Weighted new quasi Lindley distribution with Properties and Applications

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Abstract - In this paper, we have introduced a new generalization of two parameter new quasi lindley distribution using the weighted technique called as weighted new quasi lindley distribution. The different statistical and mathematical properties of the newly proposed distribution have been derived and discussed. The maximum likelihood estimation of the parameters and the Fisher information matrix have also been discussed. Finally, an application two real life data sets are used to investigate the suitability of the newly proposed distribution in modelling lifetime data.

Keywords - Weighted distribution, new quasi lindley distribution, Order statistics, Entropies, Maximum likelihood estimation.

I. INTRODUCTION

The theory of weighted distributions provides a collective access for the problems of model specification and data interpretation. it provides a technique for fitting models to the unknown weight functions when samples can be taken both from the original distribution and the developed distribution. Weighted distributions take into account the method of ascertainment, by adjusting the probabilities of the actual occurrence of events to arrive at a specification of the probabilities of those events as observed and recorded. The weighted distributions occur frequently in the studies related to reliability, analysis of family data, Meta analysis and analysis of intervention data, biomedicine, ecology and other areas, for the improvement of proper statistical models. The concept of weighted distributions was provided firstly by Fisher (1934), Fisher (1934) studied how the methods of ascertainment can influence the form of the distribution of recorded observations and later Rao (1965) introduced and formulated it in general terms in connection with modelling statistical data when the usual practice of using standard distributions were found to be unsuitable. The statistical interpretation of weighted and size biased distributions was originally identified by Buckland and Cox (1964) in the context of renewal theory. Many authors have employed the concept of weighted distribution for different purposes. Warren (1975) was the first to apply the weighted distributions in connection with sampling wood cells. Patil and Rao (1978) inspected some general models leading to weighted distributions with weight functions and studied length biased (size biased) sampling with applications to wildlife populations and human families. As a result, weighted models were formulated in such situations to record the observations according to some weighted function. Different authors have reviewed and studied the various weighted probability models and illustrated their applications in different fields. Weighted distributions are utilized to modulate the probabilities of the events as observed and transcribed. There are two types of weighted distributions: length biased and size biased distributions. Weighted distributions were applied in various research areas related to reliability, biomedicine, ecology and branching processes. For survival data analysis, Jing (2010) introduced the weighted inverse Weibull distribution and beta-inverse Weibull distribution as a new lifetime models. Ghitany, Alqallaf, Al-Mutairi and Husain (2011) introduced a two-parameter weighted Lindley distribution with applications to analyse survival data. Ajami and Jahanshahi (2017) introduced weighted rayleigh Distribution as a new generalization of rayleigh distribution and discussed its parameter estimation in broad. Ayesha, (2017) discussed the Size Biased Lindley Distribution as a new life time distribution and discussed its various statistical properties.Para and Jan (2018) introduced the Weighted Pareto type II Distribution as a new model for handling medical science data and studied its statistical properties and applications. Shanker & Shukla (2018) discussed a generalized size-biased Poisson-Lindley distribution and Its Applications to model size distribution of freely forming small group. Rather and Subramanian (2018) discussed the characterization and estimation of length biased weighted generalized uniform distribution.

A new quasi lindley distribution is a newly proposed two parameteric probability distribution which is a particular case of lindley distribution was introduced by shanker and Ghebretsadik (2013) and calculated its various mathematical and statistical properties its moments, mean residual life function, failure rate function and stochastic ordering. The newly proposed two parametric probability distribution called as a new quasi lindley distribution has better flexibility in handling life time data as compared to lindley and quasi lindley distribution. In this paper we use the weighted technique for a new quasi lindley distribution called as weighted new quasi lindley distribution.

II. WEIGHTED NEW QUASI LINDLEY (WNQL) DISTRIBUTION

The probability density function of new quasi lindley distribution is given by

$$f(x;\theta,\alpha) = \frac{\theta^2}{\theta^2+\alpha}(\theta+\alpha x)e^{-\theta x};\ x>0, \theta>0, \alpha<-\theta^2 \tag{1}$$

and the cumulative distribution function of the new quasi lindley distribution is given by

$$F(x;\theta,\alpha) = 1-\frac{\theta^2+\alpha+\theta\alpha x}{\theta^2+\alpha}e^{-\theta x};\ x>0, \theta>0, \alpha<-\theta^2 \tag{2}$$

Suppose X is a non-negative random variable with probability density function $f(x)$ .Let $w(x)$ be the non-negative weight function, then, the probability density function of the Weighted random variable X_w is given by

$$f_W(x) = \frac{w(x)f(x)}{E(w(x))},\quad x>0.$$

Where $w(x)$ be a non negative weight function and $E(w(x)) = \int w(x) f(x) dx < \infty.$ Depending upon the choice of the weight function *w(x),* we have different weighted models. Clearly when *w(x) = x*, the resulting distribution is called length biased whose pdf is given by

$$f_l(x) = \frac{xf(x)}{E(x)}, x>0$$

Weighted distributions occur frequently in research related to reliability, bio-medicine, ecology and branching process and can be seen in patil and Rao (1986). Das and Roy (2011), Gupta and Tripathi (1996) discussed the size biased weighted Generalized Rayleigh distribution with its properties, also they developed the size biased weighted weibull distribution.

The weighted new quasi lindley distribution is obtained by applying the weight function as $w(x) = x^c$ to the new quasi lindley distribution in order to obtain the weighted new quasi lindley distribution. The probability density function of weighted new quasi lindley distribution is given by

$$f_W(x;\theta,\alpha,c) = \frac{x^c f(x;\theta,\alpha)}{E(x^c)}, x>0, \theta>0, c>0, \alpha<-\theta^2 \tag{3}$$

Where $E(x^c) = \int_0^\infty x^c f(x;\theta,\alpha)dx$

$$E(x^c) = \frac{\theta^2\Gamma(c+1)+\alpha\Gamma(c+2)}{\theta^c(\theta^2+\alpha)} \qquad (4)$$

By applying the equation (1) and (4) in equation (3), we will obtain the probability density function of weighted new quasi lindley distribution which is given by

$$f_W(x;\theta,\alpha,c) = \frac{x^c\theta^{c+2}}{\theta^2\Gamma(c+1)+\alpha\Gamma(c+2)}(\theta+\alpha x)e^{-\theta x}, x,\theta,c>0, \alpha<-\theta^2 \qquad (5)$$

and the cumulative distribution function of the weighted new quasi lindley distribution is given by

$$F_W(x) = \int_0^x f_W(x;\theta,\alpha,c)dx$$

$$F_W(x;\theta,\alpha,c) = \int_0^x \frac{x^c\theta^{c+2}}{\theta^2\Gamma(c+1)+\alpha\Gamma(c+2)}(\theta+\alpha x)e^{-\theta x}dx,\ x>0,\theta>0,c>0,\alpha<-\theta^2$$

$$F_W(x;\theta,\alpha,c) = \frac{\theta^{c+2}}{\theta^2\Gamma(c+1)+\alpha\Gamma(c+2)}\int_0^x x^c(\theta+\alpha x)e^{-\theta x}dx \qquad (6)$$

After the simplification of equation (6), we obtain the cumulative distribution function of the weighted new quasi lindley distribution

$$F_W(x;\theta,\alpha,c) = \frac{1}{\theta^2\Gamma(c+1)+\alpha\Gamma(c+2)}\left(\theta^2\gamma(c+1,\theta x)+\alpha\gamma(c+2,\theta x)\right), x,\theta,c>0,\alpha<-\theta^2 \qquad (7)$$

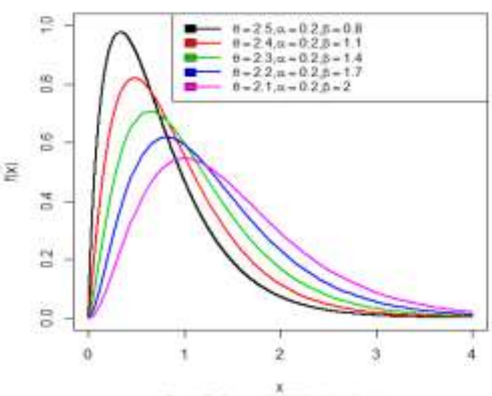

Fig 1 Pdf plot of WNQL distribution

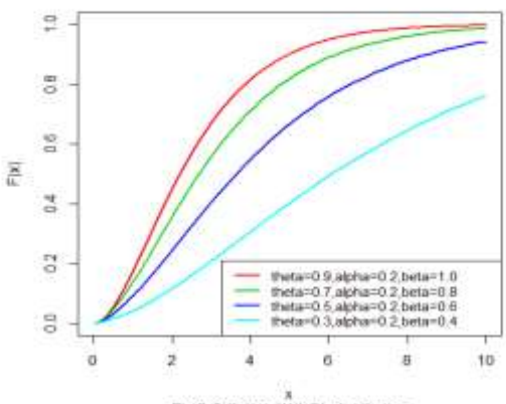

Fig 2 Cdf plot of WNQL distribution

III. RELIABILITY ANALYSIS

In this sub-section, we present the Reliability function, hazard function and the Reverse hazard function for the proposed weighted new quasi lindley distribution.

3.1 Reliability function -

The reliability function is defined as the probability that a system survives beyond a specified time. It is also referred to as survival or survivor function of the distribution. it can be computed as complement of the cumulative distribution function of the model. The reliability function or the survival function of weighted new quasi lindley distribution is calculated as

$$R(x) = 1 - F_W(x;\theta,\alpha,c)$$

$$R(x) = 1 - \frac{1}{\theta^2\Gamma(c+1) + \alpha\Gamma(c+2)}\left(\theta^2\gamma(c+1,\theta x) + \alpha\gamma(c+2,\theta x)\right)$$

3.2 Hazard function-

The hazard function is also known as hazard rate or instantaneous failure rate or force of mortality and is given by

$$h(x) = \frac{f_W(x;\theta,\alpha,c)}{R(x)}$$

$$h(x) = \frac{x^c\theta^{c+2}}{\theta^2\Gamma(c+1) + \alpha\Gamma(c+2) - (\theta^2\gamma(c+1,\theta x) + \alpha\gamma(c+2,\theta x))}(\theta + \alpha x)e^{-\theta x}$$

3.2 Reverse hazard function-

The reverse hazard function of weighted new quasi lindley distribution is given by

$$h_r(x) = \frac{f_W(x;\theta,\alpha,c)}{F_W(x;\theta,\alpha,c)}$$

$$h_r(x;\theta,\alpha,c) = \frac{x^c\theta^{c+2}}{(\theta^2\gamma(c+1,\theta x) + \alpha\gamma(c+2,\theta x))}(\theta + \alpha x)e^{-\theta x}$$

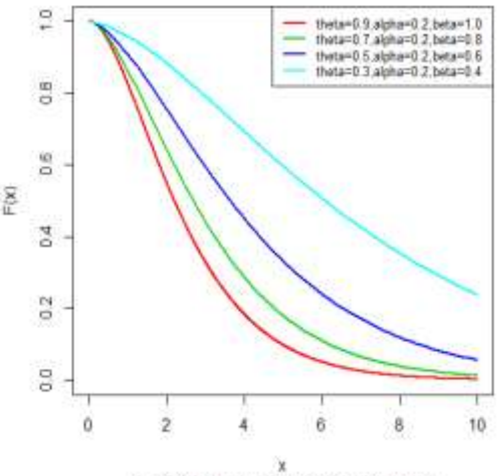

Fig 3 Reliability function of WNQL distribution

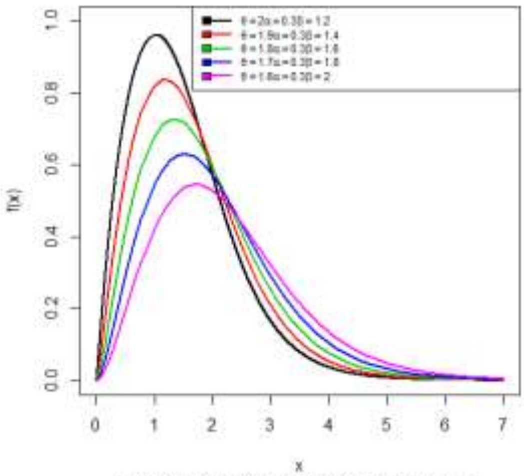

Fig 4 Showing Hazard function of WNQL distribution

IV. STATISTICAL PROPERTIES

In this section we shall discuss the Structural properties of weighted new quasi lindley distribution.

4.1 Moments

Suppose X denotes the weighted new quasi lindley distribution random variable with parameters θ, α and c then

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$$E(X^r) = \mu_r' = \int_0^\infty x^r f_w(x;\theta,\alpha,c)dx$$

$$= \int_0^\infty x^r \frac{x^c \theta^{c+2}}{\theta^2\Gamma(c+1) + \alpha\Gamma(c+2)}(\theta + \alpha x)e^{-\theta x} dx$$

$$= \frac{\theta^{c+2}}{\theta^2\Gamma(c+1) + \alpha\Gamma(c+2)} \int_0^\infty x^{c+r} (\theta + \alpha x)e^{-\theta x} dx$$

$$= \frac{\theta^{c+2}}{\theta^2\Gamma(c+1) + \alpha\Gamma(c+2)} \left(\theta \int_0^\infty x^{(c+r+1)-1} e^{-\theta x} dx + \alpha \int_0^\infty x^{(c+r+2)-1} e^{-\theta x} dx \right)$$

$$\Rightarrow E(X^r) = \mu_r' = \frac{\theta^2\Gamma(c+r+1) + \alpha\Gamma(c+r+2)}{\theta^r (\theta^2\Gamma(c+1) + \alpha\Gamma(c+2))} \tag{8}$$

Substitute r = 1,2,3,4 in equation (8) we get the first four moments of weighted new quasi lindley distribution

$$E(X) = \mu_1' = \frac{\theta^2\Gamma(c+2) + \alpha\Gamma(c+3)}{\theta(\theta^2\Gamma(c+1) + \alpha\Gamma(c+2))}$$

$$E(X^2) = \mu_2' = \frac{\theta^2\Gamma(c+3) + \alpha\Gamma(c+4)}{\theta^2 (\theta^2\Gamma(c+1) + \alpha\Gamma(c+2))}$$

$$E(X^3) = \mu_3' = \frac{\theta^2\Gamma(c+4) + \alpha\Gamma(c+5)}{\theta^3 (\theta^2\Gamma(c+1) + \alpha\Gamma(c+2))}$$

$$E(X^4) = \mu_4' = \frac{\theta^2\Gamma(c+5) + \alpha\Gamma(c+6)}{\theta^4 (\theta^2\Gamma(c+1) + \alpha\Gamma(c+2))}$$

$$\text{Variance } (\mu_2) = \frac{\theta^2\Gamma(c+3) + \alpha\Gamma(c+4)}{\theta^2 (\theta^2\Gamma(c+1) + \alpha\Gamma(c+2))} - \left(\frac{\theta^2\Gamma(c+2) + \alpha\Gamma(c+3)}{\theta(\theta^2\Gamma(c+1) + \alpha\Gamma(c+2))} \right)^2$$

$$\Rightarrow S.D(\sigma) = \sqrt{\left(\frac{\theta^2\Gamma(c+3) + \alpha\Gamma(c+4)}{\theta^2 (\theta^2\Gamma(c+1) + \alpha\Gamma(c+2))} - \frac{(\theta^2\Gamma(c+2) + \alpha\Gamma(c+3))^2}{\theta^2 (\theta^2\Gamma(c+1) + \alpha\Gamma(c+2))^2} \right)}$$

$$\text{Coefficient of Variation } (C.V) = \frac{\sigma}{\mu_1'} = \sqrt{\left(\frac{\theta^2\Gamma(c+3)+\alpha\Gamma(c+4)}{\theta^2(\theta^2\Gamma(c+1)+\alpha\Gamma(c+2))} - \frac{(\theta^2\Gamma(c+2)+\alpha\Gamma(c+3))^2}{\theta^2(\theta^2\Gamma(c+1)+\alpha\Gamma(c+2))^2}\right)} \times \frac{\theta(\theta^2\Gamma(c+1)+\alpha\Gamma(c+2))}{\theta^2\Gamma(c+2)+\alpha\Gamma(c+3)}$$

4.2 Harmonic mean

The harmonic mean of the proposed weighted new quasi lindley distribution is obtained as

$$H.M = E\left(\frac{1}{x}\right) = \int_0^\infty \frac{1}{x} f_W(x;\theta,\alpha,c)dx$$

$$= \int_0^\infty \frac{\theta^{c+2}}{\theta^2\Gamma(c+1)+\alpha\Gamma(c+2)} x^{c-1}(\theta+\alpha x)e^{-\theta x}dx$$

$$= \frac{\theta^{c+2}}{\theta^2\Gamma(c+1)+\alpha\Gamma(c+2)}\left(\theta\int_0^\infty x^{c-1}e^{-\theta x}dx + \alpha\int_0^\infty x^c e^{-\theta x}dx\right)$$

$$= \frac{\theta^{c+2}}{\theta^2\Gamma(c+1)+\alpha\Gamma(c+2)}\left(\theta\int_0^\infty e^{-\theta x}x^{(c+1)-2}dx + \alpha\int_0^\infty e^{-\theta x}x^{(c+1)-1}dx\right)$$

$$\Rightarrow H.M = \frac{\theta^{c+2}}{\theta^2\Gamma(c+1)+\alpha\Gamma(c+2)}\left(\theta\gamma(c+1,\theta x)+\alpha\gamma(c+1,\theta x)\right)$$

4.3 Moment Generating Function and Characteristics Function

In this sub section we derive the moment generating function and the characteristics function of weighted new quasi lindley distribution. We begin with the well-known definition of the moment generating function given by

$$M_X(t) = E\left(e^{tx}\right) = \int_0^\infty e^{tx} f_W(x;\theta,\alpha,c)dx$$

$$= \int_0^\infty \left[1 + tx + \frac{(tx)^2}{2!} + \ldots.\right] f_W(x;\theta,\alpha,c)dx$$

$$= \int_0^\infty \sum_{j=0}^\infty \frac{t^j}{j!} x^j f_W(x;\theta,\alpha,c)dx$$

$$= \sum_{j=0}^\infty \frac{t^j}{j!}\mu_j'$$

$$= \sum_{j=0}^\infty \frac{t^j}{j!}\left[\frac{\theta^2\Gamma(c+j+1)+\alpha\Gamma(c+j+2)}{\theta^j(\theta^2\Gamma(c+1)+\alpha\Gamma(c+2))}\right]$$

$$\Rightarrow M_X(t) = \frac{1}{(\theta^2\Gamma(c+1) + \alpha\Gamma(c+2))} \sum_{j=0}^{\infty} \frac{t^j}{j!\theta^j} \left(\theta^2\Gamma(c+j+1) + \alpha\Gamma(c+j+2)\right)$$

Similarly, the Characteristics Function of weighted new quasi Lindley distribution can be obtained as:

$$\varphi_X(t) = M_X(it)$$

$$\Rightarrow M_X(it) = \frac{1}{(\theta^2\Gamma(c+1) + \alpha\Gamma(c+2))} \sum_{j=0}^{\infty} \frac{(it)^j}{j!\theta^j} \left(\theta^2\Gamma(c+j+1) + \alpha\Gamma(c+j+2)\right)$$

V. ORDER STATISTICS

Order statistics make their appearance in many statistical theory and practice. We know that if $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ denotes the order statistics of a random sample $X_1, X_2, \ldots, X_n$ from a Continuous population with cdf $F_X(x)$ and pdf $f_x(x)$, then the pdf of rth order statistics $X_{(r)}$ is given by

$$f_{x(r)}(x) = \frac{n!}{(r-1)!(n-r)!} f_X(x)\left(F_X(x)\right)^{r-1}\left(1 - F_X(x)\right)^{n-r}$$

Using equation (5) and (7) the expression of the rth order statistics X_r of weighted new quasi lindley distribution is given by

$$f_{x(r)}(x) = \frac{n!}{(r-1)!(n-r)!}\left(\frac{x^c\theta^{c+2}}{\theta^2\Gamma(c+1) + \alpha\Gamma(c+2)}(\theta + \alpha x)e^{-\theta x}\right)$$
$$\times\left(\frac{1}{\theta^2\Gamma(c+1) + \alpha\Gamma(c+2)}\left(\theta^2\gamma(c+1,\theta x) + \alpha\gamma(c+2,\theta x)\right)\right)^{r-1}$$
$$\times\left(1 - \frac{1}{\theta^2\Gamma(c+1) + \alpha\Gamma(c+2)}\left(\theta^2\gamma(c+1,\theta x) + \alpha\gamma(c+2,\theta x)\right)\right)^{n-r}$$

Therefore, the expression of the higher order statistics $X_{(n)}$ of weighted new quasi lindley distribution is given by

$$f_{x(n)}(x) = \frac{nx^c\theta^{c+2}}{\theta^2\Gamma(c+1) + \alpha\Gamma(c+2)}(\theta + \alpha x)e^{-\theta x}\left(\frac{1}{\theta^2\Gamma(c+1) + \alpha\Gamma(c+2)}\left(\theta^2\gamma(c+1,\theta x) + \alpha\gamma(c+2,\theta x)\right)\right)^{n-1}$$

and the expression of the first order statistics $X_{(1)}$ of weighted new quasi lindley distribution is given by

$$f_{x(1)}(x) = \frac{nx^c\theta^{c+2}}{\theta^2\Gamma(c+1) + \alpha\Gamma(c+2)}(\theta + \alpha x)e^{-\theta x}\left(1 - \frac{1}{\theta^2\Gamma(c+1) + \alpha\Gamma(c+2)}\left(\theta^2\gamma(c+1,\theta x) + \alpha\gamma(c+2,\theta x)\right)\right)^{n-1}$$

VI. LIKELIHOOD RATIO TEST

Let $X_1, X_2, X_3 \ldots \ldots X_n$ be a random sample from the new quasi lindley distribution or weighted new quasi lindley distribution. We test the hypothesis

$H_o : f(x) = f(x;\theta,\alpha)$ against $H_1 : f(x) = f_w(x;\theta,\alpha,c)$

For testing whether the random sample of size n comes from new quasi lindley distribution or weighted new quasi lindley distribution, the following test statistic is used

$$\Delta = \frac{L_1}{L_o} = \prod_{i=1}^{n} \frac{f_w(x;\theta,\alpha,c)}{f(x;\theta,\alpha)}$$

$$\Delta = \frac{L_1}{L_o} = \prod_{i=1}^{n} \left[\frac{\theta^c x_i^{\,c} (\theta^2 + \alpha)}{\theta^2 \Gamma(c+1) + \alpha\Gamma(c+2)} \right]$$

$$\Delta = \frac{L_1}{L_o} = \left(\frac{\theta^c (\theta^2 + \alpha)}{\theta^2 \Gamma(c+1) + \alpha\Gamma(c+2)} \right)^n \prod_{i=1}^{n} x_i^{\,c}$$

We reject the null hypothesis if

$$\Delta = \left(\frac{\theta^c (\theta^2 + \alpha)}{\theta^2 \Gamma(c+1) + \alpha\Gamma(c+2)} \right)^n \prod_{i=1}^{n} x_i^{\,c} > k$$

Equivalently, we reject the null hypothesis

$$\Delta^* = \prod_{i=1}^{n} x_i^{\,c} > k \left(\frac{\theta^2 \Gamma(c+1) + \alpha\Gamma(c+2)}{\theta^c (\theta^2 + \alpha)} \right)^n$$

$$\Delta^* = \prod_{i=1}^{n} x_i^{\,c} > k^*, \text{Where } k^* = k \left(\frac{\theta^2 \Gamma(c+1) + \alpha\Gamma(c+2)}{\theta^c (\theta^2 + \alpha)} \right)^n$$

For a large sample of size *n*, *2log Δ* is distributed as chi-square distribution with one degree of freedom and also p- value is obtained from the chi-square distribution. Also, we reject the null hypothesis, when the probability value is given by

$p\left(\Delta^* > \beta^*\right)$, Where $\beta^* = \prod_{i=1}^{n} x_i^{\,c}$ is less than a specified level of significance and $\prod_{i=1}^{n} x_i^{\,c}$ is the observed value of the statistic Δ^*.

VII. BONFERRONI AND LORENZ CURVES

The Bonferroni and the Lorenz curves are used not only in economics to study income and poverty, but it is also being used in other fields like reliability, medicine, insurance and demography. The Bonferroni and Lorenz curves are given by

$$B(p)=\frac{1}{p\mu_1'}\int_0^q x f_W(x;\theta,\alpha,c)dx$$

and $$L(p)=pB(p)=\frac{1}{\mu_1'}\int_0^q x f_W(x;\theta,\alpha,c)dx$$

Where $\mu_1'=E(X)=\dfrac{\theta^2\Gamma(c+2)+\alpha\Gamma(c+3)}{\theta(\theta^2\Gamma(c+1)+\alpha\Gamma(c+2))}$ and $q=F^{-1}(p)$

$$\therefore \quad B(p)=\frac{\theta(\theta^2\Gamma(c+1)+\alpha\Gamma(c+2))}{p(\theta^2\Gamma(c+2)+\alpha\Gamma(c+3))}\int_0^q \frac{\theta^{c+2}}{\theta^2\Gamma(c+1)+\alpha\Gamma(c+2)}x^{c+1}(\theta+\alpha x)e^{-\theta x}dx$$

$$B(p)=\frac{\theta(\theta^2\Gamma(c+1)+\alpha\Gamma(c+2))}{p(\theta^2\Gamma(c+2)+\alpha\Gamma(c+3))}\frac{\theta^{c+2}}{\theta^2\Gamma(c+1)+\alpha\Gamma(c+2)}\int_0^q x^{c+1}(\theta+\alpha x)e^{-\theta x}dx$$

$$B(p)=\frac{\theta^{c+3}}{p(\theta^2\Gamma(c+2)+\alpha\Gamma(c+3))}\left(\theta\int_0^q e^{-\theta x}x^{(c+2)-1}dx+\alpha\int_0^q e^{-\theta x}x^{(c+3)-1}dx\right)$$

$$B(p)=\frac{\theta^{c+3}}{p(\theta^2\Gamma(c+2)+\alpha\Gamma(c+3))}\left(\theta\gamma(c+2,\theta q)+\alpha\gamma(c+3,\theta q)\right)$$

and $$L(p)=pB(p)=\frac{\theta^{c+3}}{(\theta^2\Gamma(c+2)+\alpha\Gamma(c+3))}\left(\theta\gamma(c+2,\theta q)+\alpha\gamma(c+3,\theta q)\right)$$

VIII. ENTROPIES

The concept of entropies is important in different areas such as probability and statistics, physics, communication theory and economics. Entropies quantify the diversity, uncertainty, or randomness of a system. Entropy of a random variable X is a measure of variation of the uncertainty.

8.1 Renyi Entropy-

The Renyi entropy is important in ecology and statistics as index of diversity. The Renyi entropy is also important in quantum information, where it can be used as a measure of entanglement. For a given probability distribution, Renyi entropy is given by

$$e(\beta)=\frac{1}{1-\beta}\log\left(\int f_W{}^{\beta}(x)dx\right)$$

Where, $\beta>0$ and $\beta\neq 1$

$$e(\beta)=\frac{1}{1-\beta}\log\int_0^\infty\left(\frac{x^c\theta^{c+2}}{\theta^2\Gamma(c+1)+\alpha\Gamma(c+2)}(\theta+\alpha x)e^{-\theta x}\right)^\beta dx$$

$$e(\beta)=\frac{1}{1-\beta}\log\left(\left(\frac{\theta^{c+2}}{\theta^2\Gamma(c+1)+\alpha\Gamma(c+2)}\right)^\beta\int_0^\infty x^{\beta c}e^{-\theta\beta x}\left(\theta+\alpha x\right)^\beta dx\right) \tag{9}$$

Using Binomial expansion in (9), we obtain

$$e(\beta)=\frac{1}{1-\beta}\log\left(\left(\frac{\theta^{c+2}}{\theta^2\Gamma(c+1)+\alpha\Gamma(c+2)}\right)^\beta\sum_{k=0}^\infty\binom{\beta}{k}\theta^{\beta-k}(\alpha x)^k\int_0^\infty x^{\beta c}e^{-\theta\beta x}dx\right)$$

$$e(\beta)=\frac{1}{1-\beta}\log\left(\left(\frac{\theta^{c+2}}{\theta^2\Gamma(c+1)+\alpha\Gamma(c+2)}\right)^\beta\sum_{k=0}^\infty\binom{\beta}{k}\theta^{\beta-k}\alpha^k\int_0^\infty x^{(\beta c+k+1)-1}e^{-\theta\beta x}dx\right)$$

$$e(\beta)=\frac{1}{1-\beta}\log\left(\left(\frac{\theta^{c+2}}{\theta^2\Gamma(c+1)+\alpha\Gamma(c+2)}\right)^\beta\sum_{k=0}^\infty\binom{\beta}{k}\theta^{\beta-k}\alpha^k\frac{\Gamma(\beta c+k+1)}{(\theta\beta)^{\beta c+k+1}}\right)$$

8.2 Tsallis Entropy-

A generalization of Boltzmann-Gibbs (B.G) statistical properties initiated by Tsallis has focused a great deal to attention. This generalization of B-G statistics was proposed firstly by introducing the mathematical expression of Tsallis entropy (Tsallis, 1988) for a continuous random variable is defined as follows

$$S_\lambda=\frac{1}{\lambda-1}\left(1-\int_0^\infty f_w{}^\lambda(x)dx\right)$$

$$S_\lambda=\frac{1}{\lambda-1}\left(1-\int_0^\infty\left(\frac{x^c\theta^{c+2}}{\theta^2\Gamma(c+1)+\alpha\Gamma(c+2)}(\theta+\alpha x)e^{-\theta x}\right)^\lambda dx\right)$$

$$S_\alpha=\frac{1}{\lambda-1}\left(1-\left(\frac{\theta^{c+2}}{\theta^2\Gamma(c+1)+\alpha\Gamma(c+2)}\right)^\lambda\int_0^\infty x^{\lambda c}e^{-\lambda\theta x}(\theta+\alpha x)^\lambda dx\right) \tag{10}$$

Using Binomial expansion in equation (10), we get

$$S_\lambda=\frac{1}{\lambda-1}\left(1-\left(\frac{\theta^{c+2}}{\theta^2\Gamma(c+1)+\alpha\Gamma(c+2)}\right)^\lambda\sum_{k=0}^\infty\binom{\lambda}{k}\theta^{\lambda-k}(\alpha x)^k\int_0^\infty x^{\lambda c}e^{-\lambda\theta x}dx\right)$$

$$S_{\lambda} = \frac{1}{\lambda - 1}\left(1 - \left(\frac{\theta^{c+2}}{\theta^2\Gamma(c+1) + \alpha\Gamma(c+2)}\right)^{\lambda} \sum_{k=0}^{\infty}\binom{\lambda}{k}\theta^{\lambda-k}\alpha^{k}\int_0^{\infty} x^{(\lambda c+k+1)-1}e^{-\lambda\theta x}dx\right)$$

$$S_{\lambda} = \frac{1}{\lambda - 1}\left(1 - \left(\frac{\theta^{c+2}}{\theta^2\Gamma(c+1) + \alpha\Gamma(c+2)}\right)^{\lambda} \sum_{k=0}^{\infty}\binom{\lambda}{k}\theta^{\lambda-k}\alpha^{k}\frac{\Gamma(\lambda c + k + 1)}{(\lambda\theta)^{\lambda c+k+1}}\right)$$

IX. PARAMETER ESTIMATION AND FISHER'S INFORMATION MATRIX

In this section, we will discuss the parameter estimation of weighted new quasi lindley distribution using maximum likelihood method. Let $x_1, x_2, x_3, \ldots x_n$ be a random sample of size n from the weighted new quasi lindley distribution and let f_x be the observed frequency in the sample corresponding to $X = x (x = 1,2,\ldots, n)$ such that $\sum_{x=1}^{n} f_x = n,$ Where n is the largest observed value having non-zero frequency. Then the likelihood function of weighted new quasi lindley distribution is given by

$$L(x;\theta,\alpha,c) = \prod_{i=1}^{n} f_w(x;\theta,\alpha,c)$$

$$L(x;\theta,\alpha,c) = \prod_{i=1}^{n}\left[\frac{x_i^{\,c}\theta^{c+2}}{(\theta^2\Gamma(c+1) + \alpha\Gamma(c+2))}(\theta + \alpha x)e^{-\theta x_i}\right]$$

$$L(x;\theta,\alpha,c) = \frac{\theta^{n(c+2)}}{(\theta^2\Gamma(c+1) + \alpha\Gamma(c+2))^n}\prod_{i=1}^{n}\left[x_i^{\,c}(\theta + \alpha x_i)e^{-\theta x_i}\right]$$

The log likelihood function is given by:

$$\log L(x;\theta,\alpha,c) = n(c+2)\log\theta - n\log(\theta^2\Gamma(c+1) + \alpha\Gamma(c+2)) + c\sum_{i=1}^{n}\log x_i + \sum_{i=1}^{n}\log(\theta + \alpha x_i) - \theta\sum_{i=1}^{n} x_i \tag{11}$$

Differentiating the equation (11) partially w.r.to θ, α and c and equating to zero, we will have the following system of equations.

$$\frac{\partial \log L}{\partial\theta} = \frac{n(c+2)}{\theta} - n\left(\frac{2\theta\Gamma(c+1)}{\theta^2\Gamma(c+1) + \alpha\Gamma(c+2)}\right) + \sum_{i=1}^{n}\left(\frac{1}{(\theta + \alpha x_i)}\right) - \sum_{i=1}^{n} x_i = 0 \tag{12}$$

$$\frac{\partial \log L}{\partial\alpha} = -n\left(\frac{\Gamma(c+2)}{(\theta^2\Gamma(c+1) + \alpha\Gamma(c+2))}\right) + \sum_{i=1}^{n}\left(\frac{x_i}{(\theta + \alpha x_i)}\right) = 0 \tag{13}$$

$$\frac{\partial \log L}{\partial c} = n\log\theta - n\psi(c+1) + \sum_{i=1}^{n}\log x_i = 0 \tag{14}$$

Where ψ (.) is the digamma function.

Because of the complicated form of the likelihood equations, algebraically it is very difficult to solve the system of non-linear equations. Therefore we use R and wolfram mathematics for estimating the required parameters.

To obtain confidence interval we use the asymptotic normality results. we have that if $\hat{\lambda} = (\hat{\theta}, \hat{\alpha}, \hat{c})$ denotes the MLE of $\lambda = (\theta, \alpha, c)$ we can state the results as follows:

$$\sqrt{n}(\hat{\lambda} - \lambda) \rightarrow N_3(0, I^{-1}(\lambda))$$

Where $I(\lambda)$ is Fisher's Information Matrix. i.e.,

$$I(\lambda) = -\frac{1}{n}\begin{pmatrix} E\left(\frac{\partial^2 \log L}{\partial \theta^2}\right) & E\left(\frac{\partial^2 \log L}{\partial \theta \partial \alpha}\right) & E\left(\frac{\partial^2 \log L}{\partial \theta \partial c}\right) \\ E\left(\frac{\partial^2 \log L}{\partial \alpha \partial \theta}\right) & E\left(\frac{\partial^2 \log L}{\partial \alpha^2}\right) & E\left(\frac{\partial^2 \log L}{\partial \alpha \partial c}\right) \\ E\left(\frac{\partial^2 \log L}{\partial c \partial \theta}\right) & E\left(\frac{\partial^2 \log L}{\partial c \partial \alpha}\right) & E\left(\frac{\partial^2 \log L}{\partial c^2}\right) \end{pmatrix}$$

Where

$$E\left(\frac{\partial^2 \log L}{\partial \theta^2}\right) = -\frac{n(c+2)}{\theta^2} - n\left(\frac{(\theta^2\Gamma(c+1) + \alpha\Gamma(c+2))(2\Gamma(c+1)) - (2\theta\Gamma(c+1))(2\theta\Gamma(c+1))}{(\theta^2\Gamma(c+1) + \alpha\Gamma(c+2))^2}\right) - \sum_{i=1}^{n}\left(\frac{1}{(\theta + \alpha x_i)^2}\right)$$

$$E\left(\frac{\partial^2 \log L}{\partial \alpha^2}\right) = n\left(\frac{\Gamma(c+2)(\Gamma(c+2))}{(\theta^2\Gamma(c+1) + \alpha\Gamma(c+2))^2}\right) - \sum_{i=1}^{n}\left(\frac{E(x_i^2)}{(\theta + \alpha x_i)^2}\right)$$

$$E\left(\frac{\partial^2 \log L}{\partial c^2}\right) = -n\psi'(c+1)$$

Also,

$$E\left(\frac{\partial^2 \log L}{\partial \theta \partial \alpha}\right) = E\left(\frac{\partial^2 \log L}{\partial \alpha \partial \theta}\right) = n\left(\frac{2\theta\Gamma(c+1)\Gamma(c+2)}{(\theta^2\Gamma(c+1) + \alpha\Gamma(c+2))^2}\right) - \sum_{i=1}^{n}\left(\frac{E(x_i)}{(\theta + \alpha x_i)^2}\right)$$

$$E\left(\frac{\partial^2 \log L}{\partial \theta \partial c}\right) = E\left(\frac{\partial^2 \log L}{\partial c \partial \theta}\right) = \frac{n}{\theta}$$

$$E\left(\frac{\partial^2 \log L}{\partial \alpha \partial c}\right) = E\left(\frac{\partial^2 \log L}{\partial c \partial \alpha}\right) = -n\psi''(c+1)$$

Where $\psi(.)'$ and $\psi(.)''$ is the first and second order derivatives of digamma function.

Since λ being unknown, we estimate $I^{-1}(\lambda)$ by $I^{-1}(\hat{\lambda})$ and this can be used to obtain asymptotic confidence intervals for θ, α and c.

X. DATA ANALYSIS

In this section, we have used two real-life data sets to show that the weighted new quasi lindley distribution can be a better model than the new quasi lindley distribution. The following two data sets are given below

Data Set 1- The following data set represent the strength data of glass of the aircraft window reported by Fuller et al (1994).The data set is provided below in table 1.

Table 1- Data regarding the strength of the glass of aircraft window reported by Fuller *et al* (1994)

18.83	20.80	21.657	23.03	23.23	24.05
24.321	25.50	25.52	25.80	26.69	26.77
26.78	27.05	27.67	29.90	31.11	33.20
33.73	33.76	33.89	34.76	35.75	35.91
36.98	37.08	37.09	39.58	44.045	45.29
45.381					

Data set 2- The following data set represents the time to failure (*10^3h*) of turbocharger of one type of engine studied by Xu *et.al*. (2003).The data set is provided below in table 2.

Table 2- Data regarding the time to failure of turbocharger (n=40) studied by Xu et.al. (2003).

1.6	3.5	4.8	5.4	6.0	6.5	7	7.3	7.7	8
8.4	2	3.9	5	5.6	6.1	6.5	7.1	7.3	7.8
8.1	8.4	2.6	4.5	5.1	5.8	6.3	6.7	7.3	7.7
7.9	8.3	8.5	3	4.6	5.3	6	8.7	8.8	9

In order to compare the performance of weighted new quasi lindley distribution with new quasi lindley distribution. We are using the Criterion values like AIC (Akaike information criterion), AICC (corrected Akaike information criterion) and BIC (Bayesian information criterion) .The better distribution corresponds to lesser AIC, AICC and BIC values. The formulae for calculation of AIC, BIC and AICC are

$$\text{AIC} = 2k - 2\log L \qquad \text{AICC} = AIC + \frac{2k(k+1)}{n-k-1} \quad \text{and} \quad \text{BIC} = k\log n - 2\log L$$

Where k is the number of parameters in the statistical model, n is the sample size and -2log$L$ is the maximized value of the log-likelihood function under the considered model. From table 3, it has been observed that the weighted new quasi lindley distribution have the lesser AIC, AICC, -2log$L$ and BIC values as compared to new quasi lindley distribution. Hence we can conclude that the weighted new quasi lindley distribution leads to a better fit than new quasi lindley distribution.

Table 3- Parameter Estimates, AIC, BIC, AICC, -2logL criterion values of the fitted distribution

Data sets	Distribution	MLE	SE	-2logL	AIC	BIC	AICC
1	**New quasi lindley**	$\hat{\alpha} = 3.2124$ $\hat{\theta} = 6.4909$	$\hat{\alpha} = 5.1853$ $\hat{\theta} = 8.2416$	**252.2344**	**256.2344**	**259.1024**	**256.6629**
	Weighted new quasi lindley	$\hat{\alpha} = 4.7687$ $\hat{\theta} = 0.6146$ $\hat{c} = 16.9412$	$\hat{\alpha} = 284.9577$ $\hat{\theta} = 0.1570$ $\hat{c} = 4.7982$	**208.2315**	**214.2315**	**218.5335**	**215.1203**

2	**New quasi lindley**	$\hat{\alpha}=3.3626$ $\hat{\theta}=3.1980$	$\hat{\alpha}=3.3560$ $\hat{\theta}=3.5756$	**201.0361**	**205.0361**	**208.4138**	**205.3604**
	Weighted new quasi lindley	$\hat{\alpha}=0.5245$ $\hat{\theta}=1.2718$ $\hat{c}=6.2509$	$\hat{\alpha}=1.8854$ $\hat{\theta}=0.2829$ $\hat{c}=1.9713$	**174.5677**	**180.5677**	**185.6343**	**181.2343**

XI. CONCLUSION

In this study, we have introduced the weighted new quasi lindley distribution. The subject distribution is generated by using the weighted technique and taking the two parameter new quasi lindley distribution as the base distribution. The moments, harmonic mean, survival function, hazard function and the maximum likelihood estimation of the parameters of the distribution have also been obtained. The application of the new distribution has also been demonstrated with two real life data sets. The results are compared with new quasi lindley distribution and the results indicate that the weighted new quasi lindley distribution provides a better fit than the new quasi lindley distribution.

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