Construction of Alpha Labeled Bipartite Graph from a Set of Alpha Labeled Bipartite Graphs

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Abstract

In this paper, we construct a separated path graph with new vertices attached to each copies of a path of length two. We give algorithms for constructing separated path graph and for its α-labeling. We give a special type of arrangement of labeled vertices of this graph and we call it High to Low scheme. We also develop a new technique which join two or more separated path graphs and we call it Barrier-Crossing scheme. We prove that this graph admits an α-labeling. Geometrical symmetry of edge labeling of the graph obtained by Barrier-Crossing is also shown.

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1 Introduction

Any graph use in this paper is simple and finite. We have had a lot to say so far about graceful bipartite graphs but what about the graphs obtained from smaller graceful bipartite graphs? After a long period of scramble over analysis and investigations, the notion of graceful labeling came into existence and therefore the credit goes to Rosa [5] and then by Golomb [3] for the first definition of graceful labeling. If there exists a bijective mapping $f : V(G) \to \mathbb{N} \cup \{0\}$ such that each edge $e \in E(G)$ has the induced label $\omega(f, V(G)) = \{|f(u) - f(v)| : u, v \in V(G)\}$ and $min\, \omega(f, V(G)) \leq \omega(e) \leq max\, \omega(f, V(G))$ and the resulting edge labels are distinct, then f is said to be graceful labeling for the graph $G = (V, E)$.

Consider a bipartite graph G with vertex set $V = V_1 \cup V_2$ such that $f(u) \leq f(w), \forall u \in V_1, w \in V_2$. If there exist positive integer λ in graceful labeling of G such that max label in $V_1 < \lambda$ and min label in $V_2 \geq \lambda$ or vice versa, then the graceful labeling with this additional property is called an alpha labeling (α-labeling [5]) and the positive integer λ is called the width. Rosa [5] and Golomb [3] proved that the complete bipartite graphs $K_{m,n}$ are graceful. Barrientos [1] showed that the one point union of any collection of the complete bipartite graphs is graceful graph. The one point union of any number of non-isomorphic complete bipartite graphs is graceful [6]. For a capacious survey on graph labeling, see the literature of Gallian [2].

2 Construction of Bipartite Graphs from a Path

In this paper, we construct a separated path graph $SP_{2,k,m}$, $m \in \mathbb{N}$, $k \in \mathbb{N}$ with m new vertices attached to each k-copies of a path $P_{2,i}$, $1 \leq i \leq k$ of length two with end vertices $v_{0,i}$ and $v_{1,i}$, $1 \leq i \leq k$. We give algorithms for constructing $SP_{2,k,m}$ and for its α-labeling. We give a special type of arrangement of labeled vertices of $SP_{2,k,m}$ and we call it High to Low scheme. We also develop a new technique which join two or more $SP_{2,k,m}$ graphs and

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we call it Barrier-Crossing scheme. We prove that this graph admits an α-labeling.

2.1 Algorithm for generating bipartite graphs

Input:

- step-1: Insert a set of $k \in \mathbb{N}$ known α-labeled bipartite graph $P_{2,i}$, $1 \leq i \leq k$.
- step-2: Insert $m \in \mathbb{N}$ new vertices $u_{i,1}, u_{i,2}, ..., u_{i,m}$ in $P_{2,i}$, $1 \leq i \leq k$
- step-3: Each $u_{i,j}$ is adjacent to $v_{0,i}$ and $v_{1,i}$, $1 \leq i \leq k$, $1 \leq j \leq m$.
- step-4: Arrange the vertices $v_{0,i}$ and $v_{1,i}$, $1 \leq i \leq k$, $1 \leq j \leq m$ in such a way that these vertices are at a two distant apart from each other but adjacent to each $u_{i,j}$, $1 \leq i \leq k$, $1 \leq j \leq m$.

The graph $SP_{2,k,m}$, $m \in \mathbb{N}$, $k \in \mathbb{N}$ obtained from the algorithm is given in figure 1.

2.2 Algorithm for generating α-labeled bipartite graphs

We give a special type of arrangement of labeled vertices of a class of α-labeled bipartite graph, and we call it High to Low arrangement. The vertex set of this class is denoted by $V = V_{1,i} \cup V_{2,i}$, $1 \leq i \leq k$, $k \in \mathbb{N}$. The High to Low labeling technique is as follow: the label 0 of $V_{1,i}$ is at the top and label 1 is at the bollom of the labeled graph. Similarly, the labels of $V_{2,i}$ are in decresing order i.e. highest label vertex is at top and the smallest label vertex is at the bottom).

Input:

- step-1: Insert the graph $SP_{2,k,m}$, $k \in \mathbb{N}$, $m \in \mathbb{N}$ obtained from algorithm 2.1.

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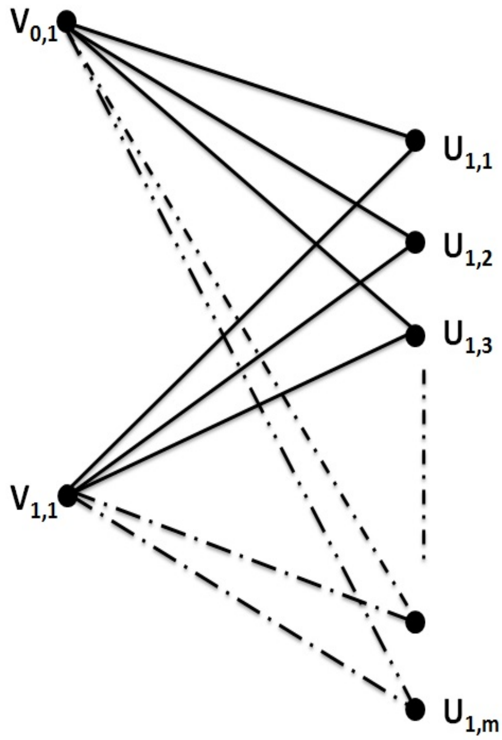

Figure 1: Labeling of $SP_{2,1,m}$

- step-2: Let θ_i $1 \leq i \leq k$ be the α-labeling of $SP_{2,k,m}$, $k \in \mathbb{N}$, $m \in \mathbb{N}$ with width $\lambda_i = 2, \forall i \in \mathbb{N}$.
- step-3: Write the size of each copy of $SP_{2,k,m}$, $k \in \mathbb{N}$, $m \in \mathbb{N}$. It will be $2m$ for each copy.
- step-4: Select the label 0 and 1 to each vertex of the copy of vertex set $V_{1,i}$, $1 \leq i \leq k$.
- step-5: Give a label $2m$ to each top vertex to each copy of vertex set $V_{2,i}$, $1 \leq i \leq k$.
- step-6: Reduce 2 to each top vertex label of $V_{2,i}$, $1 \leq i \leq k$ to give the label to each nex lower vertex.
- step-7: STOP when $i = k$.

α-labeling of $SP_{2,k,m}$, $k \in \mathbb{N}$, $m \in \mathbb{N}$ for $k = 1$ and $m = 3$ is shown in figure 2.

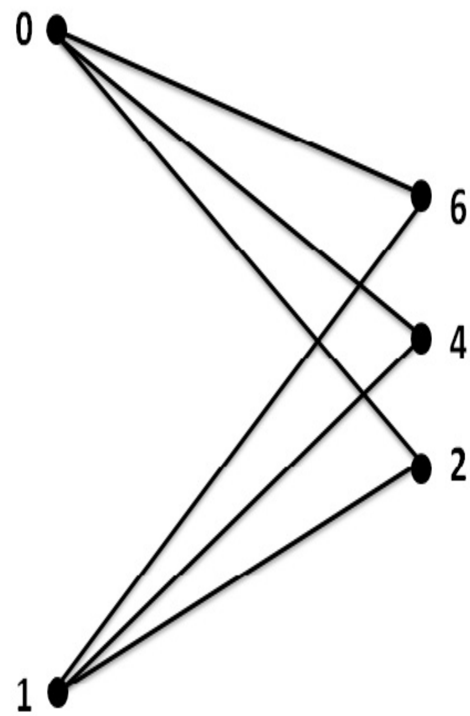

Figure 2: High to Low arrangement of α-labeled $SP_{2,1,3}$

2.3 Joining of bipartite graphs by barrier crossing

As we know that θ_i, $1 \leq i \leq k$ is an α-labeling of $SP_{2,k,m}$ with width λ_i $= 2$, $\forall i \in \mathbb{N}$. In this section, we develop a technique which join two or more bipartite graphs $SP_{2,k,m}$ and give a new bipartite graph G^* with vertex set $V = V_1 \cup V_2$. We call this technique by barrier crossing. It is a technique that join the bottom vertex of $V_{1,i}$ to the top vertex of $V_{2,i+1}$ by an edge. On the basis of the above technique, we have the following theorem:

Lemma 1. The edge labels of the graph obtained by taking $k \in \mathbb{N}$ copies of $SP_{2,k,m}$, $m \in \mathbb{N}$ are distinct and ranges from 1 to $m\lambda_i$, $1 \leq i \leq k$.

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Proof. From step 1 of algorithm 2.2 for generating α-labeled bipartite graph, we have the vertex label θ_i, $1 \leq i \leq k$ to each vertex of $SP_{2,k,m}$

$$\theta_i(v) = \begin{cases} v_{0,i} = 0 & \text{if } v_{0,i} \in V_{1,i} \text{ and } 1 \leq i \leq k \\ v_{1,i} = 1 & \text{if } v_{1,i} \in V_{1,i} \text{ and } 1 \leq i \leq k \\ u_{i,j} = 2n & \text{if } u_{i,j} \in V_{2,j} \text{ and } 1 \leq i \leq k,\ 1 \leq j \leq m,\ n \in \mathbb{N} \end{cases}$$

Let e_i be an edge of $SP_{2,k,m}$ such that $e_i = v_0 u_j$ or $e_i = v_1 u_j$, $1 \leq j \leq m$. Let $\psi(e_i)$ be the edge label of $SP_{2,k,m}$ such that

$$\begin{aligned} \psi(e_i) &= \theta_i(v_0, u_i) \text{ or } \theta_i(v_1, u_i) \\ &= |\theta_i(v_0) - \theta_i(u_i)| \text{ or } |\theta_i(v_1) - \theta_i(u_i)| \\ &= |0 - m\lambda_i| \text{ or } |1 - m\lambda_i| \end{aligned}$$

Therefore max $\psi(e_i) = m\lambda_i$.

□

Theorem 2.1. *The graph G^* obtained by Barrior-Crossing using in $SP_{2,k,m}$ admits an α-labeling.*

Proof. After using barrier crossing scheme in $SP_{2,k,m}$ with an α-labeling $\theta_i(v)$, we give a labeling $\theta(v)$ to each vertex $v \in V_1$ of the resulting graph G^* by

$$\theta(v/V_1) = \begin{cases} \theta_i(v_{0,1}) = 0 & \text{if } i = \lambda_i - 1 \\ \theta_i(v_{1,1}) = 1 & \text{if } i = \lambda_i - 1 \\ \theta_i(v_{0,2}) = 2 & \text{if } i = \lambda_i \\ \theta_i(v_{1,2}) = 3 & \text{if } i = \lambda_i \\ \theta_i(v_{0,i}) = \theta_{i-1}(v_{0,i-1}) + \lambda_i - 1 & \text{if } i > \lambda_i \\ \theta_i(v_{1,i}) = \theta_{i-1}(v_{1,i-1}) + \lambda_i - 1 & \text{if } i > \lambda_i \end{cases}$$

From the above formula, we see that each vertix of V_1 gets the distinct labels from the set $\{0, 1, 2, ..., (2k-1)$. The labels of vertices of the vertex set V_2 of G^* from bottom to top are given for each copy as:

1^{st} **cpoy:**

(2k-1)+1, (2k-1)+3, (2k-1)+5, ..., (2k-1)+(2m-1) ;

2^{nd} **cpoy:**

(2k-1)+(2m-1)+1, (2k-1)+(2m-1)+3, (2k-1)+(2m-1)+5, ..., (2k-1)+(2m-1)+(2m-1) = (2k-1)+2(2m-1) ;

:

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:
k^{th} **cpoy:**
(2k-1)+(k-1)(2m-1)+1, (2k-1)+(k-1)(2m-1)+3, (2k-1)+(k-1)(2m-1)+5, ..., (2k-1)+(k-1)(2m-1)+(2m-1) = (2k-1)+k(2m-1).

Therefore, each vertex of V_2 gets the distinct labes and the resulting edge labels of G^* also have the distinct labels.

□

There is a symmetry in the geometrical structure of edge labeling of G^* which is shown in figure 3 and the α-labeled graph G^* is shown in figure 4.

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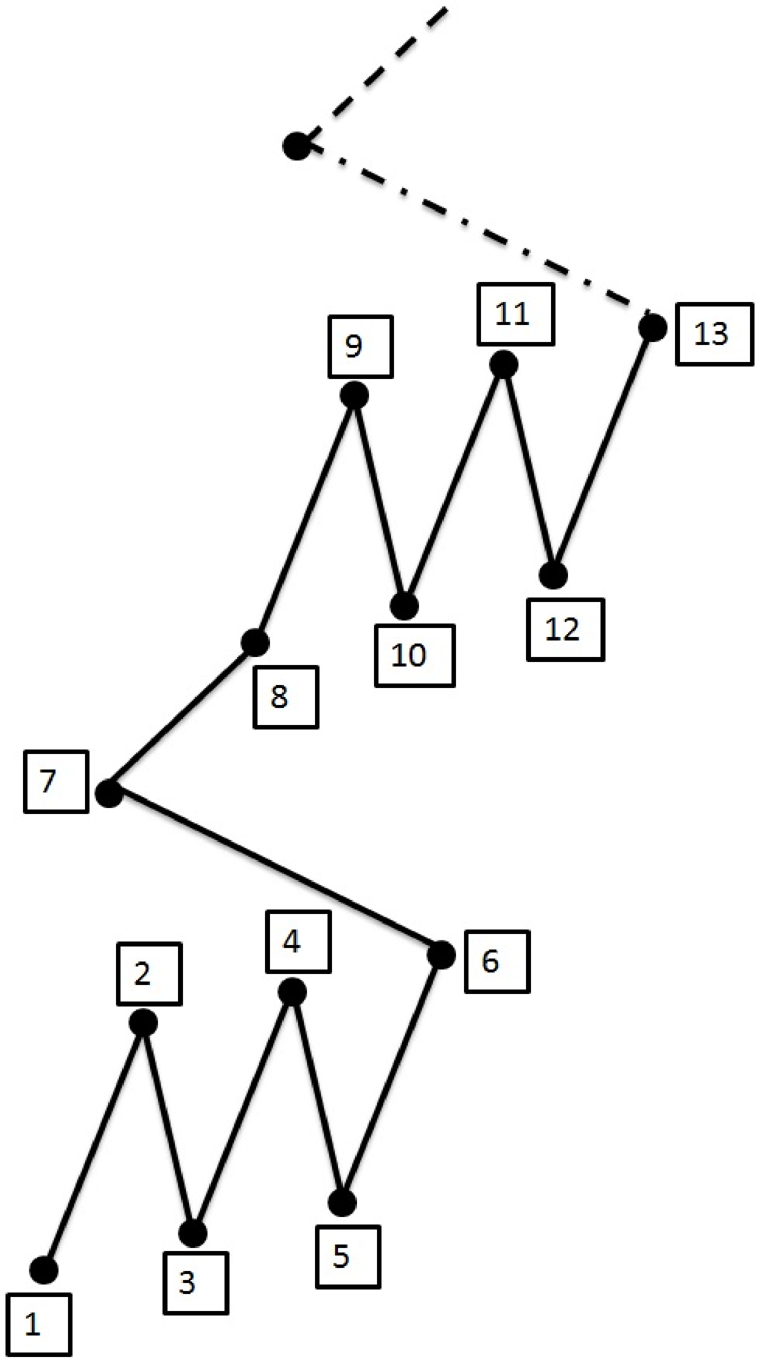

Figure 3: Symmetrical geometry of $\theta(v)$

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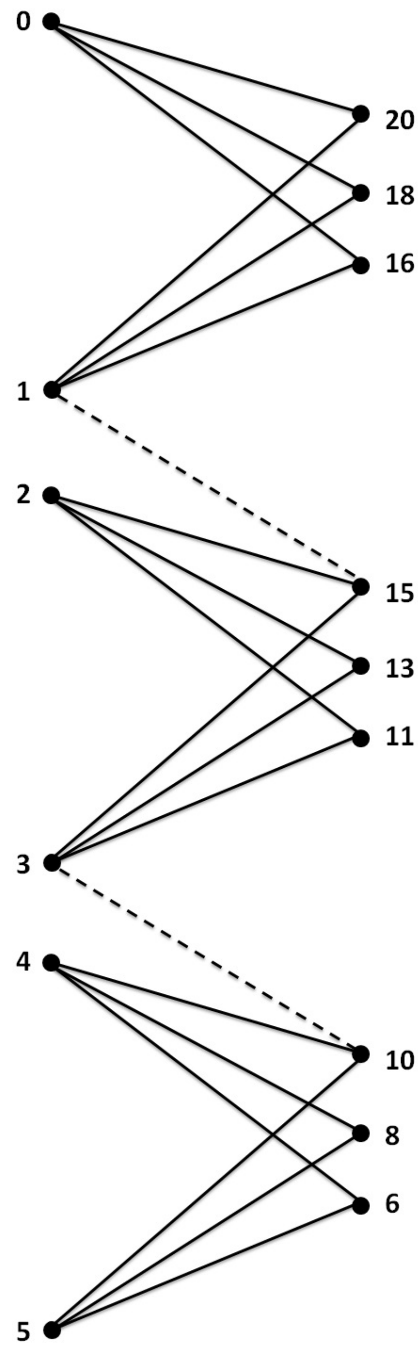

Figure 4: α-labeling of a graph obtained by barrier crossing