SOME INEQUALITIES ON THE RELATIVE DEFECTS CORRESPONDING TO THE COMMON ROOTS OF SEVERAL ENTIRE AND MEROMORPHIC FUNCTIONS ON THE BASIS OF THEIR INTEGRATED MODULI OF LOGARITHMIC DERIVATIVE

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Abstract

The purpose of this paper is to consider several entire and meromorphic functions having common roots and to estimate some new relations involving their relative defects with respect to their integrated moduli of logarithmic derivative. Some examples are provided to justify the results obtained.

1 INTRODUCTION

Let f_1,f_2, $.....and$ f_l be any l (>1) non-constant meromorphic functions defined in the open complex plane $\mathbb{C}$. Let $n_0(r,a)$ and $\bar{n_0}(r,a)$ respectively denote the number of common roots and the number of distinct common roots in the disk $|z| \leq r$ of l equations $f_1 = a$,$f_2 = a, ...and$ $f_l = a$ where $'a'$ is any complex number. For a meromorphic function f in $\mathbb{C}$, Milloux [9] introduced the concepts of absolute defect of $'a'$ with respect to the derivative f' where $a \in \mathbb{C} \cup \{\infty\}$. Later Xiong [13] extended the definition to the k-th derivative $f^{(k)}$ where $k > 1$. Singh [10] introduced the term relative defect for distinct zeros & poles and established various relations among it, relative defect and usual defect. The concept of finding out relative defects corresponding to common roots of meromorphic functions was initiated by Singh [10].

To start our paper we require the following:

Let

$$N_0(r,a) = \int_0^r \frac{n_0(t,a) - n_0(0,a)}{t} dt + n_0(0,a)\log r$$

and

$$N_{1,2,\cdots,l}(r,a) = N\left(r, \frac{1}{f_1 - a}\right) + N\left(r, \frac{1}{f_2 - a}\right) + \cdots + N\left(r, \frac{1}{f_l - a}\right) - lN_0(r,a).$$

[0]**AMS Subject Classification** (2010) : 30D30, 30D35.
Keywords and Phrases: Entire function, meromorphic function, order, relative defect, common root, integrated moduli of logarithmic derivative.
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Similarly we may define $\bar{N}_0(r,a)$ and $\bar{N}_{1,2,\cdots,l}(r,a)$ as follows:

$$\bar{N}_0(r,a) = \int_0^r \frac{\bar{n}_0(t,a) - \bar{n}_0(0,a)}{t} dt + \bar{n}_0(0,a)\log r$$

and

$$\bar{N}_{1,2,\cdots,l}(r,a) = \bar{N}\left(r, \frac{1}{f_1 - a}\right) + \bar{N}\left(r, \frac{1}{f_2 - a}\right) + \cdots + \bar{N}\left(r, \frac{1}{f_l - a}\right) - l\bar{N}_0(r,a).$$

Also let $\bar{n}_0^{(k)}(r,a)$, $\bar{N}_{1,2,\cdots,l}^{(k)}(r,a)$ etc. denote the corresponding quantities with respect to $f_1^{(k)}$, $f_2^{(k)}$,$\cdots$ and $f_l^{(k)}$ where k is any non-negative integer.

The following definition is well known.

Definition 1.1 *The order ρ_f of a meromorphic function f is defined as*

$$\rho_f = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r}.$$

If f is entire, one can easily verify that

$$\rho_f = \limsup_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log r}.$$

If $\rho_f < \infty$ then f is of finite order.

If f be a meromorphic function in the complex plane. Then the **integrated moduli of the logarithmic derivative** $I(r,f)$ is defined by

$$I(r,f) = \frac{r}{2\pi}\int_0^{2\pi} \left|\frac{f'(re^{i\theta})}{f(re^{i\theta})}\right| d\theta \text{ for } 0 < r < +\infty.$$

Also let $\bar{n}_0^{(k)}(r,a)$, $\bar{N}_{1,2,\cdots,l}^{(k)}(r,a)$ etc. denote the corresponding quantities with respect to $f_1^{(k)}$, $f_2^{(k)}$,$\cdots$ and $f_l^{(k)}$ where k is any non-negative integer.

Let $n > 1$ be any integer. We now define the following terms by using the concept of $I(r,f)$

$$_I\delta_{1,2,\cdots,l}(a) = 1 - \limsup_{r\to\infty} \frac{N_{1,2,\cdots,l}(r,a)}{I(r,f_1) + I(r,f_2) + ... + I(r,f_l)}$$

$$_I\delta_{1,2,\cdots,l}^{(k)}(a) = 1 - \limsup_{r\to\infty} \frac{N_{1,2,\cdots,l}^{(k)}(r,a)}{I(r,f_1) + I(r,f_2) + ... + I(r,f_l)}$$

$$_I\delta_0(a) = 1 - \limsup_{r\to\infty} \frac{N_0(r,a)}{I(r,f_1) + I(r,f_2) + ... + I(r,f_l)}$$

$$_I\Theta_{1,2,\cdots,l}(a) = 1 - \limsup_{r\to\infty} \frac{\bar{N}_{1,2,\cdots,l}(r,a)}{I(r,f_1) + I(r,f_2) + ... + I(r,f_l)}$$

$$_I\Theta_{1,2,\cdots,l}^{(k)}(a) = 1 - \limsup_{r\to\infty} \frac{\bar{N}_{1,2,\cdots,l}^{(k)}(r,a)}{I(r,f_1) + I(r,f_2) + ... + I(r,f_l)}$$

$$_I\Theta_0(a) = 1 - \limsup_{r\to\infty} \frac{\bar{N}_0(r,a)}{I(r,f_1) + I(r,f_2) + ... + I(r,f_l)}.$$

In the paper we establish some theorems on the relative defects corresponding to common roots of $f_1 = a$, $f_2 = a, \cdots$ and $f_l = a$ in the direction of Singh [10]. The term $S(r, f)$ denotes any quantity satisfying $S(r, f) = o\{T(r, f)\}$ as $r \to \infty$ through all values of r if f is of finite order and except possibly for a set of r of finite linear measure otherwise. Singh [10] found some relations on the relative defects corresponding to the common roots of two meromorphic functions. In the paper we further investigate the results of Singh [10] & Datta et.al. $\{cf.$ [2], [3] and [4]$\}$ and prove some new results on the relative defects of the common roots of $f_1, f_2, ..., f_l$ where $l > 1$ under the flavour of their integrated moduli of logarithmic derivative. Also relevant examples are provided in order to ensure the sharper estimation of the results obtained. We do not explain the standard definitions & notations of the value distribution and Nevanlinna's theory of entire & meromorphic functions as those are available in [12] & [8].

2 LEMMAS

In this section we present some lemmas which will be needed in the sequel.

Lemma 2.1 *[8] Let k be any positive integer and $\Psi = \sum_{i=0}^{k} a_i f^{(i)}$ where a_i are meromorphic functions such that $T(r, a_i) = S(r, f)$. Then*

$$m(r, \frac{\Psi}{f}) = S(r, f).$$

Lemma 2.2 *[8] Let a_1, a_2 and a_3 be three distinct elements in $S(f)$. Then*

$$\{1 + o(1)\} T(r, f) < \sum_{j=1}^{3} \overline{N}\left(r, \frac{1}{f - a_j}\right) + S(r, f) \quad as\ r \to \infty.$$

Lemma 2.3 $\{p.41,$ *[8]$\}$Let f be meromorphic and non-constant in $|z| \leq R_0$. Then*

$$\frac{S(r, f)}{T(r, f)} \to 0 \qquad (*)$$

as $r \to R_0$ with the following provisions :

(a) $(*)$ holds without restrictions if $R_0 = +\infty$ and f is of finite order in the plane.

(b) If f has infinite order in the plane, $(*)$ still holds as $r \to \infty$ outside a certain exceptional set E of finite length. Here E depends only on f.

(c) If $R_0 < +\infty$ and

$$\limsup_{r \to \infty} \frac{T(r, f)}{\log\{\frac{1}{R_0 - r}\}} = +\infty ,$$

then $(*)$ holds as $r \to R_0$ through a suitable sequence r_n, which depends on f only.

Lemma 2.4 *Let $f_1, f_2, ...$and f_l be l (> 1) non-constant meromorphic functions of finite order in $\mathbb{C}$. Then*

$$\lim_{r \to \infty} \frac{S(r, f_1) + S(r, f_2) + ... + S(r, f_l)}{T(r, f_1) + T(r, f_2) + ... + T(r, f_l)} = 0.$$

Proof. In view of Lemma 2.3, we get that

$$\lim_{r\to\infty}\frac{S(r,f_i)}{T(r,f_i)}=0 \quad \text{for each } i=1,2,....,l.$$

Then

$$\lim_{r\to\infty}\frac{S(r,f_1)}{T(r,f_1)}=\lim_{r\to\infty}\frac{S(r,f_2)}{T(r,f_2)}=...........=\lim_{r\to\infty}\frac{S(r,f_l)}{T(r,f_l)}=0.$$

Hence by ratio proportion formula we obtain that

$$\lim_{r\to\infty}\frac{S(r,f_1)+S(r,f_2)+\cdots+S(r,f_l)}{T(r,f_1)+T(r,f_2)+..........+T(r,f_l)}=\frac{0+0+.......+0}{1+1+......+1}=0.$$

This proves the lemma. ■

Lemma 2.5 *Let $f_1, f_2, ...$ and f_l be $l\ (>1)$ non-constant meromorphic functions of finite order in $\mathbb{C}$. Then*

$$\lim_{r\to\infty}\frac{S(r,f_1)+S(r,f_2)+......+S(r,f_l)}{I(r,f_1)+I(r,f_2)+......+I(r,f_l)}=0.$$

Proof. In view of Lemma 2.4, we obtain that

$$\lim_{r\to\infty}\frac{S(r,f_1)+S(r,f_2)+\cdots+S(r,f_l)}{T(r,f_1)+T(r,f_2)+..........+T(r,f_l)}=0.$$

Now

$$\begin{aligned}
&\lim_{r\to\infty}\frac{S(r,f_1)+S(r,f_2)+......+S(r,f_l)}{I(r,f_1)+I(r,f_2)+......+I(r,f_l)}\\
=&\lim_{r\to\infty}\left\{\frac{S(r,f_1)+S(r,f_2)+......+S(r,f_l)}{T(r,f_1)+T(r,f_2)+..........+T(r,f_l)}\cdot\frac{T(r,f_1)+T(r,f_2)+..........+T(r,f_l)}{I(r,f_1)+I(r,f_2)+......+I(r,f_l)}\right\}\\
=&\lim_{r\to\infty}\frac{S(r,f_1)+S(r,f_2)+......+S(r,f_l)}{T(r,f_1)+T(r,f_2)+..........+T(r,f_l)}\cdot\lim_{r\to\infty}\frac{T(r,f_1)+T(r,f_2)+..........+T(r,f_l)}{I(r,f_1)+I(r,f_2)+......+I(r,f_l)}\\
=&\ 0.
\end{aligned}$$

This completes the proof of Lemma 2.5. ■

Lemma 2.6 *[11] Let f be an entire function of finite order $'\rho'$ with no zeros in $\mathbb{C}$. Then*

$$\lim_{r\to\infty}\frac{I(r,f)}{T(r,f)}=\pi\rho.$$

Lemma 2.7 *Let $f_1, f_2, f_3, ...f_l$ be $l\ (>1)$ entire functions of finite order $'\rho_i\ '$ with each f_i for $i=1,2,....,l$ having no zeros in $\mathbb{C}$. Then*

$$\lim_{r\to\infty}\frac{I(r,f_1)+I(r,f_2)+\cdots+I(r,f_l)}{T(r,f_1)+T(r,f_2)+..........+T(r,f_l)}=\frac{\pi\left(\rho_1+\rho_2+...+\rho_l\right)}{l}.$$

Proof. In view of Lemma 2.6, we get that

$$\lim_{r\to\infty}\frac{I(r,f_i)}{T(r,f_i)}=\pi\rho \quad \text{for each } i=1,2,....,l.$$

Then

$$\lim_{r\to\infty}\frac{I(r,f_1)}{T(r,f_1)}=\pi\rho_1,\ \lim_{r\to\infty}\frac{I(r,f_2)}{T(r,f_2)}=\pi\rho_2,............,\ \lim_{r\to\infty}\frac{I(r,f_l)}{T(r,f_l)}=\pi\rho_l.$$

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Hence by ratio proportion formula

$$\begin{aligned}\lim_{r\to\infty}\frac{I(r,f_1)+I(r,f_2)+\cdots+I(r,f_l)}{T(r,f_1)+T(r,f_2)+.........+T(r,f_l)} &= \frac{\pi\rho_1+\pi\rho_2+.......+\pi\rho_l}{1+1+......+1}\\ &= \frac{\pi\left(\rho_1+\rho_2+...+\rho_l\right)}{l}.\end{aligned}$$

This proves the lemma. ■

3 MAIN RESULTS

In this section we present the main results of the paper.

Theorem 3.1 *Let $f_1,\ f_2,\ ...,\ f_l$ be any $l\ (>1)$ entire functions of non-zero finite order $'\rho_i\,'$ with each f_i having no zeros in $\mathbb{C}$ such that $m(r,f_i)=S(r,f_i)$ for $i=1,2,...,l$. Also let $'a'$ be a finite complex number. Then for any positive integer k and for any three finite distinct complex numbers b, c and d then*

$$\begin{aligned}&{}_I\Theta_{1,2,...,l}\left(\infty\right)\ +\ {}_I\Theta^{(k)}_{1,2,...,l}\left(b\right)+\ {}_I\Theta^{(k)}_{1,2,...,l}\left(c\right)+\ {}_I\Theta^{(k)}_{1,2,...,l}\left(d\right)+\ {}_I\delta_{1,2,...,l}\left(a\right)\\ &+l\left\{\ {}_I\Theta_0\left(\infty\right)+\ {}_I\Theta^{(k)}_0\left(b\right)+\ {}_I\Theta^{(k)}_0\left(c\right)+\ {}_I\Theta^{(k)}_0\left(d\right)+\ {}_I\delta_0\left(a\right)\right\}+\frac{l}{\pi\left(\rho_1+\rho_2+...+\rho_l\right)}\le 5\left(1+l\right).\end{aligned}$$

Proof. For any positive integer k, let us consider the following identity

$$\frac{1}{f-a}=\left[\left\{\frac{f^{(k)}}{f-a}\cdot\frac{1}{f^{(k)}}\right\}\cdot\left\{\frac{f^{(k)}}{f}-\frac{f^{(k)}-d}{f^{(k+1)}}\cdot\frac{f^{(k+1)}}{f}\right\}\right]\cdot\frac{f}{d}.$$

Since $m\left(r,\frac{f}{\alpha}\right)\le m\left(r,f\right)+O\left(1\right)$, $T\left(r,\frac{1}{f}\right)=T\left(r,f\right)+O\left(1\right)$, we get in view of Lemma 2.1 and$\{p.34,[8]\}$ that

$$\begin{aligned}m\left(r,\frac{1}{f-a}\right) &\le m\left(r,\frac{1}{f^{(k)}}\right)+m\left(r,\frac{f^{(k)}-d}{f^{(k+1)}}\right)+m\left(r,f\right)\\ &\quad+S\left(r,f\right)+O\left(1\right)\end{aligned}$$

$$\begin{aligned}i.e.,\ m\left(r,\frac{1}{f-a}\right) &\le T\left(r,\frac{1}{f^{(k)}}\right)-N\left(r,\frac{1}{f^{(k)}}\right)+T\left(r,\frac{f^{(k)}-d}{f^{(k+1)}}\right)\\ &\quad-N\left(r,\frac{f^{(k)}-d}{f^{(k+1)}}\right)+m\left(r,f\right)+S\left(r,f\right)+O\left(1\right). \qquad (1)\end{aligned}$$

In view of Lemma 2.2, we get from Equation (1) that

$$\begin{aligned}m\left(r,\frac{1}{f-a}\right) &\le \overline{N}\left(r,\frac{1}{f^{(k)}}\right)+\overline{N}\left(r,\frac{1}{f^{(k)}-b}\right)+\overline{N}\left(r,\frac{1}{f^{(k)}-c}\right)-N\left(r,\frac{1}{f^{(k)}}\right)\\ &\quad+N\left(r,\frac{f^{(k+1)}}{f^{(k)}-d}\right)-N\left(r,\frac{f^{(k)}-d}{f^{(k+1)}}\right)+m\left(r,f\right)\\ &\quad+S\left(r,f\right)+O\left(1\right). \qquad (2)\end{aligned}$$

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As $\overline{N}\left(r,\frac{1}{f^{(k)}}\right)-N\left(r,\frac{1}{f^{(k)}}\right)\leq 0$, it follows from Equation (2) that

$$\begin{aligned} m\left(r,\frac{1}{f-a}\right) &\leq \overline{N}\left(r,\frac{1}{f^{(k)}-b}\right)+\overline{N}\left(r,\frac{1}{f^{(k)}-c}\right)+N\left(r,f^{(k+1)}\right) \\ &\quad +N\left(r,\frac{1}{f^{(k)}-d}\right)-N\left(r,f^{(k)}\right)-N\left(r,\frac{1}{f^{(k+1)}}\right)+m\left(r,f\right) \\ &\quad +S\left(r,f\right)+O\left(1\right) \end{aligned}$$

$$\begin{aligned} i.e., \; m\left(r,\frac{1}{f-a}\right) &\leq \overline{N}\left(r,f\right)+\overline{N}\left(r,\frac{1}{f^{(k)}-b}\right)+\overline{N}\left(r,\frac{1}{f^{(k)}-c}\right) \\ &\quad +N\left(r,\frac{1}{f^{(k)}-d}\right)+m\left(r,f\right)+S\left(r,f\right)+O\left(1\right) \end{aligned}$$

$$\begin{aligned} i.e., \; T\left(r,f\right) &\leq \overline{N}\left(r,f\right)+\overline{N}\left(r,\frac{1}{f^{(k)}-b}\right)+\overline{N}\left(r,\frac{1}{f^{(k)}-c}\right)+N\left(r,\frac{1}{f^{(k)}-d}\right) \\ &\quad +N\left(r,\frac{1}{f-a}\right)+m\left(r,f\right)+S\left(r,f\right)+O\left(1\right). \end{aligned} \tag{3}$$

Applying Equation (3) for f_1, f_2,..., f_l we obtain that

$$\begin{aligned} &\{T(r,f_1)+T(r,f_2)+...+T(r,f_l)\} \\ &\leq \{\overline{N}(r,f_1)+\overline{N}(r,f_2)+...+\overline{N}(r,f_l)\} \\ &+\left\{\overline{N}\left(r,\frac{1}{f_1^{(k)}-b}\right)+\overline{N}\left(r,\frac{1}{f_2^{(k)}-b}\right)+...+\overline{N}\left(r,\frac{1}{f_l^{(k)}-b}\right)\right\} \\ &+\left\{\overline{N}\left(r,\frac{1}{f_1^{(k)}-d}\right)+\overline{N}\left(r,\frac{1}{f_2^{(k)}-d}\right)+...+\overline{N}\left(r,\frac{1}{f_l^{(k)}-d}\right)\right\} \\ &+\left\{N\left(r,\frac{1}{f_1-a}\right)+N\left(r,\frac{1}{f_2-a}\right)+...+N\left(r,\frac{1}{f_l-a}\right)\right\} \\ &+\{m\left(r,f_1\right)+m\left(r,f_2\right)+...+m\left(r,f_l\right)\} \\ &+\{S\left(r,f_1\right)+S\left(r,f_2\right)+...+S\left(r,f_l\right)\}+O\left(1\right). \end{aligned} \tag{4}$$

In view of $m(r,f_1)=S(r,f_1), m(r,f_2)=S(r,f_2),...,m(r,f_l)=S(r,f_l)$ and dividing both side of Equation (4) by $\{I(r,f_1)+I(r,f_2)+...+I(r,f_l)\}$ and taking limit superior we obtain in view of Lemma 2.5 and Lemma 2.7 that

$$\begin{aligned} \frac{l}{\pi\left(\rho_1+\rho_2+...+\rho_l\right)} &\leq \{1-\;_I\Theta_{1,2,...,l}\left(\infty\right)\}+l\{1-\;_I\Theta_0\left(\infty\right)\}+\left\{1-\;_I\Theta_{1,2,...,l}^{(k)}\left(b\right)\right\} \\ &\quad +l\left\{1-\;_I\Theta_0^{(k)}\left(b\right)\right\}+\left\{1-\;_I\Theta_{1,2,...,l}^{(k)}\left(c\right)\right\}+l\left\{1-\;_I\Theta_0^{(k)}\left(c\right)\right\} \\ &\quad +\left\{1-\;_I\Theta_{1,2,...,l}^{(k)}\left(d\right)\right\}+l\left\{1-\;_I\Theta_0^{(k)}\left(d\right)\right\}+\{1-\;_I\delta_{1,2,...,l}\left(a\right)\} \\ &\quad +l\{1-\;_I\delta_0\left(a\right)\} \end{aligned}$$

$$i.e., \;_I\Theta_{1,2,...,l}\left(\infty\right)+\;_I\Theta_{1,2,...,l}^{(k)}\left(b\right)+\;_I\Theta_{1,2,...,l}^{(k)}\left(c\right)+\;_I\Theta_{1,2,...,l}^{(k)}\left(d\right)+\;_I\delta_{1,2,...,l}\left(a\right)$$

$$+l\left\{\;_I\Theta_0\left(\infty\right)+\;_I\Theta_0^{(k)}\left(b\right)+\;_I\Theta_0^{(k)}\left(c\right)+\;_I\Theta_0^{(k)}\left(d\right)+\;_I\delta_0\left(a\right)\right\}+\frac{l}{\pi\left(\rho_1+\rho_2+...+\rho_l\right)}\leq 5\left(1+l\right).$$

Thus the theorem is established. ■

Remark 3.1 *The condition $\rho_1, \rho_2, ..., \rho_l > 0$ in Theorem 3.1 is necessary as we see from the following example.*

Example 1 *Let $l = 2$, $f_1 = z$, $f_2 = z^2$ and $k = 1$. Then* $N_{1,2}(r, f_1) = N_{1,2}(r, f_2) = \overline{N}_{1,2}(r, f_1) = \overline{N}_{1,2}(r, f_2) = \overline{N}^{(1)}_{1,2}(r, f_1) = \overline{N}^{(1)}_{1,2}(r, f_2) = N_0(r, f_1) = \overline{N}_0(r, f_2) = 0,$

$$I(r, f_1) = \frac{r}{2\pi}\int_0^{2\pi}\left|\frac{f'\left(re^{i\theta}\right)}{f\left(re^{i\theta}\right)}\right| d\theta = \frac{r}{2\pi}\int_0^{2\pi}\left|\frac{re^{i\theta}\cdot i}{re^{i\theta}}\right| d\theta = \frac{r}{2\pi}\cdot 2\pi = r \neq 0,$$

$$\begin{aligned} I(r, f_2) &= \frac{r}{2\pi}\int_0^{2\pi}\left|\frac{f'\left(re^{i\theta}\right)}{f\left(re^{i\theta}\right)}\right| d\theta = \frac{r}{2\pi}\int_0^{2\pi}\left|\frac{r^2\cdot 2e^{2i\theta}}{r^2\cdot e^{2i\theta}}\right| d\theta \\ &= \frac{r}{2\pi}\int_0^{2\pi}|2|\, d\theta = \frac{r}{2\pi}\cdot 2\int_0^{2\pi} d\theta = 2r \neq 0, \end{aligned}$$

$$\rho_1 = \limsup_{r\to\infty}\frac{\log^{[2]} M(r, f)}{\log r} = \limsup_{r\to\infty}\frac{\log^{[2]} r}{\log r} = \limsup_{r\to\infty}\frac{1}{\log r} = 0$$

and

$$\rho_2 = \limsup_{r\to\infty}\frac{\log^{[2]} M(r, f)}{\log r} = \limsup_{r\to\infty}\frac{\log^{[2]} r^2}{\log r} = 0.$$

Thus,

$$_I\Theta_{1,2}(\infty) = {_I\Theta^{(1)}_{1,2}}(b) = {_I\Theta^{(1)}_{1,2}}(c) = {_I\Theta^{(1)}_{1,2}}(d) = {_I\delta_{1,2}}(a) = 1$$

and

$$_I\Theta_0(\infty) = {_I\Theta^{(1)}_0}(b) = {_I\Theta^{(1)}_0}(c) = {_I\Theta^{(1)}_0}(d) = {_I\delta_0}(a) = 1.$$

Therefore,

$$\infty \leq 5(1 + l) = 15,$$

which is contrary to Theorem 3.1.

Remark 3.2 *The condition $'b$, c and d are any three distinct complex numbers in Theorem 3.1$'$ is essential as is evident from the following examples.*

Example 2 *Let $l = 2$, $f_1 = \exp(2z)$, $f_2 = \exp(-2z)$, $k = 1$ and $b = c = d = 0$. Then* $N_{1,2}(r, f_1) = N_{1,2}(r, f_2) = \overline{N}_{1,2}(r, f_1) = \overline{N}_{1,2}(r, f_2) = \overline{N}^{(1)}_{1,2}(r, f_1) = \overline{N}^{(1)}_{1,2}(r, f_2) = N_0(r, f_1) = \overline{N}_0(r, f_2) = 0,$

$$\begin{aligned} I(r, f_1) &= I(r, \exp(2z)) = \frac{r}{2\pi}\int_0^{2\pi}\left|\frac{f'\left(re^{i\theta}\right)}{f\left(re^{i\theta}\right)}\right| d\theta = \frac{r}{2\pi}\int_0^{2\pi}\left|\frac{e^{2re^{i\theta}}\cdot 2re^{i\theta}\cdot i}{e^{2re^{i\theta}}}\right| d\theta \\ &= \frac{r}{2\pi}\int_0^{2\pi}\left|2re^{i\theta}\cdot i\right| d\theta = \frac{r}{2\pi}\int_0^{2\pi}(2r)\, d\theta = \frac{r^2}{\pi}\int_0^{2\pi} d\theta = \frac{r^2}{\pi}\cdot 2\pi = 2r^2 \neq 0, \end{aligned}$$

$$\begin{aligned} I(r, f_2) &= I(r, \exp(-2z)) = \frac{r}{2\pi}\int_0^{2\pi}\left|\frac{f'\left(re^{i\theta}\right)}{f\left(re^{i\theta}\right)}\right| d\theta = \frac{r}{2\pi}\int_0^{2\pi}\left|\frac{e^{-2re^{i\theta}}\cdot 2re^{i\theta}\cdot(-i)}{e^{-2re^{i\theta}}}\right| d\theta \\ &= \frac{r}{2\pi}\int_0^{2\pi}\left|2re^{i\theta}\cdot(-i)\right| d\theta = \frac{r}{2\pi}\int_0^{2\pi}(2r)\, d\theta = \frac{r^2}{\pi}\int_0^{2\pi} d\theta = \frac{r^2}{\pi}\cdot 2\pi = 2r^2 \neq 0, \end{aligned}$$

$$\rho_1 = \lim\sup_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log r} = \lim\sup_{r\to\infty} \frac{\log^{[2]} e^{2r}}{\log r} = \lim\sup_{r\to\infty} \frac{\log(2r)}{\log r} = 1$$

and

$$\rho_2 = 1.$$

Hence,

$${}_I\Theta_{1,2}(\infty) = {}_I\Theta^{(1)}_{1,2}(0) = {}_I\delta_{1,2}(0) = 1$$

and

$${}_I\Theta_0(\infty) = {}_I\Theta^{(1)}_0(0) = {}_I\delta_0(0) = 1.$$

Therefore,

$${}_I\Theta_{1,2,...,l}(\infty) + {}_I\Theta^{(k)}_{1,2,...,l}(b) + {}_I\Theta^{(k)}_{1,2,...,l}(c) + {}_I\Theta^{(k)}_{1,2,...,l}(d) + {}_I\delta_{1,2,...,l}(a)$$

$$+l\left\{ {}_I\Theta_0(\infty) + {}_I\Theta^{(k)}_0(b) + {}_I\Theta^{(k)}_0(c) + {}_I\Theta^{(k)}_0(d) + {}_I\delta_0(a)\right\} + \frac{l}{\pi(\rho_1+\rho_2+...+\rho_l)} = 15 + \frac{1}{\pi}$$

and

$$5(1+l) = 15,$$

which contradicts Theorem 3.1.

Example 3 *Let* $l = 2$, $f_1 = \exp(2z)$, $f_2 = \exp(-2z)$, $k = 1$ *and* $b = c = d = \infty$. *Then* $N_{1,2}(r,f_1) = N_{1,2}(r,f_2) = \overline{N}_{1,2}(r,f_1) = \overline{N}_{1,2}(r,f_2) = \overline{N}^{(1)}_{1,2}(r,f_1) = \overline{N}^{(1)}_{1,2}(r,f_2) = N_0(r,f_1) = \overline{N}_0(r,f_2) = 0$,

$$I(r,f_1) = 2r^2 \neq 0,$$

$$I(r,f_2) = 2r^2 \neq 0,$$

$$\rho_1 = 1$$

and

$$\rho_2 = 1.$$

Hence,

$${}_I\Theta_{1,2}(\infty) = {}_I\Theta^{(1)}_{1,2}(\infty) = {}_I\delta_{1,2}(a) = 1$$

and

$${}_I\Theta_0(\infty) = {}_I\Theta^{(1)}_0(\infty) = {}_I\delta_0(\infty) = 1.$$

Therefore,

$${}_I\Theta_{1,2,...,l}(\infty) + {}_I\Theta^{(k)}_{1,2,...,l}(b) + {}_I\Theta^{(k)}_{1,2,...,l}(c) + {}_I\Theta^{(k)}_{1,2,...,l}(d) + {}_I\delta_{1,2,...,l}(a)$$

$$+l\left\{ {}_I\Theta_0(\infty) + {}_I\Theta^{(k)}_0(b) + {}_I\Theta^{(k)}_0(c) + {}_I\Theta^{(k)}_0(d) + {}_I\delta_0(a)\right\} + \frac{l}{\pi(\rho_1+\rho_2+...+\rho_l)} = 15 + \frac{1}{\pi}$$

and

$$5(1+l) = 15.$$

So, we arrive at a contradiction.

Remark 3.3 *The condition* $'$*each* f_i *for* $i = 1,2,...,l$ *having no zeros in* $\mathbb{C}$ *in Theorem 3.1*$'$ *is essential as we see in the following example.*

Example 4 *Let* $l=2,\ f_1=z,\ f_2=\exp z$ *and* $k=1$. *Then we get that* $N_{1,2}(r,f_1)=N_{1,2}(r,f_2)=\overline{N}_{1,2}(r,f_1)=\overline{N}_{1,2}(r,f_2)=\overline{N}^{(1)}_{1,2}(r,f_1)=\overline{N}^{(1)}_{1,2}(r,f_2)=N_0(r,f_1)=\overline{N}_0(r,f_2)=0$,

$$I(r,f_1)=\frac{r}{2\pi}\int_0^{2\pi}\left|\frac{f'\left(re^{i\theta}\right)}{f\left(re^{i\theta}\right)}\right|d\theta=\frac{r}{2\pi}\int_0^{2\pi}\left|\frac{re^{i\theta}\cdot i}{re^{i\theta}}\right|d\theta=\frac{r}{2\pi}\cdot 2\pi=r\neq 0,$$

$$\begin{aligned} I(r,f_2) &= \frac{r}{2\pi}\int_0^{2\pi}\left|\frac{f'\left(re^{i\theta}\right)}{f\left(re^{i\theta}\right)}\right|d\theta=\frac{r}{2\pi}\int_0^{2\pi}\left|\frac{e^{re^{i\theta}}\cdot re^{i\theta}\cdot i}{e^{re^{i\theta}}}\right|d\theta=\frac{r}{2\pi}\int_0^{2\pi}\left|re^{i\theta}\cdot i\right|d\theta \\ &= \frac{r}{2\pi}\int_0^{2\pi}(r)\,d\theta=\frac{r^2}{2\pi}\int_0^{2\pi}d\theta=\frac{r^2}{2\pi}\cdot 2\pi=r^2\neq 0, \end{aligned}$$

$$\rho_1=\limsup_{r\to\infty}\frac{\log^{[2]}M(r,f)}{\log r}=\limsup_{r\to\infty}\frac{\log^{[2]}r}{\log r}=\limsup_{r\to\infty}\frac{1}{\log r}=0$$

and

$$\rho_2=\limsup_{r\to\infty}\frac{\log^{[2]}M(r,f)}{\log r}=\limsup_{r\to\infty}\frac{\log^{[2]}e^r}{\log r}=1.$$

Thus,

$${}_I\Theta_{1,2}(\infty)\ =\ {}_I\Theta^{(1)}_{1,2}(b)=\ {}_I\Theta^{(1)}_{1,2}(c)=\ {}_I\Theta^{(1)}_{1,2}(d)=\ {}_I\delta_{1,2}(a)=1$$

and

$${}_I\Theta_0(\infty)=\ {}_I\Theta^{(1)}_0(b)=\ {}_I\Theta^{(1)}_0(c)=\ {}_I\Theta^{(1)}_0(d)=\ {}_I\delta_0(a)=1.$$

Therefore,

$$\begin{aligned} &{}_I\Theta_{1,2,\ldots,l}(\infty)\ +\ {}_I\Theta^{(k)}_{1,2,\ldots,l}(b)+\ {}_I\Theta^{(k)}_{1,2,\ldots,l}(c)+\ {}_I\Theta^{(k)}_{1,2,\ldots,l}(d)+\ {}_I\delta_{1,2,\ldots,l}(a) \\ &+l\left\{\ {}_I\Theta_0(\infty)+\ {}_I\Theta^{(k)}_0(b)+\ {}_I\Theta^{(k)}_0(c)+\ {}_I\Theta^{(k)}_0(d)+\ {}_I\delta_0(a)\right\}+\frac{l}{\pi(\rho_1+\rho_2+\ldots+\rho_l)} \\ &=1+1+1+1+1+10+\frac{2}{\pi}=15+\frac{2}{\pi} \end{aligned}$$

and

$$5(1+l)=15,$$

which contradicts Theorem 3.1.

Theorem 3.2 *Let* $f_1,\ f_2,\ldots,\ f_l$ *be any* $l\ (>1)$ *entire functions of non-zero finite order* $'\rho_i\ '$ *with each* f_i *for* $i=1,2,\ldots,l$ *having no zeros in* $\mathbb{C}$ *such that* $m(r,f_1)=S(r,f_1)$, $m(r,f_2)=S(r,f_2),\ldots,\ m(r,f_l)=S(r,f_l)$. *Also let* $'a'$ *be any finite complex number. Then for any positive integer* k *and for any two distinct finite complex numbers* $'b'$ *and* $'c'$,

$$\begin{aligned} &(2k+1)\ \ {}_I\Theta_{1,2,\ldots,l}(\infty)+2\ {}_I\delta_{1,2,\ldots,l}(a)+\ {}_I\delta_{1,2,\ldots,l}(b)+\ {}_I\Theta^{(k)}_{1,2,\ldots,l}(c) \\ &+l\left\{(2k+1)\ \ {}_I\Theta_0(\infty)+2\ {}_I\delta_0(a)+\ {}_I\delta_0(b)+\ {}_I\Theta^{(k)}_0(c)\right\}+\frac{l}{\pi(\rho_1+\rho_2+\ldots+\rho_l)}\leq(2k+5)(1+l). \end{aligned}$$

Proof. Let us consider the identity

$$\frac{b-a}{f-a}=\left\{\frac{f^{(k)}}{f-a}\cdot\frac{f-a}{f^{(k)}}-\frac{f^{(k)}}{f-a}\cdot\frac{f-b}{f^{(k)}}\right\}\cdot\left[\left\{\frac{f^{(k)}}{f}-\frac{f^{(k)}-c}{f^{(k+1)}}\cdot\frac{f^{(k+1)}}{f}\right\}\cdot\frac{f}{c}\right]. \qquad (5)$$

Since $m\left(r,\frac{f}{\alpha}\right)\leq m\left(r,f\right)+O\left(1\right)$, we get from Equation (5) in view of Lemma 2.1 and $\{p.34,[8]\}$ that

$$\begin{aligned} m\left(r,\frac{b-a}{f-a}\right) &\leq m\left(r,\frac{f-a}{f^{(k)}}\right)+m\left(r,\frac{f-b}{f^{(k)}}\right)+m\left(r,\frac{f^{(k)}-c}{f^{(k+1)}}\right) \\ &\quad +m\left(r,f\right)+S\left(r,f\right)+O\left(1\right). \end{aligned} \tag{6}$$

As $m\left(r,\frac{1}{f-a}\right)\leq m\left(r,\frac{b-a}{f-b}\right)+O\left(1\right)$, $T\left(r,\frac{1}{f}\right)=T\left(r,f\right)+O\left(1\right)$ and $N\left(r,\frac{1}{f^{(k)}}\right)\geq 0$, we get from Equation (6) that

$$\begin{aligned} m\left(r,\frac{1}{f-a}\right) &\leq T\left(r,\frac{f-a}{f^{(k)}}\right)-N\left(r,\frac{f-a}{f^{(k)}}\right)+T\left(r,\frac{f-b}{f^{(k)}}\right)-N\left(r,\frac{f-b}{f^{(k)}}\right) \\ &\quad +T\left(r,\frac{f^{(k)}-c}{f^{(k+1)}}\right)-N\left(r,\frac{f^{(k)}-c}{f^{(k+1)}}\right)+m\left(r,f\right)+S\left(r,f\right)+O\left(1\right) \end{aligned}$$

$$\begin{aligned} i.e.,\ m\left(r,\frac{1}{f-a}\right) &\leq T\left(r,\frac{f^{(k)}}{f-a}\right)-N\left(r,\frac{f-a}{f^{(k)}}\right)+T\left(r,\frac{f^{(k)}}{f-b}\right)-N\left(r,\frac{f-b}{f^{(k)}}\right) \\ &\quad +T\left(r,\frac{f^{(k+1)}}{f^{(k)}-c}\right)-N\left(r,\frac{f^{(k)}-c}{f^{(k+1)}}\right)+m\left(r,f\right)+S\left(r,f\right)+O\left(1\right) \end{aligned}$$

$$\begin{aligned} i.e.,\ m\left(r,\frac{1}{f-a}\right) &\leq N\left(r,\frac{f^{(k)}}{f-a}\right)-N\left(r,\frac{f-a}{f^{(k)}}\right)+N\left(r,\frac{f^{(k)}}{f-b}\right)-N\left(r,\frac{f-b}{f^{(k)}}\right) \\ &\quad +N\left(r,\frac{f^{(k+1)}}{f^{(k)}-c}\right)-N\left(r,\frac{f^{(k)}-c}{f^{(k+1)}}\right)+m\left(r,f\right)+S\left(r,f\right)+O\left(1\right) \end{aligned}$$

$$\begin{aligned} i.e.,\ m\left(r,\frac{1}{f-a}\right) &\leq N\left(r,f^{(k)}\right)+N\left(r,\frac{1}{f-a}\right)-N\left(r,f-a\right)-N\left(r,\frac{1}{f^{(k)}}\right) \\ &\quad +N\left(r,f^{(k)}\right)+N\left(r,\frac{1}{f-b}\right)-N\left(r,f-b\right)-N\left(r,\frac{1}{f^{(k)}}\right) \\ &\quad +N\left(r,f^{(k+1)}\right)+N\left(r,\frac{1}{f^{(k)}-c}\right)-N\left(r,f^{(k)}-c\right)-N\left(r,\frac{1}{f^{(k+1)}}\right) \\ &\quad +m\left(r,f\right)+S\left(r,f\right)+O\left(1\right). \end{aligned} \tag{7}$$

Since, $N\left(r,f^{(k+1)}\right)-N\left(r,f^{(k)}\right)=\overline{N}\left(r,f\right)$, $N\left(r,\frac{1}{f^{(k)}-c}\right)-N\left(r,\frac{1}{f^{(k+1)}}\right)=\overline{N}\left(r,\frac{1}{f^{(k)}-c}\right)$ and $N\left(r,\frac{1}{f^{(k)}}\right)\geq 0$, we obtain from Equation (7) that

$$\begin{aligned} m\left(r,\frac{1}{f-a}\right) &\leq N\left(r,f\right)+k\overline{N}\left(r,f\right)+N\left(r,\frac{1}{f-a}\right)-N\left(r,f\right) \\ &\quad +N\left(r,f\right)+k\overline{N}\left(r,f\right)+N\left(r,\frac{1}{f-b}\right)-N\left(r,f\right) \\ &\quad +\overline{N}\left(r,f\right)+\overline{N}\left(r,\frac{1}{f^{(k)}-c}\right)+m\left(r,f\right)+S\left(r,f\right)+O\left(1\right) \end{aligned}$$

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$$i.e.,\ m\left(r,\frac{1}{f-a}\right) \leq (2k+1)\overline{N}(r,f)+N\left(r,\frac{1}{f-a}\right)+N\left(r,\frac{1}{f-b}\right) +\overline{N}\left(r,\frac{1}{f^{(k)}-c}\right)+m(r,f)+S(r,f)+O(1)$$

$$i.e.,\ T(r,f) \leq (2k+1)\overline{N}(r,f)+2N\left(r,\frac{1}{f-a}\right)+N\left(r,\frac{1}{f-b}\right) +\overline{N}\left(r,\frac{1}{f^{(k)}-c}\right)+m(r,f)+S(r,f)+O(1). \quad (8)$$

Applying Equation (8) for $f_1,\ f_2, ...,\ f_l$ we get that

$$\{T(r,f_1)+T(r,f_2)+...+T(r,f_l)\}$$

$$\begin{aligned}\leq\ & (2k+1)\left\{\overline{N}(r,f_1)+\overline{N}(r,f_2)+...+\overline{N}(r,f_l)\right\}\\ &+2\left\{N\left(r,\frac{1}{f_1-a}\right)+N\left(r,\frac{1}{f_2-a}\right)+...+N\left(r,\frac{1}{f_l-a}\right)\right\}\\ &+\left\{N\left(r,\frac{1}{f_1-b}\right)+N\left(r,\frac{1}{f_2-b}\right)+...+N\left(r,\frac{1}{f_l-b}\right)\right\}\\ &+\left\{\overline{N}\left(r,\frac{1}{f_1^{(k)}-c}\right)+\overline{N}\left(r,\frac{1}{f_2^{(k)}-c}\right)+...+\overline{N}\left(r,\frac{1}{f_l^{(k)}-c}\right)\right\}\\ &+\{m(r,f_1)+m(r,f_2)+...+m(r,f_l)\}\\ &+\{S(r,f_1)+S(r,f_2)+...+S(r,f_l)\}+O(1). \end{aligned} \quad (9)$$

In view of $m(r,f_1)=S(r,f_1)$, $m(r,f_2)=S(r,f_2)$, ..., $m(r,f_l)=S(r,f_l)$ and dividing both side of Equation (9) by $\{I(r,f_1)+I(r,f_2)+...+I(r,f_l)\}$ and taking limit superior we obtain in view of Lemma 2.5 and Lemma 2.7 that

$$\begin{aligned}\frac{l}{\pi(\rho_1+\rho_2+...+\rho_l)} \leq\ & (2k+1)\left[\{1-\ _I\Theta_{1,2,...,l}(\infty)\}+l\{1-\ _I\Theta_0(\infty)\}\right]\\ &+2\left[\{1-\ _I\delta_{1,2,...,l}(a)\}+l\{1-\ _I\delta_0(a)\}\right]\\ &+\{1-\ _I\delta_{1,2,...,l}(b)\}+l\{1-\ _I\delta_0(b)\}\\ &+\left\{1-\ _I\Theta^{(k)}_{1,2,...,l}(c)\right\}+l\left\{1-\ _I\Theta^{(k)}_0(c)\right\}\end{aligned}$$

$$i.e.,\ (2k+1)\ _I\Theta_{1,2,...,l}(\infty)+2\ _I\delta_{1,2,...,l}(a)+\ _I\delta_{1,2,...,l}(b)+\ _I\Theta^{(k)}_{1,2,...,l}(c)$$

$$+l\left\{(2k+1)\ _I\Theta_0(\infty)+2\ _I\delta_0(a)+\ _I\delta_0(b)+\ _I\Theta^{(k)}_0(c)\right\}+\frac{l}{\pi(\rho_1+\rho_2+...+\rho_l)}\leq(2k+5)(1+l).$$

This proves the theorem. ■

Remark 3.4 *The condition $\rho_1,\rho_2,...,\rho_l>0$ in Theorem 3.2 is necessary as we see from the following example.*

Example 5 *Let $l=2$, $f_1=z$, $f_2=z^2$ and $k=1$. Then* $N_{1,2}(r,f_1)=N_{1,2}(r,f_2)=\overline{N}_{1,2}(r,f_1)=\overline{N}_{1,2}(r,f_2)=\overline{N}^{(1)}_{1,2}(r,f_1)=\overline{N}^{(1)}_{1,2}(r,f_2)=N_0(r,f_1)=\overline{N}_0(r,f_2)=0$,

$$I(r,f_1)=r\neq 0,$$

$$I\left(r,f_2\right)=2r\neq 0,$$

$$\rho_{_1}=0$$

and

$$\rho_{_2}=0.$$

Thus,

$${}_I\Theta_{1,2}\left(\infty\right)={}_I\delta_{1,2}\left(a\right)={}_I\delta_{1,2}\left(b\right)={}_I\Theta_{1,2,}^{(1)}\left(c\right)=1$$

and

$${}_I\Theta_{0}\left(\infty\right)={}_I\delta_{0}\left(a\right)={}_I\delta_{0}\left(b\right)={}_I\Theta_{0}^{(1)}\left(c\right)=1$$

Therefore,

$$\infty\leq\left(2k+5\right)\left(1+l\right)=21,$$

which is contradictory to the conclusion of Theorem 3.2.

Remark 3.5 *The condition*$'a$, b *and* c *are any three distinct complex numbers in Theorem 3.2*$'$ *is essential as is evident from the following examples.*

Example 6 *Let* $l=2$, $f_1=\exp\left(2z\right)$, $f_2=\exp\left(-2z\right)$, $k=1$ *and* $a=b=c=\infty$. *Then we see that* $N_{1,2}\left(r,f_1\right)=N_{1,2}\left(r,f_2\right)=\overline{N}_{1,2}\left(r,f_1\right)=\overline{N}_{1,2}\left(r,f_2\right)=\overline{N}_{1,2}^{(1)}\left(r,f_1\right)=\overline{N}_{1,2}^{(1)}\left(r,f_2\right)=N_0\left(r,f_1\right)=\overline{N}_0\left(r,f_2\right)=0,$

$$T(r,f_1)=\frac{r}{\Pi},$$

$$T(r,f_2)=T\left(r,\frac{1}{f_1}\right)=T(r,f_1)+O(1)=\frac{r}{\Pi}+O(1),$$

$$I\left(r,f_1\right)=r^2\neq 0,$$

$$I\left(r,f_2\right)=r^2\neq 0,$$

$$\rho_{_1}=1$$

and

$$\rho_{_2}=1.$$

Thus,

$${}_I\Theta_{1,2}\left(\infty\right)={}_I\delta_{1,2}\left(\infty\right)={}_I\Theta_{1,2,}^{(1)}\left(\infty\right)=1$$

and

$${}_I\Theta_{0}\left(\infty\right)={}_I\delta_{0}\left(\infty\right)={}_I\Theta_{0}^{(1)}\left(\infty\right)=1.$$

Therefore,

$$\begin{aligned}&\left(2k+1\right){}_I\Theta_{1,2,\ldots,l}\left(\infty\right)+2\,{}_I\delta_{1,2,\ldots,l}\left(a\right)+{}_I\delta_{1,2,\ldots,l}\left(b\right)+{}_I\Theta_{1,2,\ldots,l}^{(k)}\left(c\right)\\&+l\left\{\left(2k+1\right){}_I\Theta_{0}\left(\infty\right)+2\,{}_I\delta_{0}\left(a\right)+{}_I\delta_{0}\left(b\right)+{}_I\Theta_{0}^{(k)}\left(c\right)\right\}+\frac{l}{\pi\left(\rho_1+\rho_2+\ldots+\rho_l\right)}\\=&\;3+2+1+1+6+4+2+2+\frac{2}{2\pi}=21+\frac{1}{\pi}\end{aligned}$$

and

$$\left(2k+5\right)\left(1+l\right)=21,$$

which contradicts Theorem 3.2.

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Example 7 *Let* $l = 2,\ f_1 = \exp(z),\ f_2 = \exp(-z),\ k = 1$ *and* $a = b = c = 0$. *Then* $N_{1,2}(r, f_1) = N_{1,2}(r, f_2) = \overline{N}_{1,2}(r, f_1) = \overline{N}_{1,2}(r, f_2) = \overline{N}^{(1)}_{1,2}(r, f_1) = \overline{N}^{(1)}_{1,2}(r, f_2) = N_0(r, f_1) = \overline{N}_0(r, f_2) = 0$,

$$T(r, f_1) = N(r,f) + m(r,f) = \frac{1}{2\Pi}\int_0^{2\Pi} \log^+ \left|f(re^{i\theta})\right| d\theta = \frac{1}{2\Pi}\int_0^{2\Pi} \log^+ \left|e^{re^{i\theta}}\right|$$

$$= \frac{1}{2\Pi}\int_0^{2\Pi} \log^+(e^{r\cos\theta})d\theta = \frac{1}{2\Pi}\int_{-\frac{\Pi}{2}}^{\frac{\Pi}{2}} r\cos\theta d\theta = \frac{r}{\Pi},$$

$$T(r, f_2) = T\left(r, \frac{1}{f_1}\right) = T(r, f_1) + O(1) = \frac{r}{\Pi} + O(1),$$

$$I(r, f_1) = \frac{r}{2\pi}\int_0^{2\pi} \left|\frac{f'(re^{i\theta})}{f(re^{i\theta})}\right| d\theta = \frac{r}{2\pi}\int_0^{2\pi} \left|\frac{e^{re^{i\theta}} \cdot re^{i\theta} \cdot i}{e^{re^{i\theta}}}\right| d\theta = \frac{r}{2\pi}\int_0^{2\pi} \left|re^{i\theta} \cdot i\right| d\theta$$

$$= \frac{r}{2\pi}\int_0^{2\pi} (r)\, d\theta = \frac{r^2}{2\pi}\int_0^{2\pi} d\theta = \frac{r^2}{2\pi} \cdot 2\pi = r^2 \neq 0,$$

$$I(r, f_2) = \frac{r}{2\pi}\int_0^{2\pi} \left|\frac{f'(re^{i\theta})}{f(re^{i\theta})}\right| d\theta = \frac{r}{2\pi}\int_0^{2\pi} \left|\frac{e^{-re^{i\theta}} \cdot re^{i\theta} \cdot (-i)}{e^{-re^{i\theta}}}\right| d\theta = \frac{r}{2\pi}\int_0^{2\pi} \left|re^{i\theta} \cdot (-i)\right| d\theta$$

$$= \frac{r}{2\pi}\int_0^{2\pi} (r)\, d\theta = \frac{r^2}{2\pi}\int_0^{2\pi} d\theta = \frac{r^2}{2\pi} \cdot 2\pi = r^2 \neq 0,$$

$$\rho_1 = \limsup_{r\to\infty} \frac{\log T(r; f)}{\log r} = \limsup_{r\to\infty} \frac{\log \frac{r}{\pi}}{\log r} = 1$$

and

$$\rho_2 = \limsup_{r\to\infty} \frac{\log T(r; f)}{\log r} = \limsup_{r\to\infty} \frac{\log \frac{r}{\pi} + O(1)}{\log r} = 1.$$

Thus,

$${}_I\Theta_{1,2}(\infty) = {}_I\delta_{1,2}(0) = {}_I\Theta^{(1)}_{1,2,}(0) = 1$$

and

$${}_I\Theta_0(\infty) = {}_I\delta_0(0) = {}_I\Theta^{(1)}_0(0) = 1.$$

Therefore,

$$(2k+1)\ {}_I\Theta_{1,2,\ldots,l}(\infty) + 2\ {}_I\delta_{1,2,\ldots,l}(a) + {}_I\delta_{1,2,\ldots,l}(b) + {}_I\Theta^{(k)}_{1,2,\ldots,l}(c)$$

$$+l\left\{(2k+1)\ {}_I\Theta_0(\infty) + 2\ {}_I\delta_0(a) + {}_I\delta_0(b) + {}_I\Theta^{(k)}_0(c)\right\} + \frac{l}{\pi(\rho_1 + \rho_2 + \ldots + \rho_l)}$$

$$= 3 + 2 + 1 + 1 + 6 + 4 + 2 + 2 + \frac{2}{2\pi} = 21 + \frac{1}{\pi}$$

and

$$(2k+5)(1+l) = 21,$$

which is contrary to Theorem 3.2.

Remark 3.6 *The condition 'each f_i for $i = 1, 2, ..., l$ having no zeros in $\mathbb{C}$ in Theorem 3.2' is essential as is evident from the following example.*

Example 8 *Let $l = 2$, $f_1 = z$, $f_2 = \exp\left(z^2\right)$ and $k = 1$. Then $N_{1,2}(r, f_1) = N_{1,2}(r, f_2) = \overline{N}_{1,2}(r, f_1) = \overline{N}_{1,2}(r, f_2) = \overline{N}^{(1)}_{1,2}(r, f_1) = \overline{N}^{(1)}_{1,2}(r, f_2) = N_0(r, f_1) = \overline{N}_0(r, f_2) = 0$,*

$$I(r, f_1) = \frac{r}{2\pi}\int_0^{2\pi}\left|\frac{f'\left(re^{i\theta}\right)}{f\left(re^{i\theta}\right)}\right|d\theta = \frac{r}{2\pi}\int_0^{2\pi}\left|\frac{re^{i\theta}\cdot i}{re^{i\theta}}\right|d\theta = \frac{r}{2\pi}\cdot 2\pi = r \neq 0,$$

$$\begin{aligned} I(r, f) &= \frac{r}{2\pi}\int_0^{2\pi}\left|\frac{f'\left(re^{i\theta}\right)}{f\left(re^{i\theta}\right)}\right|d\theta = \frac{r}{2\pi}\int_0^{2\pi}\frac{\left|e^{r^2e^{2i\theta}}\right|\cdot\left|2ir^2e^{2i\theta}\right|}{\left|e^{r^2e^{2i\theta}}\right|}d\theta \\ &= \frac{r}{2\pi}\cdot 2r^2\int_0^{2\pi}\frac{e^{r^2\cos 2\theta}\cdot e^{c\cos 2\theta}}{e^{r^2\cos 2\theta}}d\theta = \frac{r^3}{\pi}\int_0^{2\pi}e^{\cos 2\theta}d\theta \\ &= \frac{r^3}{\pi}\cdot\frac{1}{2}\int_0^{4\pi}e^{\cos\eta}d\eta = \frac{r^3}{2\pi}\cdot 4\pi I_0(1) = \frac{2r^3}{\pi}\cdot I_0(1) \neq 0, \end{aligned}$$

where $I_n(z)$ is the Modified Bessel Function of the first kind such that

$$I_n(z) = \frac{1}{\pi}\int_0^{\pi}e^{z\cos\theta}\cdot\cos n\theta d\theta,$$

$$\rho_1 = 0$$

and

$$\rho_2 = \limsup_{r\to\infty}\frac{\log^{[2]}M(r, f)}{\log r} = \limsup_{r\to\infty}\frac{\log^{[2]}e^{r^2}}{\log r} = \limsup_{r\to\infty}\frac{2\log r}{\log r} = 2.$$

Thus,

$${}_I\Theta_{1,2}(\infty) = {}_I\delta_{1,2}(a) = {}_I\delta_{1,2}(b) = {}_I\Theta^{(1)}_{1,2,}(c) = 1$$

and

$${}_I\Theta_0(\infty) = {}_I\delta_0(a) = {}_I\delta_0(b) = {}_I\Theta^{(1)}_0(c) = 1$$

Therefore,

$$\begin{aligned} &(2k+1)\ {}_I\Theta_{1,2,...,l}(\infty) + 2\ {}_I\delta_{1,2,...,l}(a) + {}_I\delta_{1,2,...,l}(b) + {}_I\Theta^{(k)}_{1,2,...,l}(c) \\ &+l\left\{(2k+1)\ {}_I\Theta_0(\infty) + 2\ {}_I\delta_0(a) + {}_I\delta_0(b) + {}_I\Theta^{(k)}_0(c)\right\} + \frac{l}{\pi(\rho_1 + \rho_2 + ... + \rho_l)} \\ = &\ 3 + 2 + 1 + 1 + 6 + 4 + 2 + 2 + \frac{2}{2\pi} = 21 + \frac{1}{\pi} \end{aligned}$$

and

$$(2k+5)(1+l) = 21,$$

which contradicts Theorem 3.2.

Future Prospect : In the line of the works as carried out in the paper one may think of finding out relative deficiencies of higher index in case of meromorphic functions with respect to another one on the basis of sharing of values of them and this treatment can be done under the flavour of bicomplex analysis. As a consequence, the derivation of relevant results is still open to the future workers of this branch.

Acknowledgement

The first author sincerely acknowledges the financial support rendered by DST-FIST 2019-2020 running at the Department of Mathematics, University of Kalyani, P.O.: Kalyani, Dist: Nadia, Pin: 741235, West Bengal, India.

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