Some Algebraic Properties of Partial Multiplicative Arithmetic Functions with Respect to Partial Basic Sequence on the Set of Square-free Integers

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ABSTRACT

In this paper, we consider arithmetic functions from a set of positive integers which are Square-free to Real numbers and also introduce Partial basic sequence. We establish some basic algebraic result of Partial Multiplicative Function with respect to partial basic sequences.

KEY WORDS

Arithmetic functions, Square-free integers, Convolution, Partial basic sequence, Partial multiplicative functions.

1. INTRODUCTION

A real or complex valued function defined of the set of all natural numbers or the set of all positive integers is called an arithmetical function. Their various properties were investigated by several authors and they represent an important research topic [1, 2, 3, 4]. Properties of $\mathcal{B}$-multiplicative and quasi $\mathcal{B}$-multiplicative functions are studied in [5, 6].

In this paper, first we consider the set of square-free integers and define partial multiplication with respect to partial operator $*$ on $Å \times Å$. We introduce a partial basic sequence of $Å \times Å$ satisfying three properties. We introduce convolution operator $\circ$ on partial basic sequence. Finally we prove some general properties of partial multiplicative arithmetic functions and define related the convolution. Using this convolution we establish some algebraic results on partial multiplicative functions.

2. PRELIMINARIES

Let $\mathbb{Z}^+$ Be the set of all positive integers.

Define $Å = \{n \in \mathbb{Z}^+ | n \text{ is squarefree}\}$ (i.e., p is a prime, $p|n \implies p^2 \dagger n$). Clearly $1 \in Å$.

Let $*$ be the partial binary operator defined on Å as follows.

For $m, n \in Å$, $m * n = mn$ is not always defined but it is defined only when $(m, n) = 1$.

(i.e., gcd of $m, n = 1$).

Let F be the sub set of $Å \times Å$ such that $(m, n) \in \mathrm{F}$ if $(m, n) = 1$.

Thus $\mathrm{F} = \{(m, n) | m, n \in Å, (m, n) = 1\}$.

We observe that $m * n$ is defined if $(m, n) \in \mathrm{F}$.

F has the following properties:

(i) $(a, b) \in \mathrm{F} \Leftrightarrow (b, a) \in \mathrm{F}$

(ii) Suppose $a, b, c \in Å$. Then $(a, bc) \in \mathrm{F} \Leftrightarrow (a, b) \in \mathrm{F}, (a, c) \in \mathrm{F}$ and $(b, c) = 1$

(iii) $(1, a) \in \mathrm{F}$ for all $a \in Å$.

F is called partial basic sequence on Å.

Note: Observe that

1. $m * n = n * m$ if $m, n \in$ Å and $(m, n) = 1$.
2. $(l * m) * n = l * (m * n)$ if $l, m, n \in$ Å whenever one side is meaningful.

Here $(l, m) = 1$ and $(lm, n) = 1 \iff (m, n) = 1$ and $(l, mn) = 1$.

We define the convolution operator $\circ$ for functions defined on Å.

Suppose $f :$ Å $\longrightarrow \mathbb{R}$, $g :$ Å $\longrightarrow \mathbb{R}$ *and* $h :$ Å $\longrightarrow \mathbb{R}$.

Define $f \circ g :$ Å $\longrightarrow \mathbb{R}$ by

$$(f \circ g)(n) = \sum_{d|n} f(d)\, g\left(\frac{n}{d}\right), \text{ for all } n \in \text{Å}$$

3. RESULTS

3.1 Lemma: For all $n \in$ Å, $d \mid n \implies \left(d, \frac{n}{d}\right) = 1$.

3.2 Lemma: If m, n are positive integers and $(m, n) = 1$ then $d \mid mn \implies$

$\exists$ unique pair (δ_1, δ_2) such that $\delta_1 \mid m$, $\delta_2 \mid n$ and $\delta_1 \delta_2 = d$.

3.3 Definition: Define $I_{\text{Å}} :$ Å $\longrightarrow \mathbb{R}$ by

$$I_{\text{Å}} = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{otherwise} \end{cases}$$

Write $\mathcal{F} = \{f \mid f :$ Å $\longrightarrow \mathbb{R}\}$,

We observe that $I_{\text{Å}} \in \mathcal{F}$ and $f, g \in \mathcal{F} \implies f \circ g \in \mathcal{F}$.

3.4 Theorem: $f \in \mathcal{F} \implies f \circ I_{\text{Å}} = f$.

Proof:

$$(f \circ I_{\text{Å}})(1) = \sum_{d|1} f(d)\, I_{\text{Å}}\left(\frac{1}{d}\right) = f(1) I_{\text{Å}}(1) = f(1)$$

$$(f \circ I_{\text{Å}})(2) = \sum_{d|2} f(d)\, I_{\text{Å}}\left(\frac{2}{d}\right) = f(1) I_{\text{Å}}(2) + f(2) I_{\text{Å}}(1) = f(2)$$

$$(f \circ I_{\text{Å}})(\text{n}) = \sum_{d|\text{n}} f(d)\, I_{\text{Å}}\left(\frac{n}{d}\right) = f(n) I_{\text{Å}}(1) + \sum_{\substack{d|\text{n} \\ 1 \le d < n}} f(d)\, I_{\text{Å}}\left(\frac{n}{d}\right) = f(n).1 + 0 = f(n), for\ n \ge 2$$

Therefore $f \circ I_{\text{Å}} = f$.

3.5 Definition: $I_{\text{Å}}$ is called the identity function of $\mathcal{F}$.

3.6 Theorem: For all $f, g \in \mathcal{F}$, $f \circ g = g \circ f$.

Proof: For $n \in$ Å,

$$(f\circ g)(n)=\sum_{d|n} f(d)\,g\left(\frac{n}{d}\right)$$

Now,

$$(g\circ f)(n)=\sum_{d|n} g(d)f\left(\frac{n}{d}\right)$$

$$=\sum_{d|n} f\left(\frac{n}{d}\right)g(d)$$

$$=\sum_{\delta|n} f(\delta)g\left(\frac{n}{\delta}\right)\quad where\ \ \delta=\frac{n}{d}$$

$$=(f\circ g)(n)$$

So that $f\circ g=g\circ f\ \ \ for\ all\ n\in Å$.

3.7 Theorem: For all $f,g,h\in Å$, we have $f\circ(g\circ h)=(f\circ g)\circ h$.

Proof: For $n\in Å$

$$\big((f\circ g)\circ h\big)(n)=\sum_{d|n}(f\circ g)(d)\,h\left(\frac{n}{d}\right)$$

$$=\sum_{d|n}\left(\sum_{\delta|d} f(\delta)g\left(\frac{d}{\delta}\right)\right)h\left(\frac{n}{d}\right)\qquad \ldots\ldots\ldots.\ldots.. \ (7.1)$$

If $\frac{n}{d}=\eta$ and $\frac{d}{\delta}=\lambda$ then $n=\frac{n}{d}d=\eta\lambda\delta$.

Hence

$$(7.1)=\sum_{d|n} h\left(\frac{n}{d}\right)\sum_{\delta|d} f(\delta)g\left(\frac{d}{\delta}\right)=\sum_{n=\eta\lambda\delta} h(\eta)f(\delta)g(\lambda)\ \ldots\ldots\ldots\ldots\ (7.2)$$

$$\big(f\circ(g\circ h)\big)(n)=\sum_{d|n} f(d)\,(g\circ h)\left(\frac{n}{d}\right)$$

$$=\sum_{d|n} f(d)\,(g\circ h)(\eta)$$

$$=\sum_{d|n} f(d)\left(\sum_{\lambda|\eta} g(\lambda)h\left(\frac{\eta}{\lambda}\right)\right)$$

$$=\sum_{d|n} f(d)\sum_{\lambda|\eta} g(\lambda)h(\delta)$$

$$=\sum_{n=\eta\lambda\delta} h(\delta)f(\eta)g(\lambda)$$

Therefore, $f\circ(g\circ h)=(f\circ g)\circ h$

3.8 Theorem: Suppose $f\in\mathcal{F}\ and\ f(1)\neq 0$. Then there exists unique $g\in\mathcal{F}$ such that $f\circ g=I_{Å}$.

Proof: Define

$g(1) = \frac{1}{f(1)}$ (since $f(1) \neq 0$).

So that $(f \circ g)(1) = f(1)g(1) = I_{Å}(1)$.

Define $g(2) = -\frac{f(2)g(1)}{f(1)} = -\frac{f(2)}{(f(1))^2}$

So that $(f \circ g)(2) = f(1)g(2) + f(2)g(1) = 0 = I_{Å}(2)$.

If $n > 1$, for $n \in Å$, we define g inductively by

$$g(n) = -\frac{1}{f(1)}\left(f(n)g(1) + \sum_{\substack{d|n \\ 1<d<n}} f(d)\, g\left(\frac{n}{d}\right)\right)$$

So that

$$(f \circ g)(n) = \sum_{d|n} f(d)\, g\left(\frac{n}{d}\right)$$

$$= f(1)g(n) + f(n)g(1) + \sum_{\substack{d|n \\ 1<d<n}} f(d)\, g\left(\frac{n}{d}\right) = 0 = I_{Å}(n)$$

Therefore

$$f(1)g(n) = -f(n)g(1) - \sum_{\substack{d|n \\ 1<d<n}} f(d)\, g\left(\frac{n}{d}\right).$$

Thus $(f \circ g)(n) = I_{Å}(n) \quad \forall\, n \in Å$.

So that $f \circ g = I_{Å}$.

3.9 Note: If h is such that $foh = I_{Å}$ then

$$h = h \circ I_{Å} = h \circ (f \circ g) = (h \circ f) \circ g = (f \circ h) \circ g = I_{Å} \circ g = g$$

Hence such g is unique and is written as f^{-1}.

3.10 Observations: Theorems 3.4, 3.6 and 3.7 show that $(\mathcal{F}_1, \circ)$ is a commutative semi group with identity. Further, by Theorem 3.8, $\mathcal{F}_1 = \{f \in \mathcal{F}_1, f(1) \neq 0\}$ is a commutative group.

4. MAIN RESULTS

4.1 Definition: Suppose $f \in \mathcal{F}$, we say that f is partially multiplicative function if f is not identically zero and $f(m * n) = f(m)f(n)$ whenever $(m, n) \in \mathcal{F}$.

4.2 Theorem: If f is a partially multiplicative function then $f(1) = 1$.

Proof: Suppose f is partially multiplicative. Then

$$f(1) = f(1.1) = f(1)f(1) \qquad \text{since } (1,1) \in \mathcal{F}.$$

Therefore $f(1) = 0$ or 1.

Suppose $f(1) = 0$. Then for any $n \in$ Å, we have

$$f(n) = f(1.n) = f(1)f(n) \qquad \text{since } (1,n) \in \mathcal{F}.$$

Therefore, $f(n) = 0 \ \ \forall\, n \in$ Å.

i.e., f is identically zero on Å, a contradiction.

Therefore, $f(1) = 1$.

4.3 Theorem: If f, g are partially multiplicative then $f \circ g$ is also a partially multiplicative function.

Proof: Clearly $(f \circ g)(1) = f(1)g(1) = I_{Å}(1) = 1$.

Suppose $m, n \in$ Å and $(m,n) = 1$ so that $mn \in$ Å.

By Lemma 1, there exist a unique pair (δ_1, δ_2) such that $d \,\big|\, mn \Leftrightarrow d = \delta_1\delta_2$

(infact $\delta_1 = (d,m), \delta_2 = (d,n)$ since $(m,n) = 1$), $\delta_1 \,\big|\, m,\ \ \delta_2 \,\big|\, n$.

Now,

$$(f \circ g)(m * n) = (f \circ g)(mn) = \sum_{d \mid mn} f(d)\, g\left(\frac{mn}{d}\right)$$

$$= \sum_{\delta_1\delta_2 \mid mn} f(\delta_1\delta_2) g\left(\frac{mn}{\delta_1\delta_2}\right)$$

where $\delta_1 = (d,m), \delta_2 = (d,n)$.

$$= \sum_{\delta_1\delta_2 \mid mn} f(\delta_1) f(\delta_2)\, g\left(\frac{m}{\delta_1}\right) g\left(\frac{n}{\delta_2}\right)$$

$$= \sum_{\delta_1 \mid m,\ \delta_2 \mid n} f(\delta_1) f(\delta_2) g\left(\frac{m}{\delta_1}\right) g\left(\frac{n}{\delta_2}\right)$$

$$= \sum_{\delta_1 \mid m,\ \delta_2 \mid n} f(\delta_1) g\left(\frac{m}{\delta_1}\right) f(\delta_2) g\left(\frac{n}{\delta_2}\right)$$

$$= \left(\sum_{\delta_1 \mid m} f(\delta_1) g\left(\frac{m}{\delta_1}\right)\right)\left(\sum_{\delta_2 \mid n} f(\delta_2) g\left(\frac{n}{\delta_2}\right)\right)$$

$$= (f \circ g)(m)(f \circ g)(n).$$

Therefore $(f \circ g)(m * n) = (f \circ g)(m)(f \circ g)(n)$.

Therefore $f \circ g$ is partially multiplicative.

4.4 Theorem: If f is partially multiplicative then f^{-1} is also partially multiplicative.

Proof : If $m = 1$ then

$$f^{-1}(mn) = f^{-1}(1.n) = f^{-1}(n) = 1.f^{-1}(n) = f^{-1}(1)f^{-1}(n) = f^{-1}(m)f^{-1}(n).$$

Therefore $f^{-1}(mn) = f^{-1}(m)f^{-1}(n)$.

If $m = 1, n = 2$ then $(1,2) \in \mathcal{F}$

$$f^{-1}(mn) = f^{-1}(1.2) = f^{-1}(2) = f^{-1}(1)f^{-1}(2).$$

Therefore $f^{-1}(mn) = f^{-1}(m)f^{-1}(n)$.

Suppose this result is true for $(m', n') \in \mathcal{F},\ m'n' < mn\ and\ (m,n) \in \mathcal{F}$.

Then $0 = (f \circ f^{-1})(m * n)$

$$= \sum_{d|mn} f(d)\, f^{-1}\left(\frac{mn}{d}\right)$$

$$= \sum_{\delta_1\delta_2|mn} f(\delta_1\delta_2)\, f^{-1}\left(\frac{mn}{\delta_1\delta_2}\right)$$

$$= \sum_{\delta_1|m,\ \delta_2|n} f(\delta_1\delta_2)\, f^{-1}\left(\frac{mn}{\delta_1\delta_2}\right)$$

$$= f(1\cdot 1)f^{-1}(mn) + f(mn)f^{-1}(1\cdot 1) + \sum_{\substack{\delta_1|m,\ \delta_2|n \\ 1<\delta_1\delta_2<mn}} f(\delta_1\delta_2)f^{-1}\left(\frac{mn}{\delta_1\delta_2}\right)$$

$$= f(1)f^{-1}(mn) + f(mn)f^{-1}(1) + \sum_{\substack{\delta_1|m,\ \delta_2|n \\ 1<\delta_1\delta_2<mn}} f(\delta_1\delta_2)f^{-1}\left(\frac{m}{\delta_1}\cdot\frac{n}{\delta_2}\right)$$

$$= f(mn) + \sum_{\substack{\delta_1|m,\ \delta_2|n \\ 1<\delta_1\delta_2<mn}} f(\delta_1)\, f(\delta_2)\, f^{-1}\left(\frac{m}{\delta_1}\right)f^{-1}\left(\frac{n}{\delta_2}\right) + f^{-1}(mn)$$

$$= f(m)f(n) + \sum_{\substack{\delta_1|m,\ \delta_2|n \\ 1<\delta_1\delta_2<mn}} f(\delta_1)\, f(\delta_2)\, f^{-1}\left(\frac{m}{\delta_1}\right)f^{-1}\left(\frac{n}{\delta_2}\right)$$

$$+ \left(f^{-1}(mn) - f^{-1}(m)f^{-1}(n) + f^{-1}(m)f^{-1}(n)\right)$$

$$= \sum_{\substack{\delta_1|m,\ \delta_2|n \\ 1\le\delta_1\delta_2\le mn}} f(\delta_1)\, f(\delta_2)\, f^{-1}\left(\frac{m}{\delta_1}\right)f^{-1}\left(\frac{n}{\delta_2}\right) + f^{-1}(mn) - f^{-1}(m)f^{-1}(n)$$

$$= \left(\sum_{\delta_1|m} f(\delta_1)f^{-1}\left(\frac{m}{\delta_1}\right)\right)\left(\sum_{\delta_2|n} f(\delta_2)f^{-1}\left(\frac{n}{\delta_2}\right)\right) + f^{-1}(mn) - f^{-1}(m)f^{-1}(n)$$

$$= (f \circ f^{-1})(m)\cdot(f \circ f^{-1})(n) + f^{-1}(mn) - f^{-1}(m)f^{-1}(n)$$

$$= I_{\text{Å}}(m)\cdot I_{\text{Å}}(n) + f^{-1}(mn) - f^{-1}(m)f^{-1}(n)$$

$$= f^{-1}(mn) - f^{-1}(m)f^{-1}(n)$$

Therefore $f^{-1}(m * n) = f^{-1}(m)f^{-1}(n)$.

Therefore, f is partially multiplicative implies f^{-1} is also a partially multiplicative.

4.5 Theorem: If $f\ and\ f \circ g$ are partially multiplicative then g is partially multiplicative.

Proof : Suppose $(m,n) \in \mathcal{F}$. Then

$$g(m*n) = g(mn)$$

$$= \left(f^{-1} \circ (f \circ g)\right)(mn)$$

$$= \sum_{d \mid mn} f^{-1}(d)(f \circ g)\left(\frac{mn}{d}\right)$$

$$= \sum_{\delta_1 \mid m,\ \delta_2 \mid n} f^{-1}(\delta_1\delta_2)(f \circ g)\left(\frac{mn}{\delta_1\delta_2}\right)$$

$$= f^{-1}(mn)(f \circ g)(1) + \sum_{\substack{\delta_1 \mid m,\ \delta_2 \mid n \\ 1<\delta_1\delta_2<mn}} f^{-1}(\delta_1\delta_2)(f \circ g)\left(\frac{mn}{\delta_1\delta_2}\right) + f^{-1}(1)(f \circ g)(mn)$$

$$= f^{-1}(mn) + (f \circ g)(mn) + \sum_{\substack{\delta_1 \mid m,\ \delta_2 \mid n \\ 1<\delta_1\delta_2<mn}} f^{-1}(\delta_1\delta_2)(f \circ g)\left(\frac{m}{\delta_1}\right)(f \circ g)\left(\frac{n}{\delta_2}\right)$$

$$= f^{-1}(mn) + (f \circ g)(mn) + f^{-1}(1)(f \circ g)(m)(f \circ g)(n) - f^{-1}(1)(f \circ g)(m)(f \circ g)(n)$$
$$+ f^{-1}(mn)(f \circ g)(1)(f \circ g)(1) - f^{-1}(mn)(f \circ g)(1)(f \circ g)(1)$$
$$+ \sum_{\substack{\delta_1 \mid m,\ \delta_2 \mid n \\ 1<\delta_1\delta_2<mn}} f^{-1}(\delta_1\delta_2)(f \circ g)\left(\frac{m}{\delta_1}\right)(f \circ g)\left(\frac{n}{\delta_2}\right)$$

$$= f^{-1}(mn) + (f \circ g)(mn) - (f \circ g)(m)(f \circ g)(n) - f^{-1}(mn)$$
$$+ \sum_{\substack{\delta_1 \mid m,\ \delta_2 \mid n \\ 1<\delta_1\delta_2<mn}} f^{-1}(\delta_1)f^{-1}(\delta_2)(f \circ g)\left(\frac{m}{\delta_1}\right)(f \circ g)\left(\frac{n}{\delta_2}\right)$$

$$= \left(\sum_{\delta_1 \mid m} f(\delta_1)(f \circ g)\left(\frac{m}{\delta_1}\right)\right)\left(\sum_{\delta_2 \mid n} f(\delta_2)(f \circ g)\left(\frac{n}{\delta_2}\right)\right)$$

$$= \left(f^{-1} \circ (f \circ g)\right)(m)\left(f^{-1} \circ (f \circ g)\right)(n)$$

$$= g(m)g(n)$$

Therefore $g(m*n) = g(m)g(n)$.

Therefore, if f *and* $f \circ g$ are partially multiplicative then g is partially multiplicative.

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