On the generalization of Enström-Kakeya theorem for entire functions

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Abstract

The prime concern of this paper is to extend the classical Enström-Kakeya theorem for entire functions of non zero finite order having lacunary type power series expansion. A few examples with related figures are given here to justify the results obtained.

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1 Introduction, Definitions and Notations.

The classical Enström-Kakeya theorem{cf.[5]} states that if $P(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n with real coefficients satisfying $0 \leq a_0 \leq a_1 \leq ... \leq a_n$, then all the zeros of $P(z)$ lie in the unit disk $|z| \leq 1$ in the complex plane $\mathbb{C}$. Many results on generalization of Enström-Kakeya theorem by putting various conditions on the coefficients of the polynomials exist {cf.[2],[3],[4],[5],[7] & [8]}. Aziz and Zargar first generalized Ensröm-Kakeya theorem in a different direction by considering the even indexed and odd indexed coefficients separately [1] and consequently a lot of improvements of their results for polynomials and analytic functions can be found in the literature {cf. [5],[9] & [11]}. We recall that an entire function f of one complex variable z is a function analytic in the finite complex plane $\mathbb{C}$ and therefore it can be represented by an every where convergent power series like

$$f(z) = c_0 + c_1 z + ... + c_n z^n + ...$$

where $c_i, i = 0, 1, ..., n, ...$ are real or complex constants. Thus entire functions can be thought of as the natural generalization of polynomials.

The aim of this paper is to derive zero free region for some entire functions of non zero finite order with lacunary type power series expansion putting various conditions on the coefficients c_n's. We do not explain the standard theories, notations and definitions of entire functions as those are available in [10] & [12].

Some well known definitions are given below.

Definition 1. *[10] The order ρ of an entire function $f(z)$ is defined as*

$$\rho = \inf\{k > 0 : M_f(r) < e^{r^k}, r > r_0\} \text{ where } M(r,f) := M_f(r) = \max_{|z|=r} |f(z)|.$$

Definition 1 can be alternatively stated as:

Definition 2. *[10] The order ρ of an entire function $f(z)$ is defined as*

$$\rho = \limsup_{r \longrightarrow \infty} \frac{\log^{[2]} M(r,f)}{\log r}$$

where $\log^{[k]} x = \log(\log^{[k-1]} x)$ *for* $k = 1, 2, 3, ...$ *and* $\log^{[0]} x = x$.

If $\rho < \infty$ then $f(z)$ is said to be of finite order. Also $\rho = 0$ means that $f(z)$ is of order zero.

2 Lemma.

In this section we present a lemma which will be needed in the sequel.

Lemma 2.1. *[6] Let $\{f_n(z)\}, n = 1, 2, ...$ be a sequence of functions that are analytic in a region D and that converge uniformly to a function $f(z)$ in every closed sub region of D. Let z_0 be an interior point of D. If z_0 is a limit point of the zeros of $f_n(z)$, then z_0 is a zero of $f(z)$. Conversely, if z_0 is an m-fold zero of $f(z)$, every sufficiently small neighborhood of z_0 contains exactly m zeros (counted with their multiplicities) of each f_n with $n > N$ for a sufficiently large integer N.*

Remark 2.1. *Lemma 2.1 is known as Hurwitz theorem in $\mathbb{C}$.*

3 Theorems.

In this section we present the main results of the paper.

Theorem 3.1. *Let $f(z) = a_0 + a_{n_1} z^{n_1} + a_{n_2} z^{n_2} + ...$ be an entire function of non zero finite order ρ with $a_0 \neq 0$ and $n_1, n_2, ...$ are positive integers such that $1 \leq n_1 < n_2 < ...$. Also let*

$$\rho^{n_1} |a_{n_1}| \geq \rho^{n_2} |a_{n_2}| \geq ... .$$

Then no zeros of $f(z)$ lie in

$$|z| < \frac{\rho |a_0|}{|a_0| + \rho^{n_1} |a_{n_1}|} .$$

Proof. Let

$$f_k(z) = a_0 + a_{n_1} z^{n_1} + a_{n_2} z^{n_2} + ... + a_{n_k} z^{n_k}$$

and

$$F(z) = z^{n_k} f_k(\frac{\rho}{z}) .$$

Then,

$$|F(z)| = |a_0 z^{n_k} + a_{n_1} \rho^{n_1} z^{n_k - n_1} + a_{n_2} \rho^{n_2} z^{n_k - n_2} + ... + a_{n_k} \rho^{n_k}|$$

$$\text{i.e, } |F(z)| \geq |a_0||z|^{n_k} - |a_{n_1} \rho^{n_1} z^{n_k - n_1} + a_{n_2} \rho^{n_2} z^{n_k - n_2} + ... + a_{n_k} \rho^{n_k}| . \tag{1}$$

Now for $|z| = R (> 1)$, it follows that

$$\begin{aligned}
&|a_{n_1} \rho^{n_1} z^{n_k - n_1} + a_{n_2} \rho^{n_2} z^{n_k - n_2} + ... + a_{n_k} \rho^{n_k}| \\
\leq &|a_{n_1}| \rho^{n_1} R^{n_k - n_1} + |a_{n_2}| \rho^{n_2} R^{n_k - n_2} + ... + |a_{n_k}| \rho^{n_k} \\
\leq &|a_{n_1}| \rho^{n_1} R^{n_k} \left\{ \frac{1}{R^{n_1}} + \frac{1}{R^{n_2}} + ... + \frac{1}{R^{n_k}} \right\} \\
\leq &|a_{n_1}| \rho^{n_1} R^{n_k} \sum_{j=1}^{\infty} \frac{1}{R^k} \\
= &|a_{n_1}| \rho^{n_1} R^{n_k} \frac{1}{R - 1} .
\end{aligned}$$

Hence we get from (1) for $|z| = R(> 1)$ that

$$|F(z)| \geq |a_0|R^{n_k} - |a_{n_1}|\rho^{n_1} R^{n_k} \frac{1}{R-1} > 0 \text{ if } R > \frac{|a_0| + |a_{n_1}|\rho^{n_1}}{|a_0|} .$$

Therefore,

$$|F(z)| > 0 \text{ if } |z| > \frac{|a_0| + |a_{n_1}|\rho^{n_1}}{|a_0|} .$$

Consequently,

$$|f_k(z)| > 0 \text{ if } |z| < \frac{|a_0|\rho}{|a_0| + |a_{n_1}|\rho^{n_1}} .$$

Hence by Lemma 2.1, it follows that

$$|f(z)| > 0 \text{ if } |z| < \frac{|a_0|\rho}{|a_0| + |a_{n_1}|\rho^{n_1}} .$$

This proves the theorem. □

Remark 3.1. *The following example with related figure ensures the validity of Theorem 3.1.*

Example 3.1. *Let* $f(z) = \cos z + 1$.

Now the Taylor's series expansion of $f(z)$ *is*

$$f(z) = 2 - \frac{z^2}{2!} + \frac{z^4}{4!} \dots .$$

Here, $\rho = 1, a_0 = 2$ *and* $a_{n_1} = -\frac{1}{2}$.

Also, it follows that

$$\rho^{n_1}|a_{n_1}| \geq \rho^{n_2}|a_{n_2}| \geq \dots .$$

Hence by Theorem 3.1, no zeros of $f(z)$ *lie in*

$$|z| < 0.8 .$$

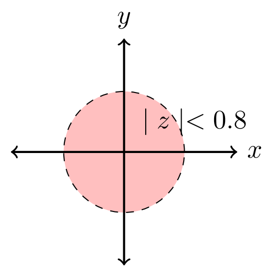

Figure 1: Zero free region of $f(z) = \cos z + 1$

Theorem 3.2. *Let* $f(z) = a_0 + a_{n_1}z^{n_1} + \dots + a_{n_l}z^{n_l} + a_{n_m}z^{n_m} + \dots$ *be an entire function of finite order* $\rho(> 0)$ *with* $a_0 \neq 0$ *and* $n_1, n_2, \dots, n_l, n_m, \dots$ *are positive integers such that* $1 \leq n_1 < \dots < n_l < n_m < \dots$. *Also let*

$$\rho^{n_l}|a_{n_l}| \geq \rho^{n_m}|a_{n_m}| \geq \dots .$$

Then no zeros of $f(z)$ *lie in*

$$|z| < \frac{|a_0|\rho}{|a_0| + M}$$

where $M = \max\{\rho^{n_1}|a_{n_1}|, \rho^{n_2}|a_{n_2}|, \dots, \rho^{n_l}|a_{n_l}|\}$.

Proof. Let

$$f_k(z) = a_0 + a_{n_1}z^{n_1} + ... + a_{n_l}z^{n_l} + a_{n_m}z^{n_m} + ... + a_{n_k}z^{n_k} \ .$$

Also, let

$$F(z) = z^{n_k} f_k(\frac{\rho}{z})$$

i.e, $F(z) = a_0 z^{n_k} + a_{n_1}\rho^{n_1}z^{n_k-n_1} + ... + a_{n_l}\rho^{n_l}z^{n_k-n_l} + a_{n_m}\rho^{n_m}z^{n_k-n_m} + ... + a_{n_k}\rho^{n_k}$.

Now for $|z| = R(> 1)$, we get that

$$\begin{aligned}
&|a_{n_1}\rho^{n_1}z^{n_k-n_1} + a_{n_2}\rho^{n_2}z^{n_k-n_2} + ... + a_{n_l}\rho^{n_l}z^{n_k-n_l} + a_{n_m}\rho^{n_m}z^{n_k-n_m} + ... + a_{n_k}\rho^{n_k}| \\
&\leq |a_{n_1}|\rho^{n_1}R^{n_k-n_1} + |a_{n_2}|\rho^{n_2}R^{n_k-n_2} + ... + |a_{n_l}|\rho^{n_l}R^{n_k-n_l} + |a_{n_m}|\rho^{n_m}R^{n_k-n_m} + ... + |a_{n_k}|\rho^{n_k} \\
&\leq MR^{n_k}\left\{\frac{1}{R^{n_1}} + \frac{1}{R^{n_2}} + ... + \frac{1}{R^{n_l}} + \frac{1}{R^{n_m}} + ... + \frac{1}{R^{n_k}}\right\} \\
&\text{where } M = \max\{\rho^{n_1}|a_{n_1}|, \rho^{n_2}|a_{n_2}|, ..., \rho^{n_l}|a_{n_l}|\} \\
&\leq MR^{n_k}\sum_{j=1}^{\infty}\frac{1}{R^k} \\
&= MR^{n_k}\frac{1}{R-1} \ .
\end{aligned}$$

Hence for $|z| = R(> 1)$, it follows that

$$|F(z)| \geq |a_0|R^{n_k} - MR^{n_k}\frac{1}{R-1} > 0 \text{ if } R > \frac{|a_0| + M}{|a_0|} \ .$$

Therefore,

$$|F(z)| > 0 \text{ if } |z| > \frac{|a_0| + M}{|a_0|} \ .$$

Consequently,

$$|f_k(z)| > 0 \text{ if } |z| < \frac{|a_0|\rho}{|a_0| + M} \ .$$

Finally by Lemma 2.1, we have

$$|f(z)| > 0 \text{ if } |z| < \frac{|a_0|\rho}{|a_0| + M} \ .$$

Thus the theorem is established. □

Remark 3.2. *The following example with related figure justifies the validity of Theorem 3.2.*

Example 3.2. *Let* $f(z) = \cos z + 1 + z + z^3$.

Then the Taylor's series expansion of $f(z)$ *is*

$$f(z) = 2 + z - \frac{z^2}{2!} + z^3 + \frac{z^4}{4!}... \ .$$

Here, $\rho = 1, a_0 = 2, a_{n_1} = 1, a_{n_2} = -\frac{1}{2}, a_{n_3} = 1$

and

$$\rho^{n_3}|a_{n_3}| \geq \rho^{n_4}|a_{n_4}| \geq ... \ .$$

Hence by Theorem 3.2, no zeros of $f(z)$ *lie in*

$$|z| < 0.67 \ .$$

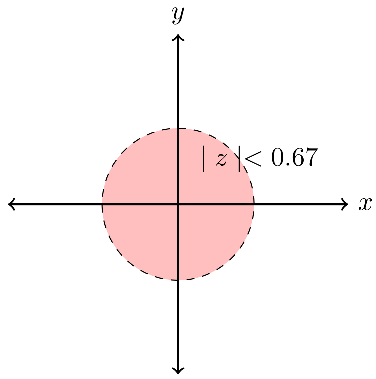

Figure 2: Zero free region of $f(z) = \cos z + 1 + z + z^3$

Theorem 3.3. *Let $f(z) = a_0 + a_1 z + a_2 z^2 + ...$ be an entire function of non zero finite order ρ with $a_0 \neq 0$. Also, let*

$$|a_0| \geq \rho^2 |a_2| \geq \rho^4 |a_4| \geq ...$$

and

$$\rho |a_1| \geq \rho^3 |a_3| \geq \rho^5 |a_5| \geq ... .$$

Then $f(z)$ does not vanish in

$$|z| < \frac{\rho |a_0|}{|a_0| + 2M}$$

where $M = \max\{|a_0|, |a_1|\rho\}$.

Proof. Let

$$f_n(z) = a_0 + a_1 z + a_2 z^2 + ... + a_n z^n$$

and

$$F(z) = z^n f_n(\frac{\rho}{z}) .$$

Again, let

$$\begin{aligned} &Q(z) = (z^2 - 1)F(z) \\ &\text{i.e, } Q(z) = (z^2 - 1)(a_0 z^n + a_1 \rho z^{n-1} + a_2 \rho^2 z^{n-2} + ... + a_n \rho^n) \\ &\text{i.e, } Q(z) = a_0 z^{n+2} + a_1 \rho z^{n+1} + (a_2 \rho^2 - a_0) z^n + (a_3 \rho^3 - a_1 \rho) z^{n-1} + ... - a_{n-1} \rho^{n-1} z - a_n \rho^n \\ &\text{i.e, } Q(z) = a_0 z^{n+2} + P(z) . \end{aligned} \tag{2}$$

Now for $|z| = R(> 1)$, we get that

$$\begin{aligned} |P(z)| &= |a_1 \rho z^{n+1} + (a_2 \rho^2 - a_0) z^n + (a_3 \rho^3 - a_1 \rho) z^{n-1} + ... - a_{n-1} \rho^{n-1} z - a_n \rho^n| \\ &\leq 2MR^{n+2} \left\{ \frac{1}{R} + \frac{1}{R^2} + ... + \frac{1}{R^n} \right\} \text{ where } M = \max\{|a_0|, |a_1|\rho\} \\ &\leq 2MR^{n+2} \sum_{k=1}^{\infty} \frac{1}{R^k} \\ &= 2MR^{n+2} \frac{1}{R-1} . \end{aligned}$$

Hence for $|z| = R(> 1)$, it follows from (2) that

$$|Q(z)| \geq |a_0| R^{n+2} - 2MR^{n+2} . \frac{1}{R-1} > 0 \text{ if } R > \frac{|a_0| + 2M}{|a_0|}$$

$$\text{i.e, } |Q(z)| > 0 \text{ if } |z| > \frac{|a_0| + 2M}{|a_0|} \ .$$

Therefore,

$$|F(z)| > 0 \text{ if } |z| > \frac{|a_0| + 2M}{|a_0|}$$

$$\text{i.e, } |f_n(\frac{\rho}{z})| > 0 \text{ if } |z| > \frac{|a_0| + 2M}{|a_0|}$$

$$\text{i.e, } |f_n(z)| > 0 \text{ if } |z| < \frac{|a_0|\rho}{|a_0| + 2M} \ .$$

Thus, it follows by Lemma 2.1 that

$$|f(z)| > 0 \text{ if } |z| < \frac{|a_0|\rho}{|a_0| + 2M} \ .$$

This completes the proof of the theorem. □

Remark 3.3. *The following example with related figure ensures the validity of Theorem 3.3.*

Example 3.3. *Let* $f(z) = \sin 2z + \cos z$.

Then the Taylor's series expansion of $f(z)$ *is*

$$f(z) = 1 + 2z - \frac{z^2}{2} - \frac{4}{3}z^3 + \frac{z^4}{24} + \frac{4}{15}z^5 - \dots \ .$$

Here, $\rho = 1, a_0 = 1$ *and* $a_1 = 2$.

Also, it follows that

$$|a_0| \geq \rho^2|a_2| \geq \rho^4|a_4| \geq \dots$$

and

$$\rho|a_1| \geq \rho^3|a_3| \geq \rho^5|a_5| \geq \dots \ .$$

Hence by Theorem 3.3, $f(z)$ *does not vanish in*

$$|z| < 0.2 \ .$$

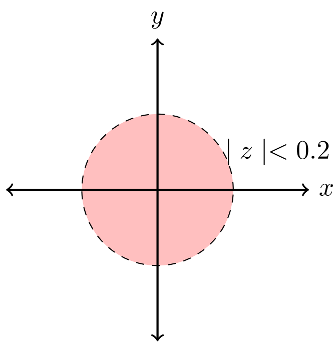

Figure 3: Zero free region of $f(z) = \sin 2z + \cos z$

Theorem 3.4. *Let $f(z) = a_0 + a_1 z + a_2 z^2 + ...$ be an entire function of finite order $\rho(> 0)$ with $a_0 \neq 0$. Also, let for some positive integer l*

$$\rho^{2l}|a_{2l}| \geq \rho^{2l+2}|a_{2l+2}| \geq \rho^{2l+4}|a_{2l+4}| \geq ...$$

and

$$\rho^{2l+1}|a_{2l+1}| \geq \rho^{2l+3}|a_{2l+3}| \geq \rho^{2l+5}|a_{2l+5}| \geq ... \ .$$

Then $f(z)$ does not vanish in

$$|z| < \frac{|a_0|\rho}{|a_0| + 2M}$$

where $M = \max\{|a_0|, \rho|a_1|, ..., \rho^{2l}|a_{2l}|, \rho^{2l+1}|a_{2l+1}|\}$.

The proof of Theorem 3.4 can be carried out in the line of Theorem 3.3 and therefore its proof is omitted.

Remark 3.4. *The following example with related figure justifies the validity of Theorem 3.4.*

Example 3.4. *Let $f(z) = \sin 2z + \cos z + 1 - 2z^2 - z^3$.*

Now the Taylor's series expansion of $f(z)$ is

$$f(z) = 2 + 2z - \frac{5}{2}z^2 - \frac{7}{3}z^3 + \frac{z^4}{24} + \frac{4}{15}z^5 - ... \ .$$

Here, $\rho = 1, a_0 = 2, a_1 = 2, a_2 = -\frac{5}{2}$ and $a_3 = -\frac{7}{3}$.

Again, it follows that

$$\rho^2|a_2| \geq \rho^4|a_4| \geq \rho^6|a_6| \geq ...$$

and

$$\rho^3|a_3| \geq \rho^5|a_5| \geq \rho^7|a_7| \geq ... \ .$$

Hence by Theorem 3.4, $f(z)$ does not vanish in

$$|z| < 0.29 \ .$$

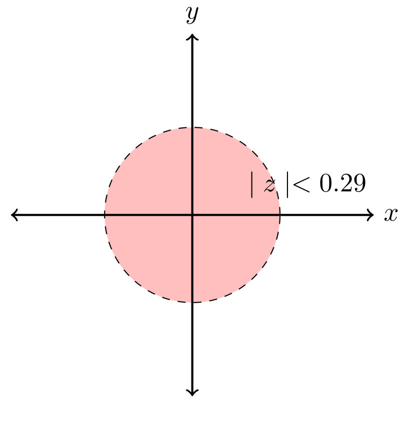

Figure 4: Zero free region of $f(z) = \sin 2z + \cos z + 1 - 2z^2 - z^3$

Future prospect. In the line of the works as carried out in the paper one may think of more generalization of Enström-Kakeya theorem for entire functions with infinite order.

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