# Generalization of some fixed point theorems in $b$ −metric spaces.

**Zainab Hussein Sabri and Zeana Zaki Jamil**

**University of Baghdad, College of Science, Dept. of Math., IRAQ**

**Abstract**

In this paper, we proved generalization Berinde`s theory using the 6-dimansional and the diameter of the orbit of a self-mapping.

**Keywords:** Fixed point, b-metric space, generalized $\varphi$ −contraction, orbit of a mapping, diameter of the set.

**2010 MSC:** 47H10, 54H25.

**1 Introduction**

One topic to develop nonlinear analysis is fixed point theory which is effectively in many branch of science, like: biology, engineering, economics …etc.

In 1922, Banach [1] proved the contraction principle theorem which is one of the fundamental theorems in fixed point theory. It has followed by several authors regarding different spaces such as: Banach spaces [2], quasi-matric spaces [3], partial metric spaces [4] and G-matric spaces [5] … etc.

Bakhtin [6] and Czerwik [7] introduce a $b$ −metric space, and argued the Banach's [1] contraction principle theorem in it.

Definition 1.1 [6], [7]:

Let $X$ be a nonempty set and $s \geq 1$ be a given real number. A function $d : X \times X \to \mathbb{R}_+ = [0, \infty)$ is a $b$ −metric on $X$, if for all $x, y, z \in X$, the following conditions hold:

1. $d(x, y) = 0$ if and only if $x = y$,
2. $d(x, y) = d(y, x)$,
3. $d(x, z) \leq s[d(x, y) + d(y, z)]$, ($b$ −triangular inequality).

In this case, the triple $(X, d, s)$ is called a $b$ −metric space

When $s = 1$, $b$-metric space is metric space while the converse is false e.g. [8],

The space $l_p (0 < p < 1)$ ,

$$l_p = \left\{ \{x_n\} : x_n \in \mathbb{R}, \sum_{n=1}^{\infty} |x_n|^p < \infty \right\},$$

together with the function $d: l_p \times l_p \to \mathbb{R}_+$

$$d(x, y) = \left[ \sum_{n=1}^{\infty} |x_n - y_n|^p \right]^{\frac{1}{p}},$$

where $x = \{x_n\}$ , $y = \{y_n\}$ in $l_p$ and $s = 2^{\frac{1}{p}}$ is a $b$ −matric space.

Thus the concept of a $b$ −matric space is wider than the concept of a matric space.

Boriceanu et al [9] presented the concept of the complete $b$ −matric space.

**Definition 1.2 [9]**: Let $(X, d, s)$ be a $b$-metric space and $\{x_n\}$ is a sequence in $X$ . Then

1) $\{x_n\}$ is called $b$-convergent (for simplicity we call it convergent) if there exists $x \in X$ such that $d(x_n, x) \to 0$ as $n \to \infty$ , we write $\lim_{n\to\infty} x_n = x$:
2) $\{x_n\}$ is called $b$-Cauchy (for simplicity we call it Cauchy) if $d(x_n, x_m) \to 0$ as $n, m \to \infty$.
3) A $b$-metric space $(X, d, s)$ is said to be a $b$- complete $b$-metric space (for simplicity we call it complete $b$-metric space) if every Cauchy sequence in $X$ is convergent.

Banach has shown that the contraction mapping $T$ under the distance $d(x, y)$ in a metric space has unique fixed point (Banach`s contraction mapping principle Theorem). Recently, several authors, as Kannan`s [10], Zamfiescu`s [11] and Ciric`s [12], have developed Banach`s theorem [13] by using different kinds of the displacements of $x$, $y$ or their image, as $d(x, Ty), d(y, Tx)$ , $d(x, Tx)$ and $d(y, Ty)$.

Berinde in [13], defined 5-dimenstional comparison mapping $\varphi: \mathbb{R}_+^5 \to \mathbb{R}_+$ in order to generalize all the previse theorems in one theorem.

We will develop Berinde`s theorem to a $b$-metric space with 6-dimensional comparison mapping, $\varphi: \mathbb{R}_+^6 \to \mathbb{R}_+$, and by using the diameter of the orbit of a self-mapping under Picard iteration process.

**2 The Main Result.**

In this section we argue the generalization of Berinde`s theorem in $b$ −metric space.

But first we need to define the following mappings and give many examples.

**Definition 2.1:** A mapping $\varphi: \mathbb{R}_+^6 \to \mathbb{R}_+$ is called 6-dimensional comparison (for simplify we call it comparison) if it is non-decreasing for each coordinate.

**Definition 2.2:** A conjugate mapping to a comparison mapping $\varphi$ is defined as $\psi: \mathbb{R}_+ \to \mathbb{R}_+$ such that

$\psi(t) = \varphi(t, t, t, t, t, t);\ t \in \mathbb{R}_+$ (1)

Satisfies conditions

(i) $\psi$ is a non-decreasing,
(ii) $\{\psi^n(t)\}$ converges to 0 for all $t \geq 0$ ,
(iii) $\psi(t) < t$ for all $t > 0$ and $\psi(0) = 0$ .

**Examples.2.3.** Let $\varphi: \mathbb{R}_+^6 \to \mathbb{R}_+$ be a mapping. In the following table we define comparison mapping $\varphi$ with its conjugate $\psi$.

| | Comparison mapping | Conjugate mapping |
|---|---|---|
| 1 | $\varphi(u_1, \ldots, u_6) = a\, max\{u_1, u_2, u_3, u_4, u_5, u_6\}$ ; $a \in (0,1]$ | $\psi(u) = au$ |
| 2 | $\varphi(u_1, \ldots, u_6) = au_1 + bu_2 + c max\{u_3, u_4, u_5, u_6\}$; $a, b, c \geq 0$ where $a + b + c < 1$. | $\psi(u) = (a + b + c)u$ |

| | | |
|---|---|---|
| 3 | $\varphi(u_1,\dots,u_6)=a(u_2+u_3)$, where $a\in[0,\frac{1}{2})$. | $\psi(u)=2au$ |
| 4 | $\varphi(u_1,\dots,u_6)=a\,max\left\{u_1,u_2,u_3,u_4,\frac{u_5+u_4}{2}\right\}$, $a\in(0,1]$ | $\psi(u)=au$ |
| 5 | $\varphi(u_1,\dots,u_6)=au_1+b(u_2+u_3)$, $a,b\in[0,\infty)$ where $a+2b<1$ | $\psi(u)=(a+2b)u$ |
| 6 | $\varphi(u_1,\dots,u_6)=a\,max\,\{u_2,u_3\}$ where $a\in(0,1)$. | $\psi(u)=au$ |
| 7 | $\varphi(u_1,\dots,u_6)=max\left\{\begin{matrix}au_1,\\ b(u_2+u_4)\\ ,c(u_3+u_5)\end{matrix}\right\}$ , $a\in(0,1],b,c\in[0,\frac{1}{2})$. | $\psi(u)=au$ *or* $\psi(u)=2bu$ $\psi(u)=2cu$ |

Next we introduce the $s$ −generalized $\varphi$ −contraction mapping.

**Definition 2.4**: Let $(X,d,s)$ be a b-metric space. A self-mapping $T:X\to X$ is called $s$ −generalized $\varphi$ −contraction if there exists comparison mapping $\varphi:\mathbb{R}_+^6\to\mathbb{R}_+$ such that for all $x,y\in X$

$$d(Tx,Ty)\le\frac{1}{s}\varphi\left(d(x,y),d(x,Tx),d(y,Ty),\frac{d(x,Ty)}{s},d(y,Tx),\frac{d(Tx,Ty)}{s}\right)$$

**Notation:** We will abbreviate

$$D(x,y):=\left(d(x,y),d(x,Tx),d(y,Ty),\frac{d(x,Ty)}{s},d(y,Tx),\frac{d(Tx,Ty)}{s}\right)$$

Recall that the orbit of a self-mapping $T$ on a set $X$, then the set $orbt(T,x,n)=\{x,Tx,T^2x,\dots,T^nx\};n\in\mathbb{N}_0$

While, the diameter of the orbit of a mapping $T:X\to X$ is $\delta(orbt(T,x,n):=\max\{d(T^ix,T^jx)\mid 0\le i,j\le n;n\in\mathbb{N}\}$ for all $x\in X$.

Note that if $n=0$, then $T$ has a fixed point $x_0$. Therefor we assume that $n=0$.

The following proposition is the key to showing our main result

**Proposition (2.5)**

Let $(X,d,s)$ be a $b$ – metric space. Let $T:X\to X$ be a $s-$ generalized $\varphi$- contraction with conjugate mapping $\psi$. Then for all $x_0\in X$ and $a,b\in\{1,2,\dots,n\}$ we have

$$d(T^ax_0,T^bx_0)\le\frac{1}{s}\psi\left(\delta(orbt(T,x_0,n))\right)$$

Proof: Let $x_n=T^nx_0;\ n\in\mathbb{N}_0$. Since for each $a,b\in\{1,2,\dots,n\}$ we get $\{a,a-1,b,b-1\}\subset\{1,2,\dots,n\}$. It follows that $x_a,x_{a-1},x_b,x_{b-1}\in orbt(T,x_0,n)$. Since $d(x_p,x_q)\le\delta(orbt(T,x_0,n))$ for each $p,q\in\{x_a,x_{a-1},x_b,x_{b-1}\}$. Then due from the $s-$ generalized $\varphi-$ contraction condition, the non-decreasing of φ and by Definition (2.2)

$$d(T^ax_0,T^bx_0)=d(x_a,x_b)\le\frac{1}{s}\ \varphi\big(D(x_{a-1},x_{b-1})\big)$$

$$\leq \frac{1}{s}\,\psi\,\Big(\delta\big(orbt(T,x_0,n\,)\big)\Big).$$

The following proposition measures the diameter of the orbit of the mapping

**Proposition (2.6)**

Let $(X,d,s)$ be a $b$ – metric space, $T: X \to X$ be a $s-$ generalized $\varphi$- contraction with conjugate mapping $\psi$ , then for each $n \in \mathbb{N}$, there exist $a \leq n$ such that

$$d(x_0,T^a x_0) \;=\; \delta(orbt(T,x_0,n\,))$$

Proof: If $\delta(orbt(T,x_0,n\,)) = 0$, then the result has done.

If $\delta(orbt(T,x_0,n\,)) \neq 0$, assume there exist $n \in \mathbb{N}$ such that for all $a \leq n$

$$d(x_0,T^a x_0) \neq \delta(orbt(T,x_0,n\,))$$

But

$\delta\big(orbt(T,x_0,n\,)\big) = max\{d(T^i x_0,T^j x_0\,)|\; 0 \leq i\,,j \leq n\,;\; n \in \mathbb{N}\,\}$, thus there exist $0 < b \leq n,$ such that

$$\delta(orbt(T,x_0,n\,)) \;=\; d(T^a\,x_0\,,T^b x_0)$$

Hence by proposition (2.5), we obtain

$$\delta(orbt(T,x_0,n\,)) \;\leq\; \frac{1}{s}\,\psi\,(\delta(orbt(T,x_0,n\,)))$$

$$< \tfrac{1}{s}\,\delta(orbt(T,x_0,n\,))$$

That is a contradiction. Therefor $d\,(x_0\,,T^a x_0) = \delta(orbt(T,x_0,n\;))$.

We need to define the following mapping to prove our main result

**Definition (2.7)**: Let $h: \mathbb{R}_+ \to \mathbb{R}_+$be a mapping defined as

$$h(t) = t - \psi(t);\;\; t \in \mathbb{R}_+ \qquad (3),$$

where $\psi$ is a conjugate mapping to a comparison mapping $\varphi$.

**Proposition (2.8)**

Let $(X\,,d,s)$ be a complete $b-$ metric space, $T: X \to X$ be $s$–generalized $\varphi-$contraction with conjugate mapping $\psi$ . If the mapping $h$ is define as (3), then for any $n \in \mathbb{N}_0$ and $x_0 \in X$, we have

$$h(\delta(orbt\;(T,x_0,n))) \;\leq\; s\,d(x_0,Tx_0).$$

Proof:

Let $x_0 \in X$, and $n \in \mathbb{N}$. By Proposition (2.6), that there exist $m \leq n$ such that $d(x_0,T^m\,x_0) = \delta(orbt(T,x_0,n))$, so that by Proposition (2.5) we get

$$\delta(orbt(T,x_0,n)) = d(x_0,T^m\,x_0) \qquad \leq\; s\,[d(x_0\,,Tx_0) + d(Tx_0\,,T^m\,x_0)]$$

$$\leq\; sd(x_0,Tx_0\,) + \psi\,(\delta(orbt\;(T,x_0,n))$$

This implies that

$$h\big(\delta(orbt\ (T, x_0, n))\big) \leq\ sd(x_0, Tx_0)\ .$$

Now the main result will be prove.

**Theorem. (2.9)**

Let $(X, d, s)$ be a complete $b$- metric space, and $T: X \to X$ be an $s$–generalized $\varphi$- contraction with a linear continuous conjugate mapping $\psi$. If the mapping $h$ given by (3) is a non-decreasing and bijection, then

(i) $T$ has unique fixed point.
(ii) The following estimate
$d(T^k x_0, u) \leq \frac{1}{s^k}\ \psi^k\left(h^{-1}\big(sd(x_0\ , Tx_0)\big)\right);\ k \in \mathbb{N}_0$ ,
for all $x_0\ \in X$ and $u$ is the unique fixed point of $T$.

Proof:

Let $x_0\ \in X, m\ , k \in \mathbb{N}_0$, without loss of the generally we can assume that $m > k$ , by proposition (2.5) and by putting $a = 1,\ b = m - k + 1, T^k x_0 = x_k$ , we get

$$d(x_k, x_m) =\ d(Tx_{k-1}, T^{m-k+1}\ x_{k-1}) \leq \frac{1}{s}\ \psi(\delta(orbt(T, x_{k-1}, m - k + 1))$$

By proposition (2.6), there exist $k_1$; $1 \leq k_1 \leq m - k + 1$ such that

$$d\ (Tx_{k-1}, T^{m-k+1}\ x_{k-1}) \leq \frac{1}{s}\ \psi(d\ (x_{k-1}, T^{k_1}\ x_{k-1}))$$

$$= \frac{1}{s}\ \ \psi\ (d\ (Tx_{k-2}, T^{k_1+1}\ x_{k-2}))$$

Using proposition (2.5) we obtain

$$d(Tx_{k-1}, T^{m-k+1}\ x_{k-1}) \leq \frac{1}{s}\ \psi\left(\frac{1}{s}\ \psi(\delta(orbt\ (T, x_{k-2}, k_1 + 1)))\right)$$

Since $\psi$ is linear, non-decreasing and $k_1 + 1 \leq m - k + 2$, we obtain:

$$d(x_k, x_m) \leq \frac{1}{s^2}\ \ \psi^2\big(\delta(orbt\ (T, x_{k-2}, m - k + 2))\big)$$

And, inductively

$$d(x_k, x_m) \leq \frac{1}{s^k}\ \ \psi^k\ \left(\delta\big(orbt\ (T, x_0\ , m)\big)\right).$$

Now by proposition (2.8) since $h$ is a non-decreasing and bijection we get

$$d(x_k, x_m\ ) \leq \frac{1}{s^k}\ \psi^k\ (h^{-1}\ \ (s\ d\ (x_0\ \ , Tx_0))) \qquad (5)$$

As $\psi$ is a conjugate mapping to a comparison mapping $\varphi$.

$\frac{1}{s^k}\psi^k\left(h^{-1}\big(sd(\ (x_0, Tx_0)\big)\right) \to 0 \quad as \quad k\ \to\ \infty$ We get

$\lim_{k\to\infty} d(x_k, x_m) = 0$

Hence, $\{x_k\}_{k=0}^{\infty}$ is a Cauchy sequence by the completeness of $X$; we consider that $\{x_k\}_{k=0}^{\infty}$convergent to $u \in X$.

Now, we will show that $u$ is the unique fixed point of $T$. In fact, for each $k \in N$

$$d(u,Tu) \le s\,[d(u,Tx_k) + d(Tx_k, Tu)] \le s\,d(u,Tx_k) + \varphi\big(D(x_k, u)\big) \qquad (6)$$

Where

$$D(x_k,u) = \left(d(x_k, u), d(x_k, x_{k+1}), d(u,Tu), \frac{d(x_k, Tu)}{s}, d(u, x_{k+1}), \frac{d(Tx_k, Tu)}{s}\right)$$

There are four cases

Case 1 If,

$max\ D(\mathrm{x_k}, \mathrm{u}) = d(u,Tu)$

Then using non-decreasing of $\varphi$, from (6) we obtain

$$d(u,Tu) \le sd(u,x_{k+1}) + \psi(d(u,Tu))$$

By proposition (2.8) since $h$ is a non-decreasing and bijection we get

$$d(u,Tu) \le h^{-1}(sd(u,x_{k+1})) \qquad (7)$$

Since $\psi$ is continuous and $h^{-1}(0) = 0$, this leads $h^{-1}$ is continuous at zero. Letting $k \to \infty$ to eq. (7), we get $d(u,Tu) = 0$

Hence $Tu = u$ .

Case 2: If

$max\ D(x_k, u) = \dfrac{d(x_k, Tu)}{s}$

Then, by (6) we obtain:

$$d(u,Tu) \le sd(u,x_{k+1}) + \psi\left(\frac{d(x_k, Tu)}{s}\right)$$

$$d(u,Tu) \le sd(u,x_{k+1}) + \psi\big(d(x_k, u) + d(u,Tu)\big)$$

Because $\psi$ is continuous and letting $k \to \infty$ we have

$$d(u,Tu) \le \psi(d(u,Tu))$$

therefore $h\big(d(u,Tu)\big) \le 0$, thus $Tu = u$ .

Case 3: If

$max\ D(x_k, u) = \frac{d(x_{k+1}, Tu)}{s}$

Then by (6) we have

$$d(u,Tu) \leq sd(u,x_{k+1}) + \psi\left(\frac{\mathrm{d}(\mathrm{x}_{\mathrm{k+1}},\mathrm{Tu})}{\mathrm{s}}\right)$$

$$\leq sd(u,x_{k+1}) + \psi(\mathrm{d}(\mathrm{x}_{\mathrm{k+1}},\mathrm{u}) + \mathrm{d}(\mathrm{u},\mathrm{Tu}))$$

Since $\psi$ is continuous and letting $k \to \infty$ we get

$$h\big(d(u,Tu)\big) \leq 0$$

Hence $Tu = u$.

Case 4:

If maximum taken one values $d(u,x_{k+1})$, $d(x_k, x_{k+1})$ or $d(x_k, u)$ the proof is similar, thus suppose that,

$max\ D(x_k, u) = d(x_k, u)$.

Then, by (6) we get

$$d(u,Tu) \leq sd(u,x_{k+1}) + \psi(d(x_k,u)).$$

Taking $k \to \infty$ and $\psi$ is continuous, then we have $d(u,Tu) = 0$.

Now, we want to prove the uniqueness of the fixed point

Assume that $Tw = w$, then

$$d(u,w) = d(T^k u, T^k w) \leq \frac{1}{s^k}\ \psi^k\big(\delta((orbt(T,u,m))\big)$$

$$= \frac{1}{s^k}\ \psi^k\big(\delta(\{u\})\big) = \frac{1}{s^k}\ \psi^k(0) = 0.$$

Thus $u = w$ .

In order to get the estimate (ii), we take $m \to \infty$ in (5)

$$d(x_k,u) \leq \frac{1}{s^k}\psi^k\left(h^{-1}\big(s\ d\ (x_0, Tx_0)\big)\right).$$

**Remark (2.10)**

- For $\varphi$ as in Example 2.3, part 1, from theorem (2.9) we get the fixed point theorem induced by Ciric
- For $\varphi$ as in Example 2.3, part 3, from theorem (2.9) we get the Kannan`s fixed point theorem.
- For $\varphi$ as in Example 2.3, part 5, from theorem (2.9) we get the Reich`s fixed point theorem [14].
- For $\varphi$ as in Example 2.3, part 6, from theorem (2.9) we obtain the fixed point theorem induced by Bianchini [15]
- For $\varphi$ as in Example 2.3, part 7, from theorem (2.9) we get the Zamfirescu`s fixed point theorem.

**Conflicts of Interest**

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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