A note on the bicomplex version of Enstr$\ddot{o}m$-Kakeya theorem

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Abstract

Enström-Kakeya theorem $\{cf.[4]\}$ says that if $P(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n with real coefficients satisfying $0 \leq a_0 \leq a_1 \leq ... \leq a_n$, then all the zeros of $P(z)$ lie in $|z| \leq 1$. In this paper we wish to prove the bicomplex version of Enstr$\ddot{o}m$-Kakeya theorem and some of its consequences. Some examples are provided to justify the results obtained.

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1 Introduction.

Segre $\{cf.[7]\}$ used an infinite collection of algebras whose elements are named as bicomplex numbers, tricomplex numbers,...,n-complex numbers, etc. Bicomplex algebra is the generalization of the field of complex numbers. Like complex entire function, a bicomplex entire function $f(z)$ is also represented by an everywhere convergent power series as $f(z) = \sum_{j=0}^{\infty} \alpha_j z^j$, where α_j's and z are bicomplex numbers. Thus, bicomplex entire functions are the natural generalization of bicomplex polynomials. The prime concern of this paper is to revisit the Enstr$\ddot{o}m$-Kakeya theorem with some of its consequences under the flavor of bicomplex analysis. We do not explain the standard definitions and notations in the theory of bicomplex analysis as those are available in $\{[5],[6]\}$. The paper has three sections viz, first section covering some basic knowledge about bicomplex numbers, second section containing lemmas and third & final section proving main results with relevant examples and figures.

2 Preliminary Definitions and Notations.

In this section we give some basic idea about bicomplex numbers.

The set $\mathbb{C}_2$ of bicomplex numbers is defined as $\mathbb{C}_2 = \{z : z = z_1 + jz_2, z_1, z_2 \in \mathbb{C}_1\}$ where $\mathbb{C}_1$ is the set of complex numbers with imaginary unit i such that $ij = ji = k$ and $i^2 = j^2 = -k^2 = -1$. Here k is known as a hyperbolic imaginary unit. Thus bicomplex numbers can be considered as the complex numbers with complex coefficients.

Defining addition and multiplication on $\mathbb{C}_2$ in a similar fashion as on $\mathbb{C}_1$, it is observed that multiplication is commutative, associative and distributive over addition and makes $\mathbb{C}_2$ a commutative algebra. However, $\mathbb{C}_2$ is not a field due to presence of zero-divisors, namely the set

$$\mathcal{O} = \{z_1 + jz_2 \in \mathbb{C}_2 : z_1^2 + z_2^2 = 0\} = \{a(1 \pm ij) : a \in \mathbb{C}_1\}.$$

Now writing $z_1 = x_1 + ix_2, z_2 = x_3 + ix_4, x_1, x_2, x_3, x_4 \in \mathbb{R}$, we see that $z = z_1 + jz_2 = x_1 + ix_2 + jx_3 + jix_4$. So, $\mathbb{C}_2$ can be viewed as a real vector space which is isomorphic to $\mathbb{R}^4$ via the map $x_1 + ix_2 + jx_3 + jix_4 \rightarrow (x_1, x_2, x_3, x_4)$.

2.0.1 Idempotent Representation.

Idempotent representation is one of the important presentation of a bicomplex number. The bicomplex numbers $e_1 := \frac{1+ij}{2}, e_2 := \frac{1-ij}{2}$ are linearly independent in the linear space $\mathbb{C}_2$ over $\mathbb{C}_1$ and $e_1 + e_2 = 1, e_1 - e_2 = ij, e_1.e_2 = 0, e_1^2 = e_1, e_2^2 = e_2$. Any number $z = z_1 + jz_2 \in \mathbb{C}_2$ can be written uniqely as $z = (z_1 - iz_2)e_1 + (z_1 + iz_2)e_2$. This representation is named as idempotent representation of z.

2.0.2 Norm.

The norm function $\|\| : \mathbb{C}_2 \to \mathbb{R}^+$ ($\mathbb{R}^+$ denote the set of all non negative real numbers) is defined as follows:
If $z = z_1 + jz_2 = \xi_1 e_1 + \xi_2 e_2 \in \mathbb{C}_2$, then

$$\|z\| = \{|z_1|^2 + |z_2|^2\}^{\frac{1}{2}} = \left\{\frac{|\xi_1|^2 + |\xi_2|^2}{2}\right\}^{\frac{1}{2}}.$$

2.0.3 Auxiliary Complex Spaces.

The complex spaces $\boldsymbol{A_1} = \{z_1 - iz_2 : z_1, z_2 \in \mathbb{C}_1\}$ and $\boldsymbol{A_2} = \{z_1 + iz_2 : z_1, z_2 \in \mathbb{C}_1\}$ are called the auxiliary complex spaces. The spaces A_1 and A_2 contain same elements as in $\mathbb{C}_1$. Though, convenient notations A_1 and A_2 are used for special representation of elements. Each point $z_1 + jz_2 = (z_1 - iz_2)e_1 + (z_1 + iz_2)e_2 \in \mathbb{C}_2$ associates the points $z_1 - iz_2 \in A_1$ and $z_1 + iz_2 \in A_2$. Also to each pair of points $(z_1 - iz_2, z_1 + iz_2) \in A_1 \times A_2$ there is a uniqe point in $\mathbb{C}_2$.

2.0.4 Cartesian Product.

$\mathbb{C}_2$-cartesian set determined by $X_1 \subseteq A_1$ and $X_2 \subseteq A_2$ is defined as follows:

$$X_1 \times_e X_2 := \{z_1 + jz_2 \in \mathbb{C}_2 : z_1 + jz_2 = \omega_1 e_1 + \omega_2 e_2, (\omega_1, \omega_2) \in X_1 \times X_2\}.$$

2.0.5 $\mathbb{C}_2$-Open Discus.

An open discus $D(a; r_1, r_2)$ with centre $a = a_1 e_1 + a_2 e_2$ and radii $r_1 > 0, r_2 > 0$ is defined as

$$D(a; r_1, r_2) = B_1(a_1, r_1) \times_e B_1(a_2, r_2)$$

$$= \{\omega_1 e_1 + \omega_2 e_2 \in \mathbb{C}_2 : |\omega_1 - a_1| < r_1, |\omega_2 - a_2| < r_2\}.$$

where $B_1(z, r)$ is an open ball with center $z \in \mathbb{C}_1$ and radius $r > 0$.

2.0.6 $\mathbb{C}_2$-Closed Discus.

A closed discus $\bar{D}(a; r_1, r_2)$ with centre $a = a_1 e_1 + a_2 e_2$ and radii $r_1 > 0, r_2 > 0$ is defined by

$$\bar{D}(a; r_1, r_2) = \bar{B}_1(a_1, r_1) \times_e \bar{B}_1(a_2, r_2)$$

$$= \{\omega_1 e_1 + \omega_2 e_2 \in \mathbb{C}_2 : |\omega_1 - a_1| \le r_1, |\omega_2 - a_2| \le r_2\}.$$

where $\bar{B}_1(z, r)$ is a closed ball with center $z \in \mathbb{C}_1$ and radius $r > 0$.

Geometrically, $\bar{D}(a; r_1, r_2)$ represents a duocylinder or double cylinder in 4-dimensional Euclidean space which is analogous to a cylinder in 3-space [8].

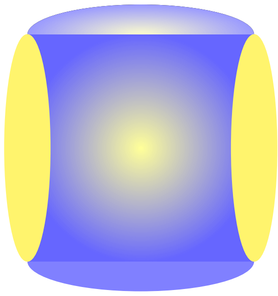

Figure 1: Perspective projection of $\bar{D}(a; r_1, r_2)$

2.0.7 $\mathbb{C}_2$-Disc.

If $r_1 > 0$, $r_2 > 0$ both are equal to r, then the discus is called a $\mathbb{C}_2$- disc and is denoted by $D(a; r, r) = D(a; r)$.

3 Lemmas.

In this section we present lemmas which will be needed in the sequel.

Lemma 3.1. *[7] Let $X = X_1e_1 + X_2e_2 := \{\xi_1e_1 + \xi_2e_2 : \xi_1 \in X_1, \xi_2 \in X_2\}$ be a domain in $\mathbb{C}_2$. A bicomplex function $F = G_1e_1 + G_2e_2 : X \to \mathbb{C}_2$ is holomorphic if and only if both the component function G_1 and G_2 are holomorphic in X_1 and X_2 respectively .*

Lemma 3.2. *[7] Let F be a bicomplex holomorphic function defined in a domain $X = X_1e_1 + X_2e_2 := \{\xi_1e_1 + \xi_2e_2 : \xi_1 \in X_1, \xi_2 \in X_2\}$ such that $F(z) = G_1(\xi_1)e_1 + G_2(\xi_2)e_2$, for all $z = \xi_1e_1 + \xi_2e_2 \in X$. Then, $F(z)$ has zero in X if and only if $G_1(\xi_1)$ and $G_2(\xi_2)$ both have zero at ξ_1 in X_1 and at ξ_2 in X_2 respectively.*

The following lemma is termed as Schwarz's lemma in $\mathbb{C}_1$.

Lemma 3.3. *[3] If $g(z)$ is holomorphic in $\mid z \mid \leq R$ in $\mathbb{C}_1$, $g(0) = 0$ and $\mid g(z) \mid \leq M$ for $\mid z \mid = R$, then*

$$|g(z)| \leq \frac{M|z|}{R}.$$

4 Theorems.

In this section we present the main results of the paper.

Theorem 4.1. *Let $f(z) = \sum_{j=0}^{\infty} a_j z^j$ be a bicomplex entire function with real positive coefficients and for some $k \geq 1, t > 0$*

$$ka_0 \geq ta_1 \geq t^2a_2 \geq \ldots .$$

Then $f(z)$ does not vanish in the open discus $D(0; t_0, t_0)$ where $t_0 = \frac{t}{2k-1}$.

Proof. Since $a_j = a_je_1 + a_je_2$ and $z = \xi_1e_1 + \xi_2e_2$, then $f(z)$ can be expressed as

$$\begin{aligned} f(z) &= \sum_{j=0}^{\infty}(a_je_1 + a_je_2)(\xi_1e_1 + \xi_2e_2)^j \\ &= \sum_{j=0}^{\infty}(a_je_1 + a_je_2)(\xi_1^je_1 + \xi_2^je_2) \\ &= \sum_{j=0}^{\infty} a_j\xi_1^je_1 + \sum_{j=0}^{\infty} a_j\xi_2^je_2 \\ &= f_1(\xi_1)e_1 + f_2(\xi_2)e_2. \end{aligned}$$

Since $f(z)$ is holomorphic in any closed discus $\bar{D}(0;t,t) \subset \mathbb{C}_2, 0 < t < \infty$, in view of Lemma 3.1, $f_1(\xi_1)$ and $f_2(\xi_2)$ both are holomorphic respectively in $X_1 = \{\xi_1 \in A_1 :| \xi_1 |\leq t\} \subset \mathbb{C}_1$ and $X_2 = \{\xi_2 \in A_2 :| \xi_2 |\leq t\} \subset \mathbb{C}_1$.

Clearly, $\lim_{j\to\infty} a_j t^j = 0$.

Now, let us consider

$$\begin{aligned}
&F(\xi_1) = (\xi_1 - t)f_1(\xi_1),\\
&\text{i.e, } F(\xi_1) = -ta_0 + (a_0 - ta_1)\xi_1 + (a_1 - ta_2)\xi_1^2 + ...\\
&\text{i.e, } F(\xi_1) = -ta_0 + (a_0 - ka_0 + ka_0 - ta_1)\xi_1 + (a_1 - ta_2)\xi_1^2 + ...\\
&\text{i.e, } F(\xi_1) = -ta_0 + (1-k)a_0\xi_1 + (ka_0 - ta_1)\xi_1 + (a_1 - ta_2)\xi_1^2 + ...\\
&\text{i.e, } F(\xi_1) = -ta_0 + (1-k)a_0\xi_1 + G(\xi_1) \text{ where } G(\xi_1) = (ka_0 - ta_1)\xi_1 + (a_1 - ta_2)\xi_1^2 + ...
\end{aligned}$$

For $|\xi_1| = t$, we have

$$\begin{aligned}
|G(\xi_1)| &= |(ka_0 - ta_1)\xi_1 + (a_1 - ta_2)\xi_1^2 + ...|\\
&\leq |ka_0 - ta_1||\xi_1| + |a_1 - ta_2||\xi_1|^2 + ...\\
&= (ka_0 - ta_1)t + (a_1 - ta_2)t^2 + ...\\
&= ka_0 t.
\end{aligned}$$

As $G(\xi_1)$ is holomorphic in $| \xi_1 |\leq t$, $G(0) = 0$ and $| G(\xi_1) |\leq ka_0 t$ for $| \xi_1 |= t$, applying Lemma 3.3, we get that

$$\begin{aligned}
| G(\xi_1) | &\leq \frac{ka_0 t|\xi_1|}{t}\\
&= ka_0|\xi_1|, \text{ for } |\xi_1| \leq t.
\end{aligned}$$

Now, for $|\xi_1| < t$, it follows that

$$\begin{aligned}
|F(\xi_1)| &\geq | - ta_0 + (1-k)a_0\xi_1| - |G(\xi_1)|\\
&\geq |ta_0 + (k-1)a_0\xi_1| - ka_0|\xi_1|\\
&\geq ta_0 - (k-1)a_0|\xi_1| - ka_0|\xi_1|\\
&= ta_0 - (2k-1)a_0|\xi_1| > 0 \text{ if } |\xi_1| < \frac{t}{2k-1}.
\end{aligned}$$

Hence for both $|\xi_1| < t$ and $|\xi_1| < t_0$, $|f_1(\xi_1)| > 0$ where $t_0 = \frac{t}{2k-1}$.

Similarly, $|f_2(\xi_2)| > 0$ if $|\xi_2| < t_0$.

Therefore $f_1(\xi_1)$ and $f_2(\xi_2)$ do not vanish respectively in $X_1' = \{\xi_1 \in X_1 :| \xi_1 |< t_0\}$ and $X_2' = \{\xi_2 \in X_2 :| \xi_2 |< t_0\}$.

Hence by Lemma 3.2, $f(z) = f_1(\xi_1)e_1 + f_2(\xi_2)e_2$ does not vanish in $X_1'e_1 + X_2'e_2 = D(0; t_0, t_0)$.

This proves the theorem.

□

Remark 4.1. *Theorem 4.1 is in fact the bicomplex version of Enström-Kakeya theorem* $\{cf.[4]\}$

Remark 4.2. *The following example with related figure ensures the validity of Theorem 4.1.*

Example 4.1. *Let* $f(z) = e^z - \frac{1}{2}$.

Then, $f(z) = \frac{1}{2} + z + \frac{z^2}{2!} + ...$.

Here, $a_0 = \frac{1}{2}, a_j = \frac{1}{j!}, j = 1, 2, ...$.

We see that all the coefficients are positive real numbers and for $k = 1, t = \frac{1}{2}$,

$$ka_0 \geq ta_1 \geq t^2 a_2 \geq ... .$$

Hence by Theorem 4.1, $f(z) = e^z - \frac{1}{2}$ *does not vanish in* $D(0; .5, .5)$.

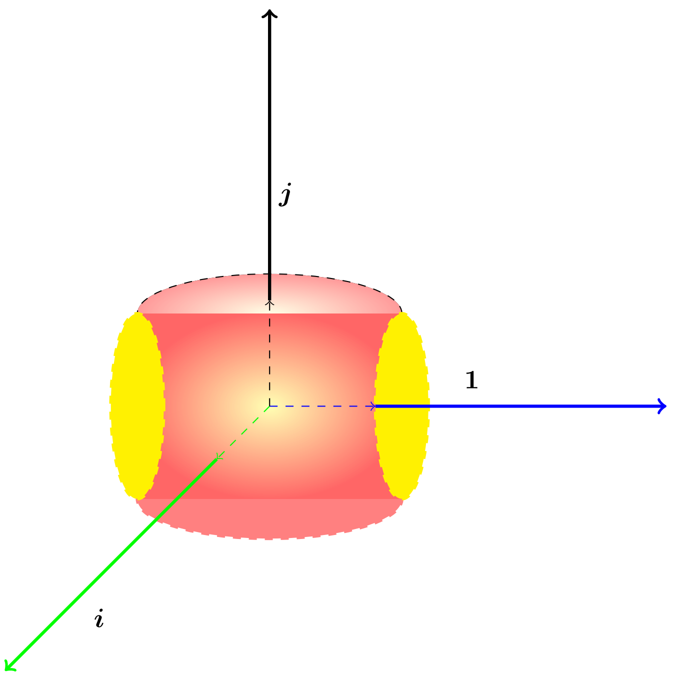

Figure 2: Perspective projection of the zero free region $D(0; .5, .5)$ of $f(z) = e^z - \frac{1}{2}$

Theorem 4.2. *Let* $f(z) = \sum_{j=0}^{\infty} a_j z^j$ *be an entire function in* $\mathbb{C}_2$ *with real positive coefficients such that for some* $k \leq 1, t > 0$ *and* $\lambda \geq 1$

$$ka_0 \leq ta_1 \leq t^2 a_2 \leq ... \leq t^\lambda a_\lambda \geq t^{\lambda+1} a_{\lambda+1} \geq ... .$$

Then $f(z)$ *does not vanish in the open discus* $D(0; r_0, r_0)$ *where* $r_0 = \frac{ta_0}{(1-2k)a_0 + 2a_\lambda t^\lambda}$.

Proof. As in Theorem 4.1, we have

$$f(z) = \sum_{j=0}^{\infty} a_j \xi_1^j e_1 + \sum_{j=0}^{\infty} a_j \xi_2^j e_2$$
$$= f_1(\xi_1) e_1 + f_2(\xi_2) e_2.$$

Clearly, $f(z)$ is holomorphic in any closed discus $\bar{D}(0; t, t) \subset \mathbb{C}_2, 0 < t < \infty$ and so by Lemma 3.1, $f_1(\xi_1)$ and $f_2(\xi_2)$ both are holomorphic respectively in $X_1 = \{\xi_1 \in A_1 :| \xi_1 |\leq t\} \subset \mathbb{C}_1$ and $X_2 = \{\xi_2 \in A_2 :| \xi_2 |\leq t\} \subset \mathbb{C}_1$. .

Also, $\lim_{j \to \infty} a_j t^j = 0$.

Let

$$F(\xi_1) = (\xi_1 - t) f_1(\xi_1),$$

i.e, $F(\xi_1) = (\xi_1 - t)(a_0 + a_1\xi_1 + a_2\xi_1^2 + ... + a_\lambda \xi_1^\lambda + a_{\lambda+1}\xi_1^{\lambda+1} + ...)$

i.e, $F(\xi_1) = -ta_0 + (a_0 - ta_1)\xi_1 + (a_1 - ta_2)\xi_1^2 + ... + (a_{\lambda-1} - ta_\lambda)\xi_1^\lambda + (a_\lambda - ta_{\lambda+1})\xi_1^{\lambda+1} + ...$

i.e, $F(\xi_1) = -ta_0 + (a_0 - ka_0 + ka_0 - ta_1)\xi_1 + (a_1 - ta_2)\xi_1^2 + ... + (a_{\lambda-1} - ta_\lambda)\xi_1^\lambda + (a_\lambda - ta_{\lambda+1})\xi_1^{\lambda+1} + ...$

i.e, $F(\xi_1) = -ta_0 + (1-k)a_0\xi_1 + (ka_0 - ta_1)\xi_1 + (a_1 - ta_2)\xi_1^2 + ... + (a_{\lambda-1} - ta_\lambda)\xi_1^\lambda + (a_\lambda - ta_{\lambda+1})\xi_1^{\lambda+1} + ...$

i.e, $F(\xi_1) = -ta_0 + (1-k)a_0\xi_1 + G(\xi_1)$,

where $G(\xi_1) = (ka_0 - ta_1)\xi_1 + (a_1 - ta_2)\xi_1^2 + ... + (a_{\lambda-1} - ta_\lambda)\xi_1^\lambda + (a_\lambda - ta_{\lambda+1})\xi_1^{\lambda+1} + ...$

Also, for $|\xi_1| = t$,

$$|G(\xi_1)| \le |ka_0 - ta_1||\xi_1| + |a_1 - ta_2||\xi_1|^2 + ... + |a_{\lambda-1} - ta_\lambda||\xi_1|^\lambda + |a_\lambda - ta_{\lambda+1}||\xi_1|^{\lambda+1} + ...$$

$$= (ta_1 - ka_0)t + (ta_2 - a_1)t^2 + ... + (ta_\lambda - a_{\lambda-1})t^\lambda + (a_\lambda - ta_{\lambda+1})t^{\lambda+1} + ...$$

$$= t^{\lambda+1}a_\lambda - ka_0t + t^{\lambda+1}a_\lambda$$

$$= (2t^\lambda a_\lambda - ka_0)t\ .$$

Now, $G(\xi_1)$ is holomorphic in $|\ \xi_1\ | \le t$. Also, $G(0) = 0$ and $|G(\xi_1)| \le (2t^\lambda a_\lambda - ka_0)t$ for $|\ \xi_1\ |= t$. So using Lemma 3.3, we get that

$$|G(\xi_1)| \le \frac{(2t^\lambda a_\lambda - ka_0)t|\xi_1|}{t}$$
$$= (2t^\lambda a_\lambda - ka_0)|\xi_1|\ , \text{ for } |\xi_1| \le t.$$

For $|\xi_1| < t$, we see that

$$|F(\xi_1)| \ge |-ta_0 + (1-k)a_0\xi_1| - |G(\xi_1)|$$
$$\ge ta_0 - (1-k)a_0|\xi_1| - (2t^\lambda a_\lambda - ka_0)|\xi_1|$$
$$= ta_0 - \{(1-2k)a_0 + 2t^\lambda a_\lambda\}|\xi_1| > 0 \text{ if } |\xi_1| < \tfrac{ta_0}{(1-2k)a_0 + 2t^\lambda a_\lambda}\ .$$

Therefore for $|\xi_1| < t$,

$|f_1(\xi_1)| > 0$ if $|\xi_1| < r_0$ where $r_0 = \frac{ta_0}{(1-2k)a_0 + 2t^\lambda a_\lambda}$.

Similarly, $|f_2(\xi_2)| > 0$ if $|\xi_2| < r_0$.

Hence both $f_1(\xi_1)$ and $f_2(\xi_2)$ have no zeros respectively in $X'_1 = \{\xi_1 \in X_1 :|\ \xi_1\ |< r_0\}$ and $X'_2 = \{\xi_2 \in X_2 :|\ \xi_2\ |< r_0\}$.

Consequently, by Lemma 3.2, $f(z) = f_1(\xi_1)e_1 + f_2(\xi_2)e_2$ has no zero in $X'_1e_1 + X'_2e_2 = D(0; r_0, r_0)$.
Thus the theorem is established.

□

Remark 4.3. *Theorem 4.2 can be regarded as the bicomplex version of the Theorem 1 of [1].*

Remark 4.4. *The following example with given figure justifies the validity of Theorem 4.2.*

Example 4.2. *Let us consider* $f(z) = e^z + \frac{z^2}{2} - \frac{z}{2} + 2$,

Then, $f(z) = 3 + \frac{z}{2} + z^2 + \frac{z^3}{3!} + ...$.

Here, $a_0 = 3, a_1 = \frac{1}{2}, a_2 = 1, a_j = \frac{1}{j!}, j = 3, 4, ...$.

So it follows that all the coefficients are positive real numbers and for $k = \frac{1}{6}, t = 1$ *and* $\lambda = 2$,

$$ka_0 \le ta_1 \le t^2a_2 \le ... \le t^\lambda a_\lambda \ge t^{\lambda+1}a_{\lambda+1} \ge ...$$

where $r_0 = \frac{ta_0}{(1-2k)a_0+2t^\lambda a_\lambda} = .75$.

Hence by Theorem 4.2, we obtain that $f(z) = e^z + \frac{z^2}{2} - \frac{z}{2} + 2$ *does not vanish in* $D(0;.75,.75)$.

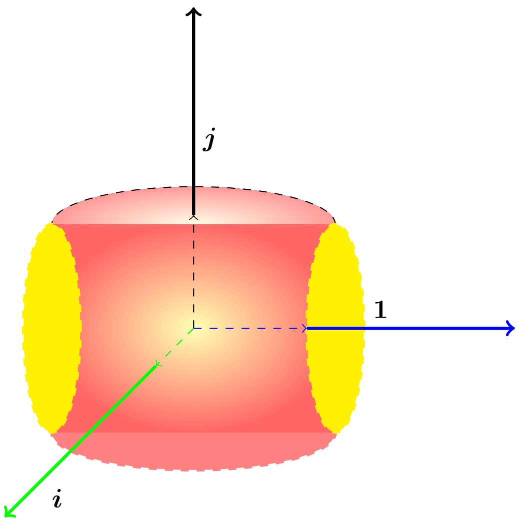

Figure 3: Perspective projection of the zero free region $D(0;.75,.75)$ of $f(z) = e^z + \frac{z^2}{2} - \frac{z}{2} + 2$

The bicomplex version of Theorem B of [2] can be seen in the next theorem.

Theorem 4.3. *Let* $f(z) = \sum_{j=0}^{\infty} a_j z^j$ *be a bicomplex entire function with complex coefficients such that* $a_0 \neq 0$ *and for some* $t > 0$

$$|a_0| \geq t|a_1| \geq t^2|a_2| \geq \ldots .$$

Then no zero of $f(z)$ *lie in the open discus* $D(0;r_0,r_0)$ *where* $r_0 = \frac{t|a_0|}{|a_0|+||a_0|-a_0|+2\sum_{j=1}^{\infty}||a_j|-a_j|t^j}$.

Proof. We can write $f(z)$ as

$$f(z) = \sum_{j=0}^{\infty} a_j \xi_1^j e_1 + \sum_{j=0}^{\infty} a_j \xi_2^j e_2$$
$$= f_1(\xi_1)e_1 + f_2(\xi_2)e_2.$$

Since $f(z)$ is holomorphic in any closed discus $\bar{D}(0;t,t) \subset \mathbb{C}_2, 0 < t < \infty$, by Lemma 3.1, $f_1(\xi_1)$ and $f_2(\xi_2)$ both are holomorphic respectively in $X_1 = \{\xi_1 \in A_1 :| \xi_1 |\leq t\} \subset \mathbb{C}_1$ and $X_2 = \{\xi_2 \in A_2 :| \xi_2 |\leq t\} \subset \mathbb{C}_1$. .

Also, $\lim_{j\to\infty} a_j t^j = 0$.

Let

$$F(\xi_1) = (\xi_1 - t)f_1(\xi_1),$$

i.e, $F(\xi_1) = -ta_0 + (a_0 - ta_1)\xi_1 + (a_1 - ta_2)\xi_1^2 + \ldots$

i.e, $F(\xi_1) = -ta_0 + G(\xi_1)$, where $G(\xi_1) = \sum_{j=1}^{\infty}(a_{j-1} - ta_j)\xi_1^j$.

For $|\xi_1| = t$,

$$|G(\xi_1)| =| \textstyle\sum_{j=1}^{\infty}(a_{j-1} - ta_j)\xi_1^j|$$
$$=| \textstyle\sum_{j=1}^{\infty}\{(|a_{j-1}| - t|a_j|) + (a_{j-1} - |a_{j-1}|) + t(|a_j| - a_j)\}\xi_1^j|$$
$$\leq \textstyle\sum_{j=1}^{\infty}||a_{j-1}| - t|a_j||t^j + \sum_{j=1}^{\infty}||a_{j-1}| - a_{j-1}|t^j + \sum_{j=1}^{\infty}||a_j| - a_j|t^{j+1}$$

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$= \sum_{j=1}^{\infty}(|a_{j-1}| - t|a_j|)t^j + \sum_{j=1}^{\infty}||a_{j-1}| - a_{j-1}|t^j + \sum_{j=1}^{\infty}||a_j| - a_j|t^{j+1}$

$= t(|a_0|+||a_0| - a_0| + 2\sum_{j=1}^{\infty}||a_j| - a_j|t^j)$.

Since $G(\xi_1)$ is holomorphic in $|\ \xi_1\ | \leq t, G(0) = 0$ and $|G(\xi_1)| \leq t(|a_0|+||a_0| - a_0| + 2\sum_{j=1}^{\infty}||a_j| - a_j|t^j)$ for $|\ \xi_1\ |= t$, by Lemma 3.3, we get

$$|G(\xi_1)| \leq \frac{t(|a_0|+||a_0| - a_0| + 2\sum_{j=1}^{\infty}||a_j| - a_j|t^j)|\xi_1|}{t}$$

$$= (|a_0|+||a_0| - a_0| + 2\sum_{j=1}^{\infty}||a_j| - a_j|t^j)|\xi_1| \text{ for } |\xi_1| \leq t\ .$$

Therefore for $|\xi_1| < t$,

$$|F(\xi_1)| \geq t|a_0| - |G(\xi_1)|$$

$$\geq t|a_0| - (|a_0|+||a_0| - a_0| + 2\sum_{j=1}^{\infty}||a_j| - a_j|t^j)|\xi_1| > 0 \text{ if } |\xi_1| < \frac{|a_0|t}{|a_0|+||a_0|-a_0|+2\sum_{j=1}^{\infty}||a_j|-a_j|t^j}.$$

Hence for $|\xi_1| < t$,

$$|f_1(\xi_1)| > 0 \text{ if } |\xi_1| < r_0 \text{ where } r_0 = \frac{|a_0|t}{|a_0|+||a_0|-a_0|+2\sum_{j=1}^{\infty}||a_j|-a_j|t^j}\ .$$

Similarly for $|\xi_1| < t$, $|f_2(\xi_2)| > 0$ if $|\xi_2| < r_0$.

Thus both $f_1(\xi_1)$ and $f_2(\xi_2)$ have no zeros respectively in $X_1' = \{\xi_1 \in X_1 :|\ \xi_1\ |< r_0\}$ and $X_2' = \{\xi_2 \in X_2 :|\ \xi_2\ |< r_0\}$.

Consequently, by Lemma 3.2, $f(z) = f_1(\xi_1)e_1 + f_2(\xi_2)e_2$ has no zero in $X_1'e_1 + X_2'e_2 = D(0; r_0, r_0)$.

This completes the proof of the theorem .

□

Remark 4.5. *The following example with related figure ensures the validity of Theorem 4.3.*

Example 4.3. *Let* $P(z) = 6 + (2 + 3i)z + (3 - i)z^2 + z^3$.

Here, $a_0 = 6, a_1 = 2 + 3i, a_2 = 3 - i, a_3 = 1, a_j = 0, j = 4, 5, ...$.

For $t = 1$*, the condition of Theorem 4.3 is satisfied.*

Now, $r_0 = \frac{|a_0|t}{|a_0|+||a_0|-a_0|+2\sum_{j=1}^{\infty}||a_j|-a_j|t^j} \approx .4$.

Hence by Theorem 4.3, the polynomial $P(z)$ *has no zero in* $D(0; .4, .4)$.

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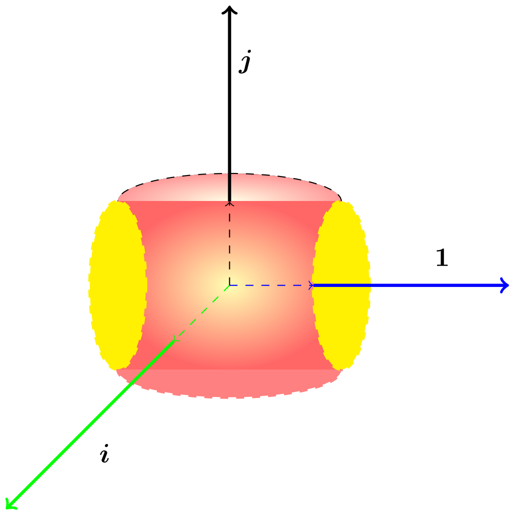

Figure 4: Perspective projection of the zero free region $D(0;.4,.4)$ of $P(z) = 6+(2+3i)z+(3-i)z^2+z^3$

The following theorem can be deduced analogously to Theorem 4 of [2] under the treatment of bi-complex analysis.

Theorem 4.4. *Let* $f(z) = \sum_{j=0}^{\infty} a_j z^j$ *be an entire function in* $\mathbb{C}_2$ *with each* $a_j \in \mathbb{C}_1$ *and* $a_0 \neq 0$ *. Also, let for some* $t > 0, k \geq 1$

$$t^k|a_k| \geq t^{k+1}|a_{k+1}| \geq t^{k+2}|a_{k+2}| \geq \ldots .$$

Then $f(z)$ *has no zero in the open discus* $D(0;r_0,r_0)$

where $r_0 = \frac{t|a_0|}{|a_0|+2|a_k|t^k+||a_k|-a_k|t^k+2\sum_{j=1}^{k-1}|a_j|t^j+2\sum_{j=k+1}^{\infty}||a_j|-a_j|t^j}$.

Proof. $f(z)$ can be written as

$$f(z) = \sum_{j=0}^{\infty} a_j \xi_1^j e_1 + \sum_{j=0}^{\infty} a_j \xi_2^j e_2$$
$$= f_1(\xi_1)e_1 + f_2(\xi_2)e_2.$$

As $f(z)$ is holomorphic in any closed discus $\bar{D}(0;t,t) \subset \mathbb{C}_2, 0 < t < \infty$, So by Lemma 3.1, $f_1(\xi_1)$ and $f_2(\xi_2)$ both are holomorphic respectively in $X_1 = \{\xi_1 \in A_1 :| \xi_1 |\leq t\} \subset \mathbb{C}_1$ and $X_2 = \{\xi_2 \in A_2 :| \xi_2 |\leq t\} \subset \mathbb{C}_1$. .

Clearly, $\lim_{j\to\infty} a_j t^j = 0$.

Let

$$F(\xi_1) = (\xi_1 - t)f_1(\xi_1),$$

i.e, $F(\xi_1) = -ta_0 + (a_0 - ta_1)\xi_1 + (a_1 - ta_2)\xi_1^2 + \ldots$

i.e, $F(\xi_1) = -ta_0 + G(\xi_1)$, where $G(\xi_1) = \sum_{j=1}^{\infty}(a_{j-1} - ta_j)\xi_1^j$.

Now, for $|\xi_1| = t$,

$$|G(\xi_1)| =| \textstyle\sum_{j=1}^{\infty}(a_{j-1} - ta_j)\xi_1^j|$$
$$\leq| \textstyle\sum_{j=1}^{k}(a_{j-1} - ta_j)\xi_1^j|+| \sum_{j=k+1}^{\infty}(a_{j-1} - ta_j)\xi_1^j|$$
$$\leq \textstyle\sum_{j=1}^{k}|a_{j-1} - ta_j|t^j+| \sum_{j=k+1}^{\infty}\{(|a_{j-1}| - t|a_j|) + (a_{j-1}-|a_{j-1}|) + t(|a_j| - a_j)\}\xi_1^j|$$
$$\leq \textstyle\sum_{j=1}^{k}(|a_{j-1}| + t|a_j|)t^j + \sum_{j=k+1}^{\infty}||a_{j-1}| - t|a_j||t^j + \sum_{j=k+1}^{\infty}||a_{j-1}| - a_{j-1}|t^j +$$

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$\sum_{j=k+1}^{\infty}||a_j|-a_j|t^{j+1}$

$= (|a_0|+|a_k|t^k)t + 2t\sum_{j=1}^{k-1}|a_j|t^j + \sum_{j=k+1}^{\infty}(|a_{j-1}|-t|a_j|)t^j + ||a_k|-a_k|t^{k+1} +$

$2t\sum_{j=k+1}^{\infty}||a_j|-a_j|t^j$

$= t(|a_0| + 2|a_k|t^k + ||a_k|-a_k|t^k + 2\sum_{j=1}^{k-1}|a_j|t^j + 2\sum_{j=k+1}^{\infty}||a_j|-a_j|t^j).$

Clearly, $G(\xi_1)$ is holomorphic in $|\xi_1| \leq t, G(0) = 0$ and $|G(\xi_1)| \leq t(|a_0| + 2|a_k|t^k + ||a_k|-a_k|t^k + 2\sum_{j=1}^{k-1}|a_j|t^j + 2\sum_{j=k+1}^{\infty}||a_j|-a_j|t^j)$ for $|\xi_1| = t$. Hence by Lemma 3.3, we get that

$|G(\xi_1)| \leq \frac{t(|a_0|+2|a_k|t^k+||a_k|-a_k|t^k+2\sum_{j=1}^{k-1}|a_j|t^j+2\sum_{j=k+1}^{\infty}||a_j|-a_j|t^j)|\xi_1|}{t}$

$= (|a_0| + 2|a_k|t^k + ||a_k|-a_k|t^k + 2\sum_{j=1}^{k-1}|a_j|t^j + 2\sum_{j=k+1}^{\infty}||a_j|-a_j|t^j)|\xi_1|$ for $|\xi_1| \leq t.$

Therefore for $|\xi_1| < t$, we get

$|F(\xi_1)| \geq t|a_0| - |G(\xi_1)|$

$\geq t|a_0| - (|a_0| + 2|a_k|t^k + ||a_k|-a_k|t^k + 2\sum_{j=1}^{k-1}|a_j|t^j + 2\sum_{j=k+1}^{\infty}||a_j|-a_j|t^j)|\xi_1| > 0$

if

$$|\xi_1| < \frac{t|a_0|}{|a_0|+2|a_k|t^k+||a_k|-a_k|t^k+2\sum_{j=1}^{k-1}|a_j|t^j+2\sum_{j=k+1}^{\infty}||a_j|-a_j|t^j} .$$

Consequently, for $|\xi_1| < t$,

$|f_1(\xi_1)| > 0$ if $|\xi_1| < r_0$ where $r_0 = \frac{t|a_0|}{|a_0|+2|a_k|t^k+||a_k|-a_k|t^k+2\sum_{j=1}^{k-1}|a_j|t^j+2\sum_{j=k+1}^{\infty}||a_j|-a_j|t^j}$.

Similarly, for $|\xi_1| < t$, $|f_2(\xi_2)| > 0$ if $|\xi_2| < r_0$.

Therefore $f_1(\xi_1)$ and $f_2(\xi_2)$ both have no zeros respectively in $X_1' = \{\xi_1 \in X_1 : |\xi_1| < r_0\}$ and $X_2' = \{\xi_2 \in X_2 : |\xi_2| < r_0\}$.

Finally by Lemma 3.2, $f(z) = f_1(\xi_1)e_1 + f_2(\xi_2)e_2$ has no zero in $X_1'e_1 + X_2'e_2 = D(0; r_0, r_0)$.

Thus the theorem is proved.

□

Remark 4.6. *The following example with respective figure justifies the validity of Theorem 4.4.*

Example 4.4. *Let* $f(z) = e^z + \frac{z^2}{2} + (-\frac{1}{2} + \frac{1}{2}i)z + 2 + 4i$.

Then, $f(z) = 3 + 4i + (\frac{1}{2} + \frac{1}{2}i)z + z^2 + \frac{z^3}{3!} + ...$.

Here, $a_0 = 3 + 4i, a_1 = \frac{1}{2} + \frac{1}{2}i, a_2 = 1, a_j = \frac{1}{j!}, j = 3, 4, ...$.

For $t = 1, k = 2$,

$$t^k|a_k| \geq t^{k+1}|a_{k+1}| \geq t^{k+2}|a_{k+2}| \geq ... .$$

Now, $r_0 = \frac{t|a_0|}{|a_0|+2|a_k|t^k+||a_k|-a_k|t^k+2\sum_{j=1}^{k-1}|a_j|t^j+2\sum_{j=k+1}^{\infty}||a_j|-a_j|t^j} \approx .59$.

Hence by Theorem 4.4 ,

$$f(z) = e^z + \frac{z^2}{2} + (-\frac{1}{2} + \frac{1}{2}i)z + 2 + 4i$$

has no zero in $D(0; .59, .59)$.

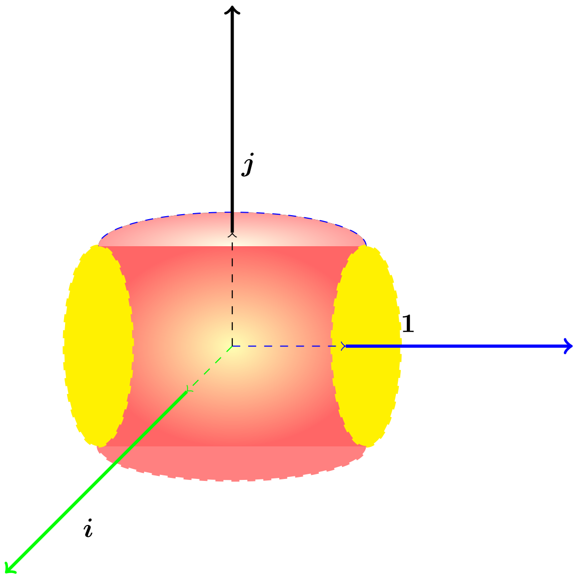

Figure 5: Perspective projection of the zero free region $D(0; .59, .59)$ of $f(z) = e^z + \frac{z^2}{2} + (-\frac{1}{2} + \frac{1}{2}i)z + 2 + 4i$

Future prospect. In the line of the works as carried out in the paper one may think of the extension of the results obtained dealing with n-dimensional bicomplex numbers with the help of the idempotents $0, 1, \frac{1 \pm i_1 i_2}{2}, \frac{1 \pm i_2 i_3}{2}, ..., \frac{1 \pm i_{n-1} i_n}{2}$ in $\mathbb{C}_n$. As a consequence, the problem of taking the co-effiecients of the power series in $\mathbb{C}_n$ is still virgin and may be considered as an open problem to the future researchers of this branch.

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