T and G, then x is unique, indeed, let y be another fixed point of G and T, then $TGy = y$ so that

$$d(x, y) = d(Gx, TGy) \leq \theta\left(\begin{matrix} d(x, Gx), d(Gy, TGy), d(x, Gy), d(y, Gy), \\ \frac{d(Gy,TGy)-sd(y,TGy)}{s}, \frac{d(y,TGy)-sd(Gy,TGy)}{s} \end{matrix}\right) \leq \theta(0,0, d(x, y), 0,0,0).$$

Hence, by Definition (1.1, part (2)), $x = y$ ■

A critical open question under which conditions the limit point of the sequence, $\{x_n\}$ define on proposition (2.2), is a common fixed point of T and G ?

Perhaps the following propositions are partial answers. First when the b-metric is continuous for one coordinate, and second if θ multiplies by $\frac{1}{s}$.

Proposition (2.3):

Let (X,d,s) be a complete $b-$metric space and let $G,T:X\to X$ be mappings satisfying the condition:

$$d(Gx,TGy) \leq \theta\left(\begin{array}{c} d(x,Gx),d(Gy,TGy),d(x,Gy),d(y,Gy) \\ \frac{d(Gy,TGy)-sd(y,TGy)}{s},\frac{d(y,TGy)-sd(Gy,TGy)}{s} \end{array}\right) \quad (3)$$

for all $x,y\in X$ and $\theta\in\Theta$ with $c\in[0,1)$ where $sc\neq 1$.Then G and T have a unique common fixed point if d is continuous for one coordinate.

Proof:

By proposition (2.2) it is enough to prove that a limit point x of the convergence sequence $\{x_n\}$ is a common fixed point of T and G.

$$d(Gx,x_{2n}) = d(Gx,TGx_{2n-2})$$

$$\leq \theta\left(\begin{array}{c} d(x,Gx),d(Gx_{2n-2},TGx_{2n-2}),d(x,Gx_{2n-2}), \\ d(x_{2n-2},Gx_{2n-2}),\frac{d(Gx_{2n-2},TGx_{2n-2})-sd(x_{2n-2},TGx_{2n-2})}{s}, \\ \frac{d(x_{2n-2},TGx_{2n-2})-sd(Gx_{2n-2},TGx_{2n-2})}{s} \end{array}\right)$$

$$= \theta\left(\begin{array}{c} d(x,Gx),d(x_{2n-1},x_{2n}),d(x,x_{2n-1}), \\ d(x_{2n-2},x_{2n-1}),\frac{d(x_{2n-1},x_{2n})-sd(x_{2n-2},x_{2n})}{s}, \\ \frac{d(x_{2n-2},x_{2n})-sd(x_{2n-1},x_{2n})}{s} \end{array}\right)$$

$$\leq \theta\left(\begin{array}{c} d(x,Gx),d(x_{2n-1},x_{2n}),d(x,x_{2n-1}), \\ d(x_{2n-2},x_{2n-1}),d(x_{2n-1},x_{2n-2}),d(x_{2n-2},x_{2n-1}) \end{array}\right)$$

When $n\to\infty$, and since d and θ are continuous, we obtain

$$d(Gx, x) = \lim_{n\to\infty} d(Gx, TGx_{2n-2}) \leq \theta(d(x, Gx), 0,0,0,0,0).$$

Thus, by Definition (1.1, part (2)), hence $Gx = x$.

On the other hand,

$$d(Tx, x_{2n+1}) = d(Tx, GTx_{2n-1})$$

$$\leq \theta \begin{pmatrix} d(x, Tx), d(Tx_{2n-1}, GTx_{2n-1}), d(x, Tx_{2n-1}), \\ d(x_{2n-1}, Tx_{2n-1}), \\ \frac{d(Tx_{2n-1}, GTx_{2n-1}) - sd(x_{2n-1}, GTx_{2n-1})}{s}, \\ \frac{d(x_{2n-1}, GTx_{2n-1}) - sd(Tx_{2n-1}, GTx_{2n-1})}{s} \end{pmatrix}$$

Hence,

$$d(Tx, x_{2n+1})$$

$$\leq \theta \begin{pmatrix} d(x, Tx), d(x_{2n}, x_{2n+1}), d(x, x_{2n}), \\ d(x_{2n-1}, x_{2n}), \frac{d(x_{2n}, x_{2n+1}) - sd(x_{2n-1}, x_{2n+1})}{s}, \\ \frac{d(x_{2n-1}, x_{2n+1}) - sd(x_{2n}, x_{2n+1})}{s} \end{pmatrix}$$

$$= \theta \begin{pmatrix} d(x, Tx), d(x_{2n}, x_{2n+1}), d(x, x_{2n}), \\ d(x_{2n-1}, x_{2n}), d(x_{2n}, x_{2n-1}), d(x_{2n-1}, x_{2n}) \end{pmatrix}$$

as $n \to \infty$, we obtain: $d(Tx, x) \leq \theta(d(x, Tx), 0,0,0,0,0)$. Thus $Tx = x$. ■

Proposition (2.4):

Let (X, d, s) be a complete b −metric space and let $G, T: X \to X$ be mappings satisfying the condition

$$d(Gx, TGy) \leq \frac{1}{s}\theta\begin{pmatrix} d(x,Gx), d(Gy,TGy), d(x,Gy), d(y,Gy) \\ \frac{d(Gy,TGy) - sd(y,TGy)}{s}, \frac{d(y,TGy) - sd(Gy,TGy)}{s} \end{pmatrix} \quad (4)$$

for all $x, y \in X$ and $\theta \in \Theta$ with $c \in [0,1)$ where $sc \neq 1$.Then G and T have a unique common fixed point.

Proof:

Since $s \geq 1$, then by proposition (2.2) the sequence $\{x_n\}$ in X converges to a limit point x in X, and it is a unique common fixed point for T, G, infact, by condition (4) we get:

$$\begin{aligned} d(Gx, x) &= sd(Gx, TGx_{2n-2}) + sd(TGx_{2n-2}, x) \\ &\leq \theta\begin{pmatrix} d(x,Gx), d(x_{2n-1}, x_{2n}), d(x, x_{2n-1}), \\ d(x_{2n-2}, x_{2n-1}), d(x_{2n-1}, x_{2n-2}), d(x_{2n-2}, x_{2n-1}) \end{pmatrix} \\ &+ sd(x_{2n}, x) \end{aligned}$$

When $n \to \infty$, we obtain $d(x, Gx) \leq \theta(d(x, Gx), 0,0,0,0,0)$.

Thus $Gx = x$.

$$\begin{aligned} d(Tx, x) &= sd(Tx, GTx_{2n-1}) + sd(GTx_{2n-1}, x) \\ &= \theta\begin{pmatrix} d(x,Tx), d(x_{2n}, x_{2n+1}), d(x, x_{2n}), \\ d(x_{2n-1}, x_{2n}), d(x_{2n}, x_{2n-1}), d(x_{2n-1}, x_{2n}) \end{pmatrix} \\ &+ sd(x,\ x_{2n+1}) \end{aligned}$$

aw $n \to \infty$, the result has done. ■

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