Derivation of relative deficiencies of wronskians on the basis of integrated moduli of logarithmic derivative of entire functions

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Abstract

In this paper we wish to derive some relative Valiron deficiencies as compared to relative Nevanlinna deficiencies from the view point of integrated moduli of logarithmic derivative of wronskians generated by transcendental entire functions. Some examples and counter examples are provided to justify the conclusion of the results.

AMS Subject Classification (2010): 30D30, 30D35.

Keywords and Phrases: Transcendental entire function, Wronskian, Integrated moduli of logarithmic derivative of entire function, Relative Nevanlinna defect with respect to $I(r, f)$, Relative Valiron defect with respect to $I(r, f)$.

1 Introduction, Definitions and Notations

Let f be a meromorphic function defined in the open complex plane $\mathbb{C}$. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $n(t, a; f)$ $(\overline{n}(t, a; f))$ the number of a-points (distinct a-points) of f in $|z| \leq t$, where an ∞-point is a pole of f . We put

$$N(r, a; f) = \int_0^r \frac{n(t, a; f) - n(0, a; f)}{t} dt + n(0, a; f) \log r$$

and

$$\bar{N}(r, a; f) = \int_0^r \frac{\overline{n}(t, a; f) - \overline{n}(0, a; f)}{t} dt + \overline{n}(0, a; f) \log r.$$

The function $N(r, a; f)(\bar{N}(r, a; f))$ is called the counting funtion of a-points (distinct a-points) of f. In many occasions $N(r, \infty; f)$ and $\bar{N}(r, \infty; f)$ are denoted by $N(r, f)$ and $\bar{N}(r, f)$ respectively. We also put

$$m(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| f(re^{i\theta}) \right| d\theta,$$

where

$$\log^+ x = \log x, \text{ if } x \geq 1$$
$$= 0, \quad \text{if } 0 \leq x < 1.$$

For $a \in \mathbb{C}$ we denote by $m(r, \frac{1}{f-a})$ by $m(r, a; f)$ and we mean by $m(r, \infty; f)$ the function $m(r, f)$,which is called the proximity function of f.

The function $T(r, f) = m(r, f) + N(r, f)$ is called the characteristic function of f. If $a \in \mathbb{C} \cup \{\infty\}$, the quantity

$$\delta(a; f) = 1 - \limsup_{r \to \infty} \frac{N(r, a; f)}{T(r, f)} = \liminf_{r \to \infty} \frac{m(r, a; f)}{T(r, f)}$$

is called the Nevanlinna deficiency of the value a.
From the second fundamental theorem it follows that the set of values of $a \in \mathbb{C} \cup \{\infty\}$ for which $\delta(a; f) > 0$ is countable and $\sum_{a \neq \infty} \delta(a; f) + \delta(\infty; f) \leq 2$ (cf. [1, p.43]). If in particular, $\sum_{a \neq \infty} \delta(a; f) + \delta(\infty; f) = 2$, we say that f has the maximum deficiency sum.
Similarly,the Valiron deficiency $\Delta(a, f)$ of the value 'a' is defined as

$$\Delta(a; f) = 1 - \liminf_{r \to \infty} \frac{N(r, a; f)}{T(r, f)} = \limsup_{r \to \infty} \frac{m(r, a; f)}{T(r, f)}.$$

Milloux [6] introduced the concept of absolute defect of 'a' with respect to f'.Later Xiong [5] extended this definition.He introduced the term

$$\delta_R^{(k)}(a; f) = 1 - \limsup_{r \to \infty} \frac{N(r, a; f^{(k)})}{T(r, f)},$$

for $k = 1, 2, 3, ...$and called it the relative Nevanlinna defect of 'a' with respect to $f^{(k)}$.Xiong [10] has shown various relations between the usual defects and the relative defects for meromorphic functions. Singh [8] introduced the term relative defect for distinct zeros and poles and established various relations between it, relative defects and the usual defects.

The following definitions are well known.

Definition 1.1 *The order ρ_f and lower order λ_f of a meromorphic function f are defined as follows:*

$$\rho_f = \limsup_{r \to \infty} \frac{\log T(r, f)}{\log r} \text{ and } \lambda_f = \liminf_{r \to \infty} \frac{\log T(r, f)}{\log r}.$$

If $\rho_f < \infty$ then f is of finite order.

The term $S(r, f)$ denotes any quantity satisfying $S(r, f) = o\{T(r, f)\}$ as $r \to \infty$ through all values of r if f is of finite order and except possibly for a set of r of finite linear measure otherwise.We do not explain the standard definitions and notations of the value distribution and the Nevanlinna theory as those are available in [4].

Definition 1.2 *A meromorphic function $a = a(z)$ is called small with respect to f if $T(r, a) = S(r, f)$.*

Let $a_1, a_2, ...a_k$ be linearly independent meromorphic functions and small with respect to f. We denote by $L(f) = W(a_1, a_2, ...a_k, f)$ the wronskian determinant of $a_1, a_2, ...a_k, f$ i.e.,

$$L(f) = \begin{vmatrix} a_1 & a_2 & .. & .. & a_k & f \\ a_1^{'} & a_2^{'} & .. & .. & a_k^{'} & f^{'} \\ .. & .. & .. & .. & .. & .. \\ .. & .. & .. & .. & .. & .. \\ .. & .. & .. & .. & .. & .. \\ a_1^{(k)} & a_2^{(k)} & .. & .. & a_k^{(k)} & f^{(k)} \end{vmatrix}.$$

We may now recall the following definition.

If f be a meromorphic function in the complex plane. Then the integrated moduli of the logarithmic derivative $I(r,f)$ is defined by

$$I(r,f)=\frac{r}{\pi}\int_0^{2\pi}\left|\frac{f'(re^{i\theta})}{f(re^{i\theta})}\right|d\theta,$$

for $0<r<+\infty$ $\{cf.$ [9]$\}$

In this paper we call the following four terms by using the concept of $I(r,f)$

$$\delta_I(a;f)=1-\limsup_{r\to\infty}\frac{N(r,a;L(f))}{I(r,f)},$$

the relative Nevanlinna defect of 'a' with respect to $I(r,f)$,

$$\Delta_I(a;f)=1-\liminf_{r\to\infty}\frac{N(r,a;f)}{I(r,f)},$$

the relative Vairon defect of 'a' with respect to $I(r,f)$,

$$\delta_I^L(a;f)=1-\limsup_{r\to\infty}\frac{N(r,a;L(f))}{I(r,f)},$$

the relative Nevanlinna defect of 'a' of $L(f)$ with respect to $I(r,f)$ and

$$\Delta_I^L(a;f)=1-\liminf_{r\to\infty}\frac{N(r,a;L(f))}{I(r,f)},$$

the relative Valiron defect 'a' of $L(f)$ with respect to $I(r,f)$.

Since the natural extension of a derivative is a differential polynomial, in this paper we prove our results for a special type of linear differential polynomials viz. the wronskians. In the present paper we establish some relationship between relative Nevanlinna's defect and relative Valiron defect under the flavour of integrated moduli of logarithmic derivative of wronskians generated by transcendental entire functions. Also relevant examples and counter examples are provided in order to ensure the sharper estimation of the results obtained.

2 Lemmas.

In this section we present some lemmas which will be needed in the sequel.

Lemma 2.1 *[5] Let f be a transcendental meromorphic function function having the maximum deficiency sum. Then*

$$\lim_{r\to\infty}\frac{T(r,L(f))}{T(r,f)}=1+k-k\delta(\infty;f).$$

Lemma 2.2 *[1] Let f be a transcendental meromorphic function having the maximum deficiency sum. Then for any α,*

$$\Delta_R^L(\alpha;f)=\{k\delta(\infty;f)-k\}+\limsup_{r\to\infty}\frac{m(r,\alpha;L(f))}{T(r,f)}$$

and

$$\delta_R^L(\alpha;f)=\{k\delta(\infty;f)-k\}+\liminf_{r\to\infty}\frac{m(r,\alpha;L(f))}{T(r,f)}.$$

Lemma 2.3 *[4] Let k be any positive integer and $\Psi=\sum\limits_{i=0}^{k}a_i f^{(i)}$, where a_i are meromorphic functions, such that $T(r,a_i)=S(r,f)$ for $i=0,1,2,....,k$. Then*

$$m(r,\frac{\Psi}{f})=S(r,f).$$

Lemma 2.4 *[9] Let f be an entire function of finite order $'\rho'$ with no zeros in $\mathbb{C}$. Then*

$$\lim_{r\to\infty}\frac{I(r,f)}{T(r,f)}=\pi\rho.$$

Lemma 2.5 *Let f be a transcendental entire function of non-zero finite order having the maximum deficiency sum. Also f has no zeros in $\mathbb{C}$. Then*

$$\delta_I(a;f)=\left(1-\frac{1}{\pi\rho}\right)+\liminf_{r\to\infty}\frac{m(r,a;f)}{I(r,f)}$$

and

$$\Delta_I(a;f)=\left(1-\frac{1}{\pi\rho}\right)+\limsup_{r\to\infty}\frac{m(r,a;f)}{I(r,f)}.$$

Proof. We know that

$$\begin{aligned}
\delta_I(a;f) &= 1-\limsup_{r\to\infty}\frac{N(r,a;f)}{I(r,f)}\\
&= 1-\limsup_{r\to\infty}\left\{\frac{N(r,a;f)}{T(r,f)}\cdot\frac{T\,(r,f)}{I(r,f)}\right\}\\
&= 1-\limsup_{r\to\infty}\frac{N(r,a;f)}{T(r,f)}\cdot\lim_{r\to\infty}\frac{T\,(r,f)}{I(r,f)}\\
&= 1-\limsup_{r\to\infty}\frac{N(r,a;f)}{T(r,f)}\cdot\frac{1}{\pi\rho}\\
&= \frac{1}{\pi\rho}\left\{1-\limsup_{r\to\infty}\frac{N(r,a;f)}{T(r,f)}\right\}+\left(1-\frac{1}{\pi\rho}\right)\\
&= \frac{1}{\pi\rho}\left\{\liminf_{r\to\infty}\frac{m(r,a;f)}{T(r,f)}\right\}+\left(1-\frac{1}{\pi\rho}\right)\\
&= \frac{1}{\pi\rho}\left\{\liminf_{r\to\infty}\frac{m(r,a;f)}{I(r,f)}\cdot\frac{I\,(r,f)}{T(r,f)}\right\}+\left(1-\frac{1}{\pi\rho}\right)\\
&= \frac{1}{\pi\rho}\left\{\liminf_{r\to\infty}\frac{m(r,a;f)}{I(r,f)}\cdot\pi\rho\right\}+\left(1-\frac{1}{\pi\rho}\right)\\
&= \left(1-\frac{1}{\pi\rho}\right)+\liminf_{r\to\infty}\frac{m(r,a;f)}{I(r,f)}
\end{aligned}$$

This proves the first part of Lemma 2.5.
Similarly we can prove the second part of the lemma. ∎

Lemma 2.6 *Let f be a transcendental entire function of non-zero finite order having the maximum deficiency sum and f has no zeros in $\mathbb{C}$. Then*

$$\delta_I^L(a;f)=1+\frac{\{k\delta(\infty;f)-(k+1)\}}{\pi\rho}+\liminf_{r\to\infty}\frac{m(r,\alpha;L(f))}{I(r,f)}$$

and

$$\Delta_I^L(a;f)=1+\frac{\{k\delta(\infty;f)-(k+1)\}}{\pi\rho}+\limsup_{r\to\infty}\frac{m(r,\alpha;L(f))}{I(r,f)}.$$

Proof. We know that

$$\begin{aligned}
\delta_I^L(a;f) &= 1-\limsup_{r\to\infty}\frac{N(r,a;L(f))}{I(r,f)}\\
&= 1-\limsup_{r\to\infty}\left\{\frac{N(r,a;L(f))}{T(r,f)}\cdot\frac{T(r,f)}{I(r,f)}\right\}\\
&= 1-\limsup_{r\to\infty}\frac{N(r,a;L(f))}{T(r,f)}\cdot\lim_{r\to\infty}\frac{T(r,f)}{I(r,f)}\\
&= 1-\limsup_{r\to\infty}\frac{N(r,a;L(f))}{T(r,f)}\cdot\frac{1}{\pi\rho}\\
&= \frac{1}{\pi\rho}\left\{1-\limsup_{r\to\infty}\frac{N(r,a;L(f))}{T(r,f)}\right\}+\left(1-\frac{1}{\pi\rho}\right)\\
&= \frac{1}{\pi\rho}\cdot\left\{\{k\delta(\infty;f)-k\}+\liminf_{r\to\infty}\frac{m(r,\alpha;L(f))}{T(r,f)}\right\}+\left(1-\frac{1}{\pi\rho}\right)\\
&= \frac{\{k\delta(\infty;f)-k\}}{\pi\rho}+\frac{1}{\pi\rho}\cdot\liminf_{r\to\infty}\frac{m(r,\alpha;L(f))}{T(r,f)}+\left(1-\frac{1}{\pi\rho}\right)\\
&= \frac{\{k\delta(\infty;f)-k\}}{\pi\rho}+\frac{1}{\pi\rho}\cdot\liminf_{r\to\infty}\left\{\frac{m(r,\alpha;L(f))}{I(r,f)}\cdot\frac{I(r,f)}{T(r,f)}\right\}+\left(1-\frac{1}{\pi\rho}\right)\\
&= \frac{\{k\delta(\infty;f)-k\}}{\pi\rho}+\frac{1}{\pi\rho}\cdot\liminf_{r\to\infty}\frac{m(r,\alpha;L(f))}{I(r,f)}\cdot\lim_{r\to\infty}\frac{I(r,f)}{T(r,f)}+\left(1-\frac{1}{\pi\rho}\right)\\
&= \frac{\{k\delta(\infty;f)-k\}}{\pi\rho}+\frac{1}{\pi\rho}\cdot\liminf_{r\to\infty}\frac{m(r,\alpha;L(f))}{I(r,f)}\cdot\pi\rho+\left(1-\frac{1}{\pi\rho}\right)\\
&= \frac{\{k\delta(\infty;f)-k\}}{\pi\rho}+\left(1-\frac{1}{\pi\rho}\right)+\liminf_{r\to\infty}\frac{m(r,\alpha;L(f))}{I(r,f)}\\
& \quad 1+\frac{\{k\delta(\infty;f)-(k+1)\}}{\pi\rho}+\liminf_{r\to\infty}\frac{m(r,\alpha;L(f))}{I(r,f)}
\end{aligned}$$

This completes the proof of the first part of Lemma 2.6.
Similarly we can establish the second part of the lemma. ▪

3 Theorems.

In this section we present the main results of the paper.

Theorem 3.1 *Let f be a transcendental entire function of non-zero finite order 'ρ' (i.e., $0<\rho<\infty$) such that f has no zeros in $\mathbb{C}$. Then for any non-zero finite complex number 'a',*

$$\delta_I(0;f)+\Delta_I^L(\infty;f)+\delta_I(a;f)+\frac{1}{\pi\rho}\le\Delta_I(\infty;f)+\Delta_I^L(0;f)+1$$

holds.

Proof. Let us consider the following identity

$$\frac{a}{f}=1-\frac{f-a}{L(f)}.\frac{L(f)}{f}$$

Since $m(r,\frac{1}{f})\le m(r,\frac{a}{f})+O(1)$, in view of Lemma 2.1 we get from the above identity that

$$m(r,\frac{1}{f})\le m(r,\frac{f-a}{L(f)})+m(r,\frac{L(f)}{f})$$

$$i.e., m(r, \frac{1}{f}) \le m(r, \frac{f-a}{L(f)}) + S(r, f). \tag{1}$$

Now by Nevanlinna's first fundamental theorem and by Lemma 2.3 it follows from Equation (1) that

$$m(r, \frac{1}{f}) \le T(r, \frac{f-a}{L(f)}) - N(r, \frac{f-a}{L(f)}) + S(r, f)$$

$$\text{i.e.}, m(r, \frac{1}{f}) \le T(r, \frac{L(f)}{f-a}) - N(r, \frac{f-a}{L(f)}) + S(r, f)$$

$$\text{i.e.}, m(r, \frac{1}{f}) \le N(r, \frac{L(f)}{f-a}) + m(r, \frac{L(f)}{f-a}) - N(r, \frac{f-a}{L(f)}) + S(r, f)$$

$$\text{i.e.}, m(r, \frac{1}{f}) \le N(r, \frac{L(f)}{f-a}) - N(r, \frac{f-a}{L(f)}) + S(r, f). \tag{2}$$

In view of $\{p.34, [4]\}$ it follows from Equation (2) that

$$m(r, \frac{1}{f}) \le N(r, L(f)) + N(r, \frac{1}{f-a}) - N(r, f-a) - N(r, \frac{1}{L(f)}) + S(r, f)$$

$$\text{i.e.}, \liminf_{r\to\infty} \frac{m(r, \frac{1}{f})}{I(r, f)} \le \liminf_{r\to\infty} \{\frac{N(r, L(f))}{I(r, f)} - \frac{N(r, f)}{I(r, f)} - \frac{N(r, \frac{1}{L(f)})}{I(r, f)}\} + \limsup_{r\to\infty} \frac{N(r, \frac{1}{f-a})}{I(r, f)}$$

$$\text{i.e.}, \liminf \frac{m(r, \frac{1}{f})}{I(r, f)} \le \liminf_{r\to\infty} \frac{N(r, L(f))}{I(r, f)} - \liminf_{r\to\infty} \frac{N(r, f)}{I(r, f)} - \liminf_{r\to\infty} \frac{N(r, \frac{1}{L(f)})}{I(r, f)} + \limsup_{r\to\infty} \frac{N(r, \frac{1}{f-a})}{I(r, f)}$$

$$\text{i.e.}, \quad \delta_I(0; f) - \left(1 - \frac{1}{\pi\rho}\right) \le \{1 - \Delta_I^L(\infty; f)\} - \{1 - \Delta_I(\infty; f)\} - \{1 - \Delta_I^L(0; f)\} + \{1 - \delta_I(a; f)\}$$

$$\text{i.e.}, \delta_I(0; f) + \Delta_I^L(\infty; f) + \delta_I(a; f) + \frac{1}{\pi\rho} \le \Delta_I(\infty; f) + \Delta_I^L(0; f) + 1.$$

This proves the theorem. ▪

Remark 3.1 *The sign* $' \le '$ *in Theorem* 3.1 *can not be replaced by* $' <'$*only. This is evident from the following example.*

Example 1 *Let* $f(z) = \exp z$. *Then* $N(r, f) = 0$ *and*

$$T(r, f) = N(r, f) + m(r, f) = \frac{1}{2\Pi} \int_0^{2\Pi} \log^+ \left| f(re^{i\theta}) \right| d\theta = \frac{1}{2\Pi} \int_0^{2\Pi} \log^+ \left| e^{re^{i\theta}} \right|$$

$$= \frac{1}{2\Pi} \int_0^{2\Pi} \log^+(e^{r\cos\theta}) d\theta = \frac{1}{2\Pi} \int_{-\frac{\Pi}{2}}^{\frac{\Pi}{2}} r\cos\theta d\theta = \frac{r}{\Pi}.$$

Now,

$$I(r, f) = \frac{r}{2\pi} \int_0^{2\pi} \left| \frac{f'(re^{i\theta})}{f(re^{i\theta})} \right| d\theta = \frac{r}{2\pi} \int_0^{2\pi} \left| \frac{e^{re^{i\theta}} \cdot re^{i\theta} \cdot i}{e^{re^{i\theta}}} \right| d\theta = \frac{r}{2\pi} \int_0^{2\pi} \left| re^{i\theta} \cdot i \right| d\theta$$

$$= \frac{r}{2\pi} \int_0^{2\pi} (r)\, d\theta = \frac{r^2}{2\pi} \int_0^{2\pi} d\theta = \frac{r^2}{2\pi} \cdot 2\pi = r^2 \ne 0$$

and

$$\rho = \limsup_{r\to\infty} \frac{\log T(r; f)}{\log r} = \limsup_{r\to\infty} \frac{\log \frac{r}{\pi}}{\log r} = 1.$$

So,

$$\delta_I(a; f) = \left(1 - \frac{1}{\pi\rho}\right) + \liminf_{r\to\infty} \frac{m(r, a; f)}{I(r, f)} = \left(1 - \frac{1}{\pi}\right) + \liminf_{r\to\infty} \left(\frac{r}{\pi} \cdot \frac{1}{r^2}\right) = \left(1 - \frac{1}{\pi}\right),$$

$$\Delta_I(\infty; f) = \Delta_I^L(0; f) = \Delta_I^L(\infty; f) = 1,$$

and

$$\delta_I(0; f) = \Delta_I(\infty; f) = 1.$$

Thus,

$$\delta_I(0; f) + \Delta_I^L(\infty; f) + \delta_I(a; f) + \frac{1}{\pi\rho} = 1 + 1 + \left(1 - \frac{1}{\pi}\right) + \frac{1}{\pi} = 3$$

and

$$\Delta_I(\infty; f) + \Delta_I^L(0; f) + 1 = 1 + 1 + 1 = 3.$$

Hence

$$\delta_I(0; f) + \Delta_I^L(\infty; f) + \delta_I(a; f) + \frac{1}{\pi\rho} = 3 = \Delta_I(\infty; f) + \Delta_I^L(0; f) + 1.$$

Remark 3.2 *The condition that 'a' is a non-zero finite complex number in Theorem 3.1 is necessary as we see in the next example.*

Example 2 *Let* $f = \exp z$ *and* $a = 0$. *Then* $N(r, f) = 0$, $T(r, f) = \frac{r}{\Pi}$, $I(r, f) = r^2 \neq 0$ *and* $\rho = 1$.
Thus,

$$\delta_I(0; f) = \delta_I(\infty; f) = 1.$$

Taking $a_1 = 1, a_2 = a_3 = ... = a_k = 0$ *in Definition* 1.2 *we get that*

$$L(f) = \begin{vmatrix} 1 & \exp z \\ 0 & \exp z \end{vmatrix} = \exp z = f.$$

Now,

$$\Delta_I(\infty; f) = 1,$$

$$\Delta_I^L(\infty; f) = \Delta_I^L(0; f) = 1$$

and

$$\delta_I(a; f) = \delta_I(0; f) = 1.$$

Therefore,

$$\delta_I(0; f) + \Delta_I^L(\infty; f) + \delta_I(a; f) + \frac{1}{\pi\rho} = 1 + 1 + 1 = 3 + \frac{1}{\pi}$$

and

$$\Delta_I(\infty; f) + \Delta_I^L(0; f) + 1 = 1 + 1 + 1 = 3,$$

which is contradictory to Theorem 3.1.

Example 3 *Considering* $f = \exp z$, $a = \infty$ *and proceeding same way as in Example 2, a contradiction arise to Theorem* ***3.1***.

Remark 3.3 *The condition* $\rho > 0$ *in Theorem 3.1 is essential as is evident from the following example.*

Example 4 *Let* $f(z) = z^2$. *Then* $N(r, f) = 0$.

So,

$$\begin{aligned} T(r,f) &= m(r,f) = \frac{1}{2\Pi}\int_0^{2\Pi} \log^+ \left|\left(r^2 e^{2i\theta}\right)\right| d\theta = \frac{1}{2\Pi}\int_0^{2\Pi} \log^+ \left|r^2\right| d\theta \\ &= \frac{1}{2\Pi}\log^+\left|r^2\right| \int_0^{2\Pi} d\theta = \frac{1}{2\Pi}\log^+\left|r^2\right| \cdot 2\pi = \log^+\left|r^2\right| \neq 0 \end{aligned}$$

and

$$\begin{aligned} I(r,f) &= \frac{r}{2\pi}\int_0^{2\pi} \left|\frac{f'\left(re^{i\theta}\right)}{f\left(re^{i\theta}\right)}\right| d\theta = \frac{r}{2\pi}\int_0^{2\pi}\left|\frac{r^2\cdot 2e^{2i\theta}}{r^2\cdot e^{2i\theta}}\right| d\theta \\ &= \frac{r}{2\pi}\int_0^{2\pi} |2|\, d\theta = \frac{r}{2\pi}\cdot 2\int_0^{2\pi} d\theta = 2r \neq 0. \end{aligned}$$

Now,

$$\rho = \limsup_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log r} = \limsup_{r\to\infty}\frac{\log^{[2]} r}{\log r} = \limsup_{r\to\infty}\frac{1}{\log r} = 0$$

and

$$\delta_I(0; f) = \delta_I(\infty; f) = 1.$$

Considering $a_1 = \frac{1}{2}, a_2 = a_3 = ... = a_k = 0$ *in Definition* 1.2 *we obtain that*

$$L(f) = \begin{vmatrix} \frac{1}{2} & z^2 \\ 0 & 2z \end{vmatrix} = z.$$

Thus,

$$\Delta_I(\infty; f) = 1,$$

$$\Delta_I^L(\infty; f) = \Delta_I^L(0; f) = 1$$

and

$$\delta_I(a; f) = \delta_I(\infty; f) = 1.$$

Therefore,

$$\infty \leq \Delta_I(\infty; f) + \Delta_I^L(0; f) + 1,$$

So, we arrive at a contradiction.

Theorem 3.2 *Let f be a transcendental entire function of finite order* $'\rho'$ *having the maximum deficiency sum with* $\delta(\infty; f) = 1$ *and also* f *has no zeros in* $\mathbb{C}$. *Then*

$$\delta_I(\infty; f) + \delta_I(0; f) \leq \Delta_I(\infty; f) + \Delta_I^L(0; f).$$

Proof. Since $f = L(f).\frac{f}{L(f)}$ we get that,

$$m(r,f) \leq m(r, L(f)) + m\left(r, \frac{f}{L(f)}\right). \tag{3}$$

Now by Nevanlinna's first fundamental theorem and by Lemma 2.3 we obtain from Equation (3) that

$$m(r,f) \leq m(r, L(f)) + T\left(r, \frac{f}{L(f)}\right) - N\left(r, \frac{f}{L(f)}\right)$$

$$\text{i.e.,} \, m(r,f) \leq m(r,L(f)) + T\left(r, \frac{L(f)}{f}\right) - N\left(r, \frac{f}{L(f)}\right) + O(1)$$

$$\begin{aligned}\text{i.e.,} \; m(r,f) \;&\leq\; m(r,L(f)) + N\left(r, \frac{L(f)}{f}\right) + m\left(r, \frac{L(f)}{f}\right) \\ &\quad -N\left(r, \frac{f}{L(f)}\right) + O(1).\end{aligned} \tag{4}$$

Now in view of $\{p.34, [4]\}$ it follows from Equation (4) that

$$m(r,f) \;\leq\; m(r,L(f)) + N(r,L(f)) + N\left(r, \frac{1}{f}\right) - N(r,f) - N\left(r, \frac{1}{L(f)}\right) + S(r,f) + O(1)$$

$$\text{i.e.,} \liminf_{r\to\infty} \frac{m(r,f)}{I(r,f)} \;\leq\; \liminf_{r\to\infty} \left\{ \frac{N(r,L(f))}{I(r,f)} - \frac{N(r,f)}{I(r,f)} - \frac{N\left(r, \frac{1}{L(f)}\right)}{I(r,f)} \right\} + \limsup_{r\to\infty} \left\{ \frac{N\left(r, \frac{1}{f}\right)}{I(r,f)} + \frac{m(r,L(f))}{I(r,f)} \right\}$$

$$\begin{aligned}\text{i.e.,} \liminf_{r\to\infty} \frac{m(r,f)}{I(r,f)} \;&\leq\; \liminf_{r\to\infty} \frac{N(r,L(f))}{I(r,f)} - \liminf_{r\to\infty} \frac{N(r,f)}{I(r,f)} - \liminf_{r\to\infty} \frac{N\left(r, \frac{1}{L(f)}\right)}{I(r,f)} \\ &\quad + \limsup_{r\to\infty} \frac{N\left(r, \frac{1}{f}\right)}{I(r,f)} + \limsup_{r\to\infty} \frac{m(r,L(f))}{I(r,f)}.\end{aligned} \tag{5}$$

Since $\delta(\infty; f) = 1$, then $\Delta(\infty; f) = 1$. We obtain from Equation (5) that

$$\begin{aligned}\delta_I(\infty; f) - \left(1 - \frac{1}{\pi\rho}\right) \;&\leq\; \{1 - \Delta_I^L(\infty; f)\} - \{1 - \Delta_I(\infty; f)\} - \{1 - \Delta_I^L(0; f)\} \\ &\quad + \{1 - \delta_I(0; f)\} + \Delta_I^L(\infty; f) - \left\{1 + \frac{\{k\delta(\infty; f) - (k+1)\}}{\pi\rho}\right\}\end{aligned}$$

$$\begin{aligned}\text{i.e.,} \, \delta_I(\infty; f) + \delta_I(0; f) \;&\leq\; \Delta_I(\infty; f) + \Delta_I^L(0; f) \\ &\quad + \left(1 - \frac{1}{\pi\rho}\right) - \left\{1 + \frac{\{k\delta(\infty; f) - (k+1)\}}{\pi\rho}\right\}\end{aligned}$$

$$\begin{aligned}\text{i.e.,} \; \delta_I(\infty; f) + \delta_I(0; f) \;&\leq\; \Delta_I(\infty; f) + \Delta_I^L(0; f) \\ &\quad - \left[\frac{1}{\pi\rho} + \frac{\{k\delta(\infty; f) - (k+1)\}}{\pi\rho}\right]\end{aligned} \tag{6}$$

$$\text{i.e.,} \; \delta_I(\infty; f) + \delta_I(0; f) \leq \Delta_I(\infty; f) + \Delta_I^L(0; f)$$

Thus the theorem is established. ▪

Remark 3.4 *For the function $f = \exp z$, the inequality '$\leq$' in Theorem* 3.2 *cannot be removed by '$<$' only .*

If we omit the condition $\delta(\infty; f) = 1$ in Theorem 3.2 then we get the alternative conclusion of the same as we see in the next theorem.

Theorem 3.3 *Let f be a transcendental entire function of non-zero finite order 'ρ' (i.e., $0 < \rho < \infty$) having the maximum deficiency sum. Also let f has no zeros in $\mathbb{C}$. Then*

$$\delta_I(\infty; f) + \delta_I(0; f) + \frac{k}{\pi\rho}\{1 - \delta(\infty; f)\} \leq \Delta_I(\infty; f) + \Delta_I^L(0; f).$$

Proof. Using the first part of Lemma 2.6 and Equation (6), it follows that

$$\text{i.e., } \delta_I(\infty; f) + \delta_I(0; f) \leq \Delta_I(\infty; f) + \Delta_I^L(0; f) - \left[\frac{1}{\pi\rho} + \frac{\{k\delta(\infty; f) - (k+1)\}}{\pi\rho}\right]$$

$$i.e., \ \delta_I(\infty; f) + \delta_I(0; f) + \frac{k}{\pi\rho} \leq \Delta_I(\infty; f) + \Delta_I^L(0; f) + \frac{k}{\pi\rho} \cdot \delta(\infty; f)$$

$$i.e., \delta_I(\infty; f) + \delta_I(0; f) + \frac{k}{\pi\rho}\{1 - \delta(\infty; f)\} \leq \Delta_I(\infty; f) + \Delta_I^L(0; f). \quad (7)$$

Thus the theorem follows from Equation (7). ▪

Remark 3.5 *The condition* $\rho > 0$ *in Theorem 3.3 is necessary as is evident from the following exmple.*

Example 5 *Let* $f(z) = z^2$. *Then* $N(r, f) = 0$, $T(r, f) = 2\log\left(\frac{r^2}{2}\right) \neq 0$, $\rho = 0$ *and* $I(r, f) = r \neq 0$. *Considering* $a_1 = z, a_2 = a_3 = ... = a_k = 0$ *in Definition* 1.2 *we obtain that*

$$L(f) = \begin{vmatrix} z & z^2 \\ 1 & 2z \end{vmatrix} = z^2.$$

Thus,

$$\delta_I(\infty; f) = \delta_I(0; f) = \delta(\infty; f) = 1$$

and

$$\Delta_I(\infty; f) = \Delta_I^L(0; f) = 1.$$

Hence,

$$\infty \leq 2,$$

which is a contradiction.

Theorem 3.4 *Let* $'a','b' \neq 0, \infty$ *be any two distinct complex numbers. Then for any transcendental entire function* f *of non-zero finite order* $'\rho'$ *(i.e.,* $0 < \rho < \infty$*) with no zeros in* $\mathbb{C}$, *the inequality*

$$2\delta(a; f) + 2\Delta_I^L(\infty; f) + \delta_I(b; f) + \frac{1}{\pi\rho} \leq 2\Delta_I(\infty; f) + 2\Delta_I^L(0; f) + 1$$

holds.

Proof. Considering the identity

$$\frac{b-a}{f-a} = \frac{L(f)}{f-a}\{\frac{f-a}{L(f)} - \frac{f-b}{L(f)}\},$$

we obtain in view of Lemma 2.3 that

$$m(r, \frac{b-a}{f-a}) \leq m(r, \frac{f-a}{L(f)}) + m(r, \frac{f-b}{L(f)}) + m(r, \frac{L(f)}{f-a})$$

$$\text{i.e.,} \, m(r, \frac{b-a}{f-a}) \leq T(r, \frac{f-a}{L(f)}) - N(r, \frac{f-a}{L(f)}) + T(r, \frac{f-b}{L(f)}) - N(r, \frac{f-b}{L(f)}) + S(r, f). \quad (8)$$

Since $m(r, \frac{1}{f-a}) \leq m(r, \frac{b-a}{f-a}) + O(1)$ and $T(r, f) = T(r, \frac{1}{f}) + O(1)$, it follows from Equation (8) that

$$m(r, \frac{1}{f-a}) \leq T(r, \frac{L(f)}{f-a}) - N(r, \frac{f-a}{L(f)}) + T(r, \frac{L(f)}{f-b}) - N(r, \frac{f-b}{L(f)}) + S(r, f) + O(1)$$

$$\text{i.e.,}\, m(r,\frac{1}{f-a}) \;\leq\; N(r,\frac{L(f)}{f-a})+m(r,\frac{L(f)}{f-a})-N(r,\frac{f-a}{L(f)}) +N(r,\frac{L(f)}{f-b})+m(r,\frac{L(f)}{f-b})-N(r,\frac{f-b}{L(f)})+S(r,f)+O(1)$$

$$\text{i.e.,}\, m(r,\frac{1}{f-a}) \;\leq\; N(r,\frac{L(f)}{f-a})-N(r,\frac{f-a}{L(f)})+N(r,\frac{L(f)}{f-b}) -N(r,\frac{f-b}{L(f)})+S(r,f)+O(1). \qquad (9)$$

In view of $\{p.34,\ [4]\}$ we get from Equation (9) that

$$m(r,\frac{1}{f-a}) \;\leq\; N(r,L(f))+N(r,\frac{1}{f-a})-N(r,f-a) -N(r,\frac{1}{L(f)})+N(r,L(f))+N(r,\frac{1}{f-b})-N(r,f-b)-N(r,\frac{1}{L(f)})+S(r,f)$$

$$\text{i.e.,}\, m(r,\frac{1}{f-a}) \;\leq\; 2N(r,L(f))-2N(r,f)-2N(r,\frac{1}{L(f)})+N(r,\frac{1}{f-a})+N(r,\frac{1}{f-b})+S(r,f)+O(1)$$

$$i.e., \liminf_{r\to\infty}\frac{m(r,\frac{1}{f-a})}{I(r,f)} \;\leq\; 2\liminf_{r\to\infty}\{\frac{N(r,L(f))}{I(r,f)}-\frac{N(r,f)}{I(r,f)}-\frac{N(r,\frac{1}{L(f)})}{I(r,f)}\}+\limsup_{r\to\infty}\{\frac{N(r,\frac{1}{f-a})}{I(r,f)}+\frac{N(r,\frac{1}{f-b})}{I(r,f)}\}$$

$$\text{i.e.,}\, \liminf_{r\to\infty}\frac{m(r,\frac{1}{f-a})}{I(r,f)} \;\leq\; 2\{\liminf_{r\to\infty}\frac{N(r,L(f))}{I(r,f)}-\liminf_{r\to\infty}\frac{N(r,f)}{I(r,f)}-\liminf_{r\to\infty}\frac{N(r,\frac{1}{L(f)})}{I(r,f)}\} +\limsup_{r\to\infty}\frac{N(r,\frac{1}{f-a})}{I(r,f)}+\limsup_{r\to\infty}\frac{N(r,\frac{1}{f-b})}{I(r,f)}$$

$$\text{i.e.,}\quad \delta_I(a;f)-\left(1-\frac{1}{\pi\rho}\right) \;\leq\; 2\{1-\Delta_I^L(\infty;f)\}-2\{1-\Delta_I(\infty;f)\}-2\{1-\Delta_I^L(0;f)\} +\{1-\delta_I(a;f)\}+\{1-\delta_I(b;f)\}$$

$$\text{i.e.,}\quad 2\delta(a;f)+2\Delta_I^L(\infty;f)+\delta_I(b;f)+\frac{1}{\pi\rho}\leq 2\Delta_I(\infty;f)+2\Delta_I^L(0;f)+1.$$

Thus the theorem is proved. ∎

Remark 3.6 *The function $f(z)=z^2$ ensures the necessity of the condition $\rho>0$ in Theorem 3.4.*

Theorem 3.5 *Let f be a transcendental entire function f of non-zero finite order $'\rho'$ (i.e., $0<\rho<\infty$) having the maximum deficiency sum. Also let f has no zeros in $\mathbb{C}$. Then*

$$\delta_I(a;f)+\delta_I^L(b;f)+\delta_I^L(c;f)+\frac{1}{\pi\rho}\leq 3,$$

where 'a' is a finite complex number and $'b'$,$'c'$ are two distinct non zero complex numbers.

Proof. Since $\frac{1}{f-a}=\frac{L(f)}{f-a}.\frac{1}{L(f)}$, by Lemma 3 we obtain that

$$m(r,\frac{1}{f-a})\leq m(r,\frac{1}{L(f)})+m(r,\frac{L(f)}{f-a})$$

$$i.e., m(r, \frac{1}{f-a}) \leq m(r, \frac{1}{L(f)}) + S(r, f). \qquad (10)$$

Applying Nevanlinna's first fundamental theorem we get from Equation (10) that

$$m(r, \frac{1}{f-a}) \leq T(r, \frac{1}{L(f)}) - N(r, \frac{1}{L(f)}) + S(r, f). \qquad (11)$$

Now by Nevanlinna's second fundamental theorem it follows from Equation (11) that

$$\begin{aligned} m(r, \frac{1}{f-a}) \ \leq \ & \bar{N}(r, \frac{1}{L(f)}) + \bar{N}(r, \frac{1}{L(f)-b}) \\ & + \bar{N}(r, \frac{1}{L(f)-c}) - N(r, \frac{1}{L(f)}) + S(r, f). \qquad (12) \end{aligned}$$

Now, in view of in view of Lemma 2.1, Lemma 2.4 & Lemma 2.6 and also as $\bar{N}(r, \frac{1}{L(f)}) - N(r, \frac{1}{L(f)}) \leq 0$, we obtain from Equation (12) that

$$m(r, \frac{1}{f-a}) \leq \bar{N}(r, \frac{1}{L(f)-b}) + \bar{N}(r, \frac{1}{L(f)-c}) + S(r, f)$$

$$\text{i.e.}, m(r, \frac{1}{f-a}) \leq N(r, \frac{1}{L(f)-b}) + N(r, \frac{1}{L(f)-c}) + S(r, f)$$

$$\text{i.e.}, m(r, \frac{1}{f-a}) \ \leq \ T(r, \frac{1}{L(f)-b}) + T(r, \frac{1}{L(f)-c}) - m(r, \frac{1}{L(f)-b}) - m(r, \frac{1}{L(f)-c}) + S(r, f)$$

$$\text{i.e.}, m(r, \frac{1}{f-a}) \ \leq \ 2T(r, L(f)) - m(r, \frac{1}{L(f)-b}) - m(r, \frac{1}{L(f)-c}) + S(r, f)$$

$$\text{i.e.}, \ \liminf_{r\to\infty} \frac{m(r, \frac{1}{f-a})}{I(r, f)} \leq 2\liminf_{r\to\infty} \frac{T(r, L(f))}{I(r, f)} - \liminf_{r\to\infty} \frac{m(r, \frac{1}{L(f)-b})}{I(r, f)} - \liminf_{r\to\infty} \frac{m(r, \frac{1}{L(f)-c})}{I(r, f)}$$

$$\text{i.e.}, \liminf_{r\to\infty} \frac{m(r, \frac{1}{f-a})}{I(r, f)} \leq 2\liminf_{r\to\infty} \frac{T(r, L(f))}{I(r, f)} - \liminf_{r\to\infty} \frac{m(r, \frac{1}{L(f)-b})}{I\,(r, f)} - \liminf_{r\to\infty} \frac{m(r, \frac{1}{L(f)-c})}{I\,(r, f)}$$

$$\text{i.e.}, \liminf_{r\to\infty} \frac{m(r, \frac{1}{f-a})}{I(r, f)} \leq 2\left\{\liminf_{r\to\infty} \frac{T(r, L(f))}{T(r, f)} \cdot \frac{T(r, f)}{I(r, f)}\right\} - \liminf_{r\to\infty} \frac{m(r, \frac{1}{L(f)-b})}{I\,(r, f)} - \liminf_{r\to\infty} \frac{m(r, \frac{1}{L(f)-c})}{I\,(r, f)}$$

$$i.e., \liminf_{r\to\infty} \frac{m(r, \frac{1}{f-a})}{I(r, f)} \leq 2\left\{\liminf_{r\to\infty} \frac{T(r, L(f))}{T(r, f)} \cdot \lim_{r\to\infty} \frac{T(r, f)}{I(r, f)}\right\} - \liminf_{r\to\infty} \frac{m(r, \frac{1}{L(f)-b})}{I\,(r, f)} - \liminf_{r\to\infty} \frac{m(r, \frac{1}{L(f)-c})}{I\,(r, f)}$$

$$i.e., \liminf_{r\to\infty} \frac{m(r, \frac{1}{f-a})}{I(r, f)} \leq 2\left\{[1 + k - k\delta(\infty; f)] \cdot \frac{1}{\pi\rho}\right\} - \liminf_{r\to\infty} \frac{m(r, \frac{1}{L(f)-b})}{I\,(r, f)} - \liminf_{r\to\infty} \frac{m(r, \frac{1}{L(f)-c})}{I\,(r, f)}$$

$$\begin{aligned} \text{i.e.}, \ \delta_I(a; f) - \left(1 - \frac{1}{\pi\rho}\right) \ \leq \ & 2\{1 + k - k\delta(\infty; f)\} \cdot \frac{1}{\pi\rho} - \delta_I^L(b; f) + 1 + \frac{k\delta\,(\infty, f) - (k+1)}{\pi\rho}\} \\ & - \delta_I^L(c; f) + 1 + \frac{k\delta\,(\infty, f) - (k+1)}{\pi\rho} \end{aligned}$$

$$i.e., \delta_I(a; f) + \delta_I^L(b; f) + \delta_I^L(c; f)\} \ \leq \ \frac{2}{\pi\rho}\left[\{1 + k - k\delta(\infty; f)\} + k\delta\,(\infty, f) - (k+1)\right] + 2 + \left(1 - \frac{1}{\pi\rho}\right)$$

$$\text{i.e.}, \delta_I(a; f) + \delta_I^L(b; f) + \delta_I^L(c; f) + \frac{1}{\pi\rho} \leq 3.$$

Thus the theorem is established. ∎

Future Prospect : In the line of the works as carried out in the paper one may think of finding out relative deficiencies of higher index in case of meromorphic functions with respect to another one on the basis of sharing of values of them. As a consequence, the derivation of relevant results is still virgin and may be posed as an open problem to the future researchers of this area.

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