Heterogeneous Arrival Tandem Fluid Model Driven by an M/M /1 Vacation Queue Subject to Catastrophe

M. Deepa
Research Scholar, Nirmala College For Women

K. Julia Rose Mary
Asso.Prof.of Mathematics, Nirmala College For Women

Abstract: Queueing models are widely used to estimate desired performance measures of the system. This paper studies heterogeneous arrival of tandem fluid model with multiple exponential vacations subject to catastrophe. By applying Laplace transform to the steady state distribution of the buffer content we obtain minimal positive solution to a crucial quadratic equation. Then the performance measures such as mean, second moment and variance of the buffer content are derived. Furthermore, we also obtain the result that mean buffer content is independent of vacation parameter θ and vacation period arrival rate λ_0 whereas second moment and variance of the buffer content are dependent on vacation parameter θ. Finally by sensitivity analysis, we illustrate the parameters effect on the mean, second moment and variance of the buffer content.

Keywords: Tandem fluid queue, Heterogeneous arrival, Multiple exponential vacation, Catastrophe, Probability generating function and Buffer content.

I. Introduction

In general, the queueing system consists of one or more queues with one or more servers and operates under a set of procedures. Due to several service points situation occur, the customers go through all service point for their service which is known as tandem queue. This paper is motivated by heterogeneous arrival tandem fluid model with catastrophe. Some examples are, customers from different parts of the country arrive and wait at reservation counter of airlines, in a car service center inorder to get completely serviced a car passes through all service points, telecommunication system and patients appointments at a health clinic etc.

Edgar Reich [1957] studied the distribution of waiting times when queues are in tandem and considered customers proceed to a second multiple counter queue after being served at a first multiple counter queue. Prabhu[1967] investigated on transient behavior of a tandem fluid queue. Nair[1971] discussed on queue length and their limiting behavior, waiting time and explored priority model treated with a single server serving between two service units in tandem. Grassmann and Steve [2000] considered tandem queues with two waiting lines and obtained joint distribution of both lines in equilibrium. Gray, Wang, Scott [2000] analyzed a model with multiple vacation and the service station subject to breakdown during operation and hence obtained the queue length distribution and the mean queue length for the model. Liu and Gong [2001] investigated on fluid queue with strict priority and obtained the result that high priority class has strict priority access to service and also obtained various analytical results for different fluid traffic models. Ke and Pearn [2004] considered the management policy of an M/M/1 queue with server breakdown and vacation where arrival rate varies according to the server's status and derived the distribution of the system size and obtained mean queue length. Thangaraj and Vanitha[2008] analysed M/M/1 feedback queue with catastrophe and obtained transient solution analytically using continued fractions and thereby computed busy period, performance measure interms of its moments. Li, Liu and Shang[2009]studied heavy tailed asymptote for fluid model and derived Laplace transform of the stationary buffer content by considering minimal positive solution to a crucial quadratic equation. Mao, Wang and Tian[2010] discussed on fluid queue with multiple exponential vacation and derived performance measure- mean of the buffer content. Isguder and Kaya[2012] analysed a tandem queueing system with blocking and computed probability of losing a customer. Vijayashree and Anjuka[2013] presented the stationary analysis of a fluid queueing model subject to catastrophes. Buffer content under steady state condition are obtained in terms of modified Bessel function. Rowan Wang, Oualid

Jouini and Saif Benjaafar[2014] considered service system with a finite number of customer's arrival and obtained various performance measures of interest, including the expected waiting time of a specific customer, arbitrary customer, the expected completion time of all customer and examined the effect of heterogeneity inter-arrival and service times. Deepa Muthu and K. Julia Rose Mary[2020] discussed a fluid model with heterogeneous arrival and exponential vacation subject to catastrophe In their paper they discussed the Laplace transform of the stationary buffer content distribution with the minimal positive solution to a crucial quadratic equation and also obtained mean of the buffer content. Deepa Muthu and K. Julia Rose Mary[2021] studied bulk arrival tandem fluid queue with multiple exponential vacations and computed the Laplace transform of buffer content and hence obtained performance measures such as mean, second moment and variance of the buffer content.

This paper aims at heterogeneous arrival tandem fluid model with multiple exponential vacations subject to catastrophe. We derive Laplace transform of the buffer content and obtain the performance measure such as mean of the buffer content which is independent of the vacation parameter θ and vacation period arrival rate λ_0. Further, second moment and variance of the buffer content are obtained which are dependent on vacation parameter θ.

This paper is organized as follows: Model descriptions is given in section II. In section III, we write system of steady state differential equations and derive Laplace transform of the buffer content. In section IV, we obtain performance measures such as mean, second moment and variance of the buffer content. In section V, by sensitivity analysis we present parameters effect on mean, second moment and variance of the buffer content.

II. Model Description

Consider a tandem fluid queue modulated by an M/M/1 multiple vacations queue subject to catastrophe in which, fluid queue arriving at the system is heterogeneous. Let the arrival rate during vacation and busy period be denoted by λ_0 and λ_1, follow a Poisson distribution. The heterogeneous arrivals are serviced at the service rate μ, follow exponential distribution whereas the arrival times and service times of the present tandem fluid model are distributed identically. When the system is non empty, catastrophe occurs with mean rate ξ follows Poisson process. The occurrence of catastrophe during busy period annihilates the system and restarts after a period of time. Therefore catastrophe affects the arrival rate as well. The customers are serviced according to their arrival and duration of each vacation is independent of arrival, service and catastrophic rate. The server goes for vacation when the system is empty. If there are no customers in the queue after vacation server takes up another vacation called as multiple vacations, where vacation times V(t) are distributed exponentially with parameter θ. Let us denote Markovian process $\{N(t), J(t), t\geq 0\}$ with state space

$$\Omega = \{(k, 0), k=0, 1, 2\ldots\ldots\} \cup \{(k, 1), k=1, 2\ldots\ldots\} \qquad (1)$$

Where N(t), J(t) represent number of customers in the queue at time t and state of the system in busy or vacation state at time t. $J(t)=\begin{cases} 0, & \text{the server is on vacation period at time } t \\ 1, & \text{the server is in busy period at time } t \end{cases}$

By defining a non-negative random variable C(t) be the buffer amount of fluid at time t, also let the output from the tandem queue of heterogeneous arrival under multiple vacations and catastrophe has a current rate R and a constant leak rate c which represents a buffer. The buffer would be empty if $\mu< c$. But we assume that $\mu > c$ always and current rate R=0, if the server is on vacation. Therefore, the net input rate of the fluid r_b to the buffer during busy period is >0 ((i.e) r_b >0) and during vacation let the net input rate be r_v ((i.e) $r_v< 0$). Hence $r_v = -c$.

We define $\pi_{kj} = P\{N = k, J = j\} = \lim_{t\to\infty} P\{N(t) = k, J(t) = j\}$ where (N,J) is stationary random vector of the stable distribution. Define, mean drift d of the Markovian process $\{C(t), t\geq 0\}$ where C(t) is the mean buffer content as $d = r_b\, P\{J=1\}+ r_v\, P\{J=0\}$. Thus the 3- dimensional

Markovian process {(N(t),C(t),J(t)): t≥0} for finite buffer is stable, if d<0. (i.e) $d=[r_v(1-(\frac{\lambda_1+\xi}{\mu}))+r_b\frac{\lambda_0}{\mu}]$ where d<0. Therefore, the Markovian process {(N(t), J(t), C(t)), t≥0} with net input rate is stable only if $\lambda_1+\xi<\mu$ and $\lambda_0<\mu$. When the process is stable we write $F_{kj}(w)=\lim_{t\to\infty}F_{kj}(t,w)$. Hence, we assume that the stability conditions are satisfied throughout the analysis.

III. SYSTEM OF DIFFERENTIAL EQUATIONS AND SOLUTION

By considering the above assumptions system of differential equations for the specified model are written as

$$r_V\frac{dF_{00}(w)}{dw}=-\lambda_0F_{00}(w)+\mu F_{11}(w) \tag{2}$$

$$r_V\frac{dF_{k0}(w)}{dw}=\lambda_0F_{k-1,0}(w)-(\lambda_0+\theta)F_{k0}(w),\qquad k\geq 1 \tag{3}$$

$$r_b\frac{dF_{11}(w)}{dw}=\theta F_{10}(w)-(\lambda_1+\mu+\xi)F_{11}(w)+\mu F_{21}(w) \tag{4}$$

$$r_b\frac{dF_{k1}(w)}{dw}=\theta F_{k0}(w)+(\lambda_1+\xi)F_{k-1,1}(w)-(\lambda_1+\mu+\xi)F_{k1}(w)+\mu F_{k+1,1}(w),\qquad k\geq 2 \tag{5}$$

$$F_{00}(0)=\frac{d}{r_V}.\frac{\theta}{\lambda_0+\theta}=a \tag{6}$$

$$F_{k0}(0)=\frac{d}{r_V}.\frac{\theta}{\lambda_0+\theta}.(\frac{\lambda_0}{\lambda_0+\theta})^k=b_k,\qquad k\geq 1 \tag{7}$$

$$F_{k1}(0)=0,\qquad k\geq 1 \tag{8}$$

$$a+\sum_{k=1}^{\infty}b_k=P\{C=0\}=\frac{d}{r_V}$$

Remark 1: Considering (6) and (7), with the Law of Conservation and by direct calculation we obtain the stationary probability of empty buffer content

$$P\{C=0\}=\frac{d}{r_V}=\frac{[r_V(1-(\frac{\lambda_1+\xi}{\mu}))+r_b\frac{\lambda_0}{\mu}]}{r_V} \tag{9}$$

Remark 2: To obtain the solution of system of steady state differential equations as

$$F_0^*(s,z)=\frac{\frac{\theta d}{\lambda_0+\theta}[\frac{\lambda_0 z}{\lambda_0(1-z)+\theta}]+\lambda_0 z\hat{F}_{00}(s)}{sr_V+\lambda_0(1-z)+\theta}$$

$$F_1^*(s,z)=\frac{\frac{\theta d}{\lambda_0+\theta}[sr_V+\lambda_0(1-z)+\theta+\frac{\lambda_0\theta z}{\lambda(1-z)+\theta}]-(sr_V+\lambda_0+\theta)(sr_V+\lambda_0(1-z))\hat{F}_{00}(s)}{[sr_b+\lambda_1(1-z)+\mu(1-1/z)+\xi(1-z)](sr_V+\lambda_0(1-z)+\theta)}$$

Proof: we consider Laplace transform of function $F_{kj}(t,w),((k,j)\in\Omega)$ as $\hat{F}_{kj}(s)=\int_0^{\infty} e^{-sw}F_{kj}(w)dw$. By taking Laplace transform on both sides of equation (2) to (5) and using (6) to (8), we obtain system of steady state differential equations as

$$(sr_V+\lambda_0)\hat{F}_{00}(s)=ar_V+\mu\hat{F}_{11}(s) \tag{10}$$

$$(sr_V+\lambda_0+\theta)\hat{F}_{k0}(s)=b_k r_V+\lambda_0\hat{F}_{k-1,0}(s) \tag{11}$$

$$(sr_b+\lambda_1+\mu+\xi)\hat{F}_{11}(s)=\theta\hat{F}_{10}(s)+\mu\hat{F}_{21}(s) \tag{12}$$

$$(sr_b+\lambda_1+\mu+\xi)\hat{F}_{k1}(s)=\theta\hat{F}_{k0}(s)+\mu\hat{F}_{k+1}(s)+(\lambda_1+\xi)\hat{F}_{k-1,1},\quad k\geq 2 \tag{13}$$

Let $F_0^*(s,z)=\sum_{k=1}^{+\infty}F_{k0}(s)z^k$ & $F_1^*(s,z)=\sum_{k=1}^{+\infty}F_{k1}(s)z^k$ (14)

From (12) & (13) we get,

$$(sr_b+\lambda_1(1-z)+\xi(1-z)+\mu(1-1/z))F_1^*(s,z)=\theta F_0^*(s,z)-\mu\hat{F}_{11}(s)$$

By using equation (10) in the above equation we get

$$[sr_b+\lambda_1(1-z)+\mu(1-1/z)+\xi(1-z)]F_1^*(s,z)+(sr_V+\lambda_0)\hat{F}_{00}(s)=ar_V+\theta F_0^*(s,z) \tag{15}$$

$$[sr_V+\lambda_0+\theta]F_0^*(s,z)=r_V\sum_{k=1}^{\infty}b_k z^k+\lambda_0 z\hat{F}_{00}(s)+\lambda_0 zF_0^*(s,z) \tag{16}$$

Considering the L.H.S of equation (15)

$$(sr_b+\lambda_1(1-z)+\xi(1-z)+\mu(1-\frac{1}{z})) \tag{17}$$

We write, the Laplace Stieltjes transform of the probability distribution function G(x) is $G^*(s)=\int_0^{+\infty}e^{-sx}dG(x)$ .A necessary condition for $z(s)=z_0(s)$ to be LST, we require $z(+\infty)=0$. Hence we take minimal positive solution to quadratic equation (17). The root of equation (17) plays an important role in the following analysis. For $s\geq 0$, equation (17) has two roots

$$z_0(s)=\frac{sr_b+\lambda_1+\xi+\mu+\sqrt{(sr_b+\lambda_1+\xi+\mu)^2-4(\lambda_1+\xi)\mu}}{2(\lambda_1+\xi)}$$ where $0<z_0(s)\leq 1$ and

$$z_1(s)=\frac{sr_b+\lambda_1+\xi+\mu-\sqrt{(sr_b+\lambda_1+\xi+\mu)^2-4(\lambda_1+\xi)\mu}}{2(\lambda_1+\xi)}$$ where $z_0(s)>1$.

Therefore, $z_0(0)=1$ & $z_1(0)=\frac{\mu}{(\lambda_1+\xi)}$

Taking derivatives (twice) on both sides of equation (17) with respect to s and letting s to be zero, we have

$$z_0'(0) = \frac{-r_b}{(\mu - (\lambda_1 + \xi))} \qquad \& \qquad z_0''(0) = \frac{2r_b^2}{\mu^2 \left(1 - \frac{(\lambda_1 + \xi)}{\mu}\right)^3} \tag{18}$$

Using boundary conditions (6) and (7) in equation (15) & (16), we get

$$F_0^*(s,z) = \frac{\frac{\theta d}{\lambda_0 + \theta}\left[\frac{\lambda_0 z}{\lambda_0(1-z)+\theta}\right] + \lambda_0 z \hat{F}_{00}(s)}{sr_v + \lambda_0(1-z) + \theta} \tag{19}$$

$$F_1^*(s,z) = \frac{\frac{\theta d}{\lambda_0 + \theta}\left[sr_v + \lambda_0(1-z) + \theta + \frac{\lambda_0 \theta z}{\lambda(1-z)+\theta}\right] - (sr_v + \lambda_0 + \theta)(sr_v + \lambda_0(1-z))\hat{F}_{00}(s)}{[sr_b + \lambda_1(1-z) + \mu(1-1/z) + \xi(1-z)](sr_v + \lambda_0(1-z) + \theta)} \tag{20}$$

Using definition (14), it is clear that $F_1^*(s,z)$ is analytic for variable z. Hence, $z_0(s)$ is the solution to equation (17) and also solution to the equation (20).Now our aim is to find $\hat{F}_{00}(s)$. Equation (20) implies that

$$\frac{\theta d}{\lambda_0 + \theta}\left[sr_v + \lambda_0(1-z) + \theta + \frac{\lambda_0 \theta z}{\lambda_0(1-z)+\theta}\right] - (sr_v + \lambda_0 + \theta)(sr_v + \lambda_0(1-z))\hat{F}_{00}(s) = 0$$

And we should always take minimal positive solution to the crucial quadratic equation(17). Then

$$\frac{\theta d}{\lambda_0 + \theta}\left[sr_v + \lambda_0(1-z_0(s)) + \theta + \frac{\lambda_0 \theta z_0(s)}{\lambda_0(1-z_0(s))+\theta}\right] = (sr_v + \lambda_0 + \theta)(sr_v + \lambda_0(1-z_0(s)))\hat{F}_{00}(s)$$

which is equivalent to

$$\hat{F}_{00}(s) = \frac{\frac{\theta d}{\lambda_0 + \theta}\left[sr_v + \lambda_0(1-z_0(s)) + \theta + \frac{\lambda_0 \theta z_0(s)}{\lambda_0(1-z_0(s))+\theta}\right]}{(sr_v + \lambda_0 + \theta)(sr_v + \lambda_0(1-z_0(s)))} \tag{21}$$

IV. THE STEADY-STATE DISTRIBUTION OF THE BUFFER CONTENT

Theorem 1. The Laplace transform of the distribution of the stationary buffer content for stable fluid queue is given by

$$\hat{H}(s) = \frac{d}{sr_b} \frac{[sr_v + \lambda_0(1-z_0(s))][\lambda_0(1-z_0(s)) + \theta](sr_b + \theta) + s(r_b - r_v)\theta^2}{(sr_v + \theta)[sr_v + \lambda_0(1-z_0(s))][\lambda_0(1-z_0(s)) + \theta]}$$

Proof: Let $H(w) = \lim_{t\to\infty} P\{C(t) \le w\}$ and denote $\hat{H}(s) = \int_0^\infty e^{-sw} H(w)dw$.

By considering the total probability $\hat{H}(s) = \hat{F}_{00}(s) + F_0^*(s,1) + F_1^*(s,1)$. Substituting z =1 in equations (19) & (20) and thereby direct calculations we obtain the above result.

Remark 3: When the vacation time V tends to zero, considering Theorem 1 and applying the Law of L'Hospital, we obtain

$$\lim_{\theta \to +\infty} \hat{H}(s) = \frac{d}{sr_b} \cdot \frac{[sr_b + \lambda_0(1 - z_0(s))]}{[sr_v + \lambda_0(1 - z_0(s))]}$$

Theorem 2. The mean and second moment of the buffer content for the stable fluid queue are given by

$$E(C) = \frac{(\lambda_1 + \xi) r_b (r_v - r_b)}{d\mu^2 (1 - (\frac{\lambda_1 + \xi}{\mu}))}$$

$$E(C^2) = \frac{2(\frac{\lambda_1 + \xi}{\mu}) r_b (r_b - r_v)[\frac{\lambda_0}{\mu} d^2\mu^2 (1 - (\frac{\lambda_1 + \xi}{\mu}))^2 + \theta^2 r_b (\frac{\lambda_0}{\mu} r_b - d\frac{\lambda_0}{\mu} - d)]}{\theta^2 d^2 \mu^2 (1 - (\frac{\lambda_1 + \xi}{\mu}))^3}$$

Proof. Now we consider the Laplace Stieltjes transform of the stable probability distribution H(w) of buffer content as $H^*(s) = \int_0^{+\infty} e^{-sw} dH(w)$ .Therefore, $H^*(s) = s\hat{H}(s)$ (22)

By using equation(22) in Theorem 1, we get

$$H^*(s) = \frac{d}{r_b} \frac{[sr_v + \lambda_0(1 - z_0(s))][(\lambda_0(1 - z_0(s)) + \theta](sr_b + \theta) + s(r_b - r_v)\theta^2}{(sr_v + \theta)[sr_v + \lambda_0(1 - z_0(s))][\lambda_0(1 - z_0(s)) + \theta]}$$

By applying L' Hospital rule and substituting s=0 in $z_0'(s)$, we get

$$\lim_{s \to 0^+} H^{*'}(s) = E(C) = \frac{(\lambda_1 + \xi) r_b (r_v - r_b)}{d\mu^2 (1 - (\frac{\lambda_1 + \xi}{\mu}))} \quad (23)$$

Again applying L'Hospital rule and using (18), we obtain

$$\lim_{s \to 0^+} H^{*''}(s) = E(C^2) = \frac{2(\frac{\lambda_1 + \xi}{\mu}) r_b (r_b - r_v)[\frac{\lambda_0}{\mu} d^2\mu^2 (1 - (\frac{\lambda_1 + \xi}{\mu}))^2 + \theta^2 r_b (\frac{\lambda_0}{\mu} r_b - d\frac{\lambda_0}{\mu} - d)]}{\theta^2 d^2 \mu^2 (1 - (\frac{\lambda_1 + \xi}{\mu}))^3} \quad (24)$$

Thus $E(C^2)$ is obtained.

Remark 4: Letting $r_b - r_v = \mu$, the expression (23) can be reduced as

$$E(C) = \frac{(\lambda_1 + \xi) r_b}{d((\lambda_1 + \xi) - \mu)}$$

Thus from the above equation we observe that mean buffer content E(C) is independent of vacation parameter θ and also independent of arrival rate during vacation period λ_0.

Remark 5: According to Theorem 2, the mean buffer content E(C) is independent of the vacation parameter θ, whereas the second moment of the buffer content $E(C^2)$ is dependent on vacation parameter θ. Based on the assumption, the current rate R(t) w.r.t. the net input rate(i.e., the input rate-the output rate) of fluid into the buffer at time t becomes

$$\underset{t\to\infty}{lt} P\{R(t)=r_b\} = P\{J=1\} = \frac{\lambda_1+\xi}{\mu},$$

$$\underset{t\to\infty}{lt} P\{R(t)=0 = P\{J=0, C=0\} = P\{C=0\} = 1-\frac{\lambda_1+\xi}{\mu}+\frac{r_b}{r_v}\frac{\lambda_0}{\mu},$$

$$\underset{t\to\infty}{lt} P\{R(t)=r_v\} = P\{J=0, C>0\} = -\frac{r_b}{r_v}\frac{\lambda_0}{\mu}.$$

Remark 6: If the fluid is stable, then the variance of the stationary buffer content $Var(C) = \sigma^2(C)$ is given by

$$\sigma^2 = \left[\frac{2(\frac{\lambda_1+\xi}{\mu})r_b(r_b-r_v)[\frac{\lambda_0}{\mu}d^2\mu^2(1-(\frac{\lambda_1+\xi}{\mu}))^2+\theta^2 r_b(\frac{\lambda_0}{\mu}r_b-d\frac{\lambda_0}{\mu}-d)]}{\theta^2 d^2\mu^2(1-(\frac{\lambda_1+\xi}{\mu}))^3} - \left[\frac{(\lambda_1+\xi)r_b(r_v-r_b)}{d\mu^2(1-(\frac{\lambda_1+\xi}{\mu}))}\right]^2\right]$$

Proof: By substituting equations (23) & (24) in $\sigma^2(C)=\left[E(C^2)-(E(C))^2\right]$ we obtain, variance of the stationary buffer content.

V. NUMERICAL ANALYSIS

In this section, to demonstrate the pertinence of above result we illustrate graph to analyze the parameters effect on mean of the buffer content and second moment of the buffer content and variance of the buffer content of the steady state system. Let λ_0 and λ_1 are the heterogeneous arrival rate during vacation and busy period. Also by letting r_v and r_b to denote net input rate of the fluid to the buffer during vacation and busy state. The heterogeneous arrivals are serviced at the service rate μ. We consider the vacation parameter to be θ and also when the system is busy catastrophe occurs with mean rate ξ . Thus E(c), $E(C^2)$ and variance become

$$E(C) = \frac{(\lambda_1+\xi)r_b(r_v-r_b)}{d\mu^2(1-(\frac{\lambda_1+\xi}{\mu}))}$$

$$E(C^2) = \frac{2(\frac{\lambda_1+\xi}{\mu})r_b(r_b-r_v)[\frac{\lambda_0}{\mu}d^2\mu^2(1-(\frac{\lambda_1+\xi}{\mu}))^2+\theta^2 r_b(\frac{\lambda_0}{\mu}r_b-d\frac{\lambda_0}{\mu}-d)]}{\theta^2 d^2\mu^2(1-(\frac{\lambda_1+\xi}{\mu}))^3}$$

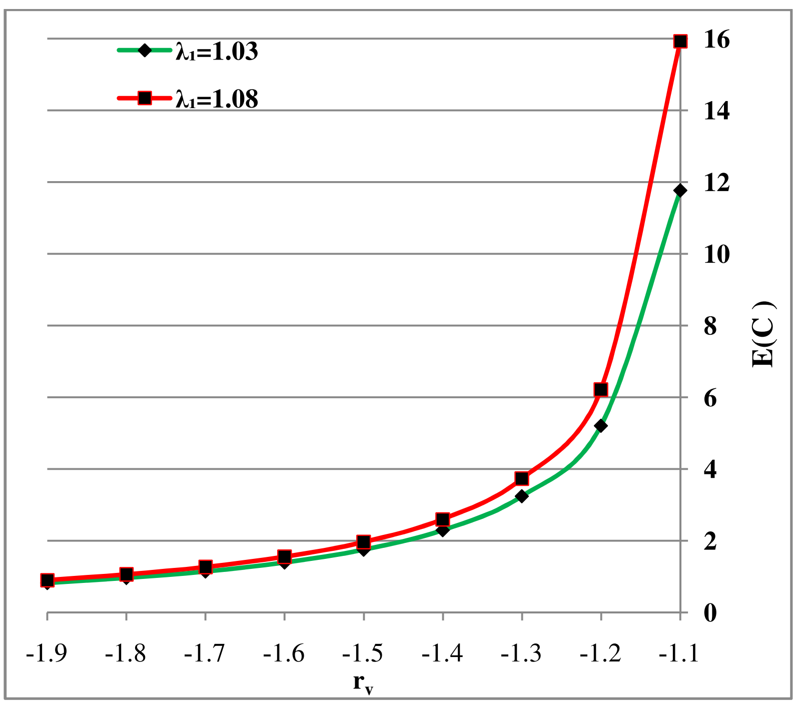
$$\text{Variance} = \left[E(C^2)-(E(C))^2\right]$$

By considering the above formula, the effect of mean buffer content E(C), second moment of the buffer content $E(C^2)$ and variance of the buffer content are evaluated. The evaluated values are tabulated in Table 1 and represented in Graph 1. .

Table 1. Effect of Mean Buffer Content E(C) with respect to λ_1 and r_v

r_v \ λ_1	1.03	1.08
-1.9	0.8259649	0.906146384
-1.8	0.9648814	1.062135708
-1.7	1.1471979	1.268125341
-1.6	1.3955153	1.550939101
-1.5	1.7510944	1.960356101
-1.4	2.2980139	2.600209083
-1.3	3.2379191	3.728755578
-1.2	5.2052052	6.214588604
-1.1	11.767282	15.93454528

Now by considering the parameters λ_0 =1, r_b=2, ξ =0.01 and by varying net input rate r_v from -1.9 to -1.1 and arrival rate during busy period λ_1 which takes the value as 1.03, 1.08, the mean of the buffer content E(C) is evaluated. We calculated the E(C) values and tabulated in table 1.



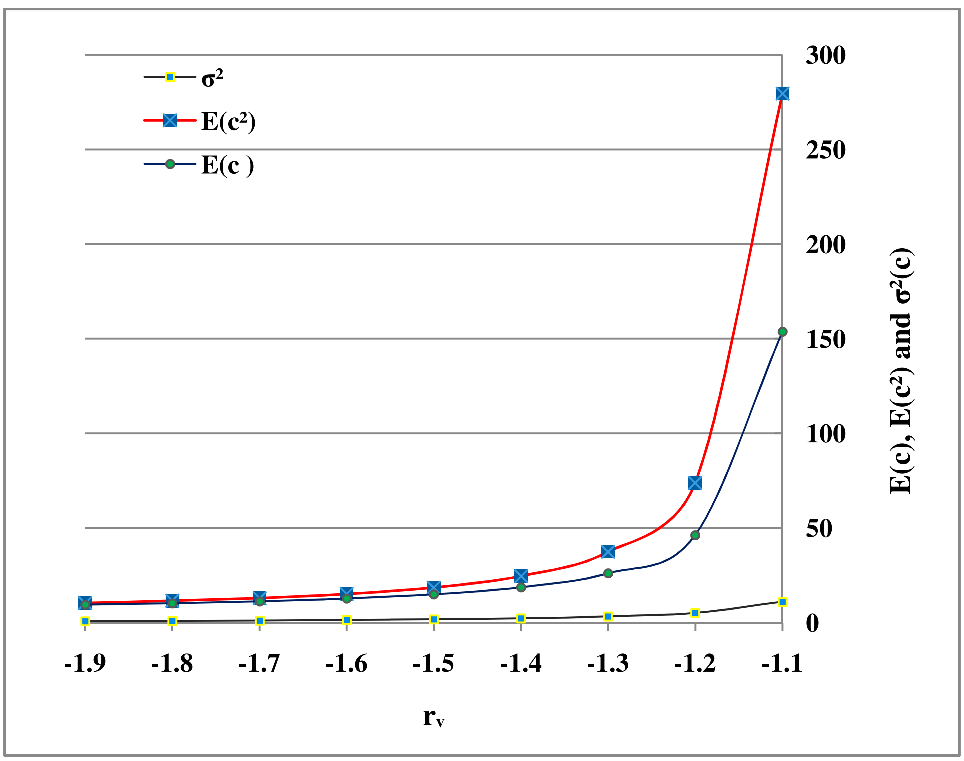
Graph 1. E(C) w.r.t λ_1 and r_v

From Graph 1 we observe that, E(C) increases gradually as r_v increases. We also notice that, in the initial stage when r_v increases there is minute difference in E(C) but when r_v increases steadily we observe that E(C) increases rapidly.

Now we compare the effect of mean, second moment and variance of the buffer content with different values of net input rate r_v ranging from -1.9 to -1.1, service rate μ ranges from 5.9 to 5.1and letting λ_0 be 1, busy period arrival rate λ_1 =1.03 , vacation parameter θ as 0.5 , catastrophe ξ as 0.01. By substituting above values in equation (23), (24) and (25) we evaluated the values and tabulated in table 2.

Table 2. Comparison of E(c), E(c^2) and σ²(c) w.r.t r_v

r_v	E(c)	E(c²)	σ²
-1.9	0.964885	10.576284	9.645281339
-1.8	1.109656	11.645713	10.41437732
-1.7	1.297402	13.122534	11.43928149
-1.6	1.549991	15.286806	12.88433253
-1.5	1.90708	18.720085	15.08313118
-1.4	2.448809	24.808449	18.81178517
-1.3	3.365141	37.606095	26.28192137
-1.2	5.241935	73.833742	46.35585428
-1.1	11.21377	279.64606	153.8974711



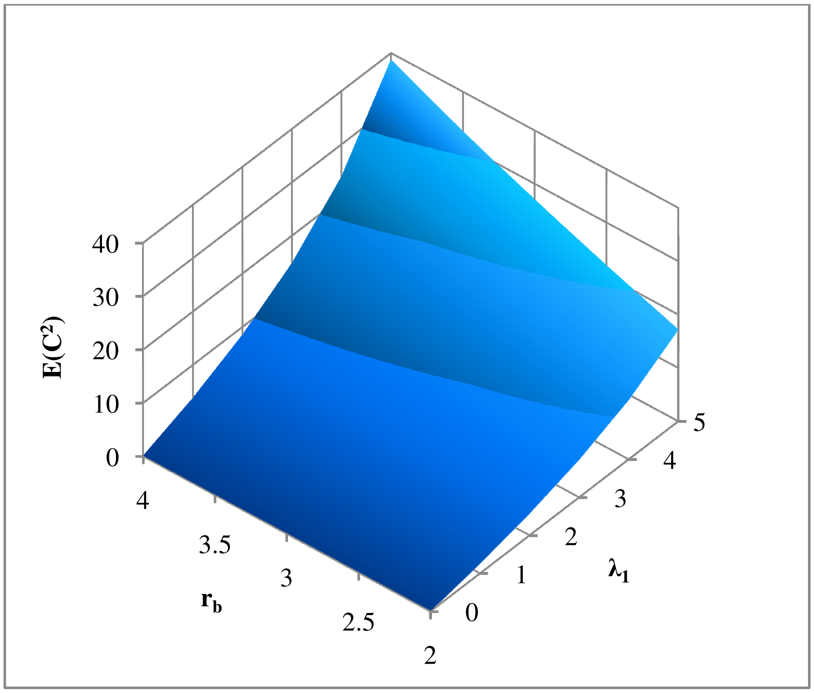
Graph 2. E(c),E(c^2) and σ²(c) w.r.t r_v

On comparing the three cases(E(C),E(C^2) and Var(C)), we notice that as r_v increases mean of the buffer content increases gradually where second moment increases very rapidly. Also we observe that there is wide difference in values between mean and second moment of the buffer content. In the beginning there is no significant difference in second moment and variance. As r_v increases widely we notice significant difference in E(C^2) and σ^2.

We now show the mutative trend of second moment of the buffer content E(C^2). By fixing arrival rate during vacation period λ_0 =1, service rate μ=10, catastrophe ξ =0.03, vacation parameter θ=0.5 and varying values of λ_1 from 0 to 5 and ranging r_b from 4 to 2, r_v from -6 to -8 and substituting in equation (24) we calculated and tabulated the following values in table 3.

Table3. Effect of Second moment of the Buffer Content E(C^2) w.r.t. r_b and λ_1

r_b \ λ_1	0	1	2	3	4	5
4	0.098322	3.782662	8.499394	14.82397	23.93597	38.75879
3.9	0.095772	3.683138	8.270645	14.40985	23.22287	37.46321
3.8	0.093231	3.58408	8.043449	13.99996	22.52124	36.20154
3.7	0.090697	3.485457	7.817702	13.59402	21.83025	34.97114
3.6	0.088171	3.387244	7.593311	13.19176	21.14916	33.76965
3.5	0.085653	3.289414	7.370191	12.79294	20.47726	32.59492
3.4	0.08314	3.191944	7.148262	12.39734	19.81394	31.44501
3.3	0.080634	3.094814	6.927451	12.00475	19.15864	30.31817
3.2	0.078134	2.998003	6.707693	11.61499	18.51081	29.21277
3.1	0.07564	2.901494	6.488924	11.22788	17.87	28.12737
3	0.073151	2.805269	6.271089	10.84327	17.23576	27.06063
2.9	0.070666	2.709313	6.054133	10.461	16.60768	26.01132
2.8	0.068187	2.613611	5.838009	10.08096	15.98539	24.97834
2.7	0.065712	2.51815	5.622671	9.702996	15.36855	23.96064
2.6	0.063241	2.422917	5.408076	9.327008	14.75685	22.95728
2.5	0.060775	2.327901	5.194186	8.952884	14.14998	21.9674
2.4	0.058312	2.23309	4.980963	8.580525	13.54769	20.99019
2.3	0.055852	2.138475	4.768373	8.209836	12.9497	20.02491
2.2	0.053397	2.044045	4.556386	7.840731	12.3558	19.07087
2.1	0.050944	1.949792	4.34497	7.473129	11.76576	18.12744
2	0.048495	1.855707	4.134099	7.106954	11.17937	17.19402



Graph 3. $E(C^2)$ w.r.t. λ_1 and r_b

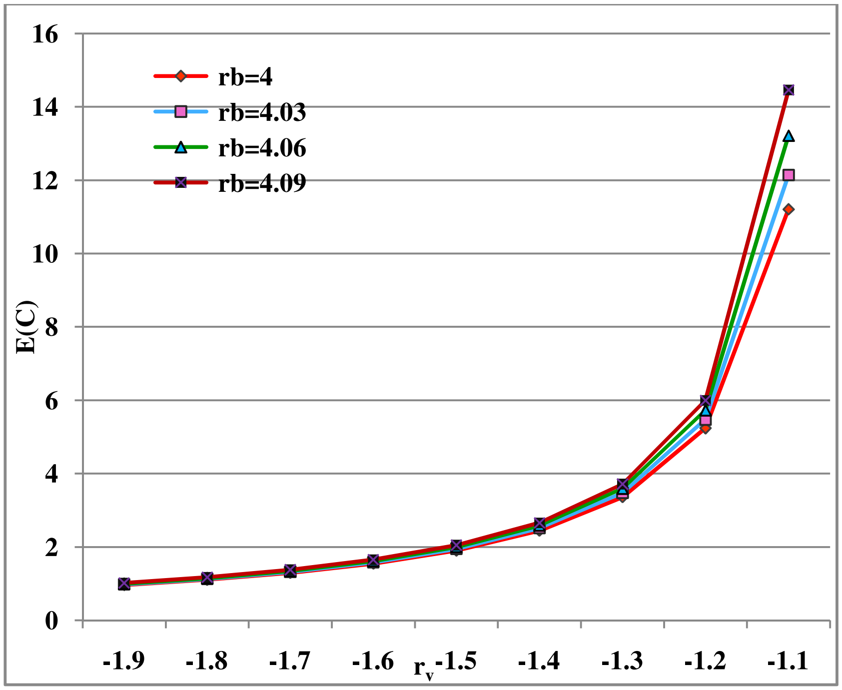
A 3-dimensional graph helps us to analyze patterns, comparisons and trends in data. The X-axis represents arrival rate λ_1 during busy period , Y-axis represents net input rate during busy period r_b and Z-axis represents second moment of the buffer content $E(C^2)$. From (Table 3 and Graph 3) we observe that $E(C^2)$ decreases as r_b decreases for fixed value λ_1, also we notice that $E(C^2)$ increases as λ_1 increases for fixed values of r_b. Hence from the illustration of Graph 3, we observe that the increasing and decreasing in the values of second moment of the buffer content $E(C^2)$ are due to the variation of λ_1 and r_b . This agrees with practical cases also.

Now we evaluate the mean of the buffer content E(C) by varying service rate μ from 5.9 to 5.1, r_v from -1.9 to -1.1 and r_b taking the values as 4, 4.03, 4.06 and 4.09. Also the parameters taken into consideration are arrival rate during vacation λ_0 as 1, arrival rate during busy period λ_1 to be 1.03, catastrophe ξ =0.01. The values are substituted in equation (23) and evaluated values are shown in table 4.

Table 4. Effect of Mean of the Buffer Content E(C) w.r.t r_v and r_b

r_v \ r_b	4	4.03	4.06	4.09
-1.9	0.964885	0.982697	1.00079	1.019169
-1.8	1.109656	1.13119	1.153098	1.175387
-1.7	1.297402	1.324141	1.351399	1.379189
-1.6	1.549991	1.584403	1.61958	1.655543
-1.5	1.90708	1.953651	2.001445	2.050502
-1.4	2.448809	2.516767	2.58693	2.659396
-1.3	3.365141	3.4774	3.594514	3.716783
-1.2	5.241935	5.477365	5.728433	5.996702
-1.1	11.21377	12.14628	13.21773	14.46139

For r_b=4 and r_v=-1.9 the mean of the buffer content is 0.964885 which increases to 11.21377 as r_v increases to -1.1. For r_b=4.03 and r_v=-1.9, the value of E(C) is 0.9827697 which increases to 12.14628 when r_v takes the value -1.1. When r_b=4.06 and r_v=-1.9, the value of E(C) is 1.00079 which increases to 13.21773 as r_v increases to -1.1. Similarly, when r_b=4.09 and r_v=-1.9, the value of E(C) is 1.019169 which keeps on increasing and the change is observed till r_v=-1.1 where r_v becomes 14.46139. On comparing the change in r_b we observe that E(C) increases for fixed r_v.



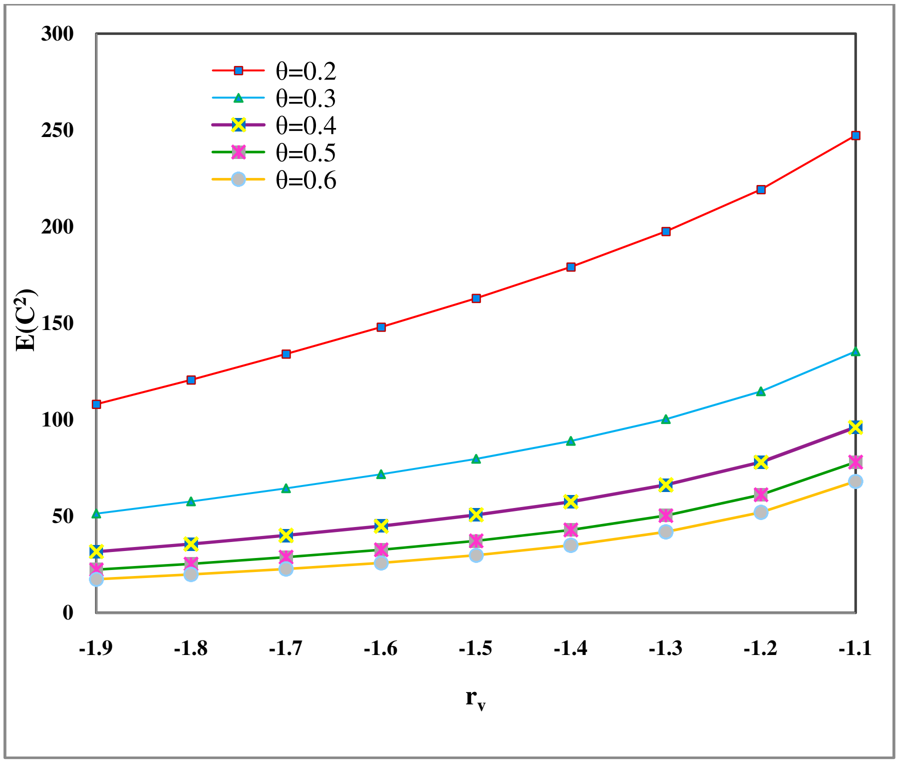
Graph 4. E(C) w.r.t r_v and r_b

Graph 4 represents the variation in E(C) with respect to r_v and r_b. For fixed r_v , we observe E(C) increases as r_b increases. We also notice that as r_b increases there is only a minute difference in E(C) till r_v takes the value -1.2 . But E(C) increases gradually as r_b increases when r_v takes the value as -1.1. Now we notice the changes in second moment of the buffer content $E(C^2)$ with different values of net input rate during vacation period r_v and vacation parameter θ. By letting arrival rate during busy period be λ_1= 2, service rate μ=10, catastrophe ξ =0.01 and varying r_v from -1.9 to -1.1, λ_0 from 1 to 1.8 and θ from 0.2 to 0.6 .By substituting the above values in equation (24) ,we calculate the following values and then the calculated values are tabulated in table 5.

Table.5 Effect of Second moment of the Buffer Content $E(C^2)$ w.r.t r_v and θ

r_v \ θ	0.2	0.3	0.4	0.5	0.6
-1.9	107.9772	51.3752	31.5645	22.39498	17.414
-1.8	120.6762	57.64532	35.58451	25.37351	19.82679
-1.7	133.9541	64.35457	39.99475	28.71963	22.59487
-1.6	147.9522	71.64436	44.9366	32.57472	25.85963
-1.5	162.8981	79.74209	50.63748	37.1662	29.84847
-1.4	179.1768	89.03288	57.48251	42.87919	34.94652
-1.3	197.4837	100.2121	66.16709	50.40909	41.84919
-1.2	219.1824	114.6434	78.05471	61.11939	51.91996
-1.1	247.2568	135.3106	96.12945	77.99417	68.1429

Table 5. shows the changes in second moment of the buffer content $E(C^2)$. For r_v = -1.9 and θ= 0.2, we observe the value of $E(C^2)$ is 107.9772 which increases rapidly to 247.2568 when r_v reaches the value of -1.1. For constant value of r_v we observe $E(C^2)$ decreases as vacation parameter θ increases. The tabulated values are depicted and represented graphically.



Graph. 5 $E(C^2)$ w.r.t r_v and θ

The X-axis represents net input rate during vacation period r_v, Y-axis represents second moment of the buffer content $E(C^2)$. For fixed values of vacation parameter θ, we notice second moment of the buffer content $E(C^2)$ increases as r_v increases. As illustrated it appears there is considerable difference in $E(C^2)$ when θ varies from 0.3 to 0.6 but there is extensive difference in $E(C^2)$ when θ= 0.2. This agrees with practical cases also.

Conclusion: In this paper, we analysed on heterogeneous arrival of tandem fluid model driven by an m/m/1 vacation queue subject to catastrophe. We set up a system of differential equations and obtained Laplace transform of steady state equations. By considering, Laplace Stieltjes transform of the probability distribution function, the steady state buffer content is expressed through the minimal positive solution to a momentous quadratic equation. Further, we generate the reliable data on the effectiveness of the mean, second moment and variance of the buffer content. From the result we observe that mean of buffer content is independent of the vacation parameter θ and also independent of vacation period arrival rate λ_0, while second moment and variance of the buffer content are dependent on vacation parameter θ. A numerical example is of indicative nature and hence all these analysis are justified by its corresponding graphical representation. The study can be further extended to analyze the model with balking also.

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