Lie ideal and Generalized (σ, τ)-Derivations in Prime Rings

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Abstract: Let R be a prime ring, I be a non-zero lie ideal of R.Suppose that $F: R \to R$ be a generalized (σ, τ)-derivation on R associated with (σ, τ)-derivation $g: R \to R$ respectively and $\tau(I) \neq 0$.In this paper, we studied the following identities in prime rings:

(i) $F(uv) \pm u\sigma(v) = 0$;then $g(U) = (0)$ and $F(u) = \mp u$ for all $u \epsilon U$

(ii) $F(uv)) \pm \sigma(vu) = 0$;then $U \subseteq Z(R)$. $g(U) = (0)$ and $F(u) = \mp u$ for all $u \epsilon U$

(iii) $F(u)F(v) \pm \sigma(uv) = 0$ $g(U) = (0)$ and $U \subseteq Z(R)$ for all $u \epsilon U$

(iv) $F(u)F(v) \pm \sigma(vu) = 0$ $g(U) = (0)$and $U \subseteq Z(R)$ for all $u \epsilon U$.

Keywords: Prime ring, Derivation, Generalized derivation, (σ, τ)-derivation, Generalized (σ, τ)-derivation.

1. INTRODUCTION

Bresar in [2], first time introduced the notion of generalized derivation. In 1992, Daif et al. in [4], proved a result which is given as let R be a semiprime ring, I be a non zero ideal of R and d be a derivation on R such that $d([x, y]) = [x, y]$, for all $x, y \in I$, then $I \subseteq Z(R)$. In 2002, Ashraf and Rehman [1] extended the result of Daif et al. [4] by replacing ideal to lie ideal. In 2003, Quadri et al. in [7] extended the result of Ashraf et al. [1] on generalized derivation given as let R be a prime ring with characteristic different from two, I be a nonzero ideal of R and F be a generalized derivation on R associated with a derivation d on R such that $F([x, y]) = [x, y]$, for all $x, y \in I$, then R is commutative. Golbasi et al. in [5] extended the result of Quadri et al. [7] by replacing ideal to lie ideal. Recently, S.K. Tiwari et al. in [8] studied Multiplicative (generalized)-derivation in semiprime rings. Further ChiragGarg et al. in [3] studied on generalized (α, β)-derivations in prime rings. In this paper we extended of S.K. Tiwari et al. in [8], we have proved some results on generalized (σ, τ)-derivations in prime rings.

2. PRELIMINARIES

Throughout this paper R denote an associative ring with center Z. Recall that a ring R is prime if $xRy = \{0\}$ implies $x = 0$or $y = 0$. For any $x, y \in R$, the symbol $[x, y]$ stands for the commutator$xy - yx$ and the symbol (xoy) denotes the anticommutator $xy + yx$. Let σ, τ be any two automorphisms of R. For any $x, y \in R$, we set $[x, y]_{\sigma,\tau} = x\sigma(y) - \tau(y)x$ and $(xoy)_{\sigma,\tau} = x\sigma(y) + \tau(y)x$. An additive mapping $d: R \to R$ is called a derivation if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. An additive mapping $d: R \to R$ is called a (σ, τ)-derivation if $d(xy) = d(x)\sigma(y) + \tau(x)d(y)$ holds for all $x, y \in R$. An additive mapping $F: R \to R$ is called a generalized derivation, if there exists a derivation $d: R \to R$ such that $F(xy) = F(x)y + xd(y)$ holds for all $x, y \in R$. An additive mapping $F: R \to R$ is said to be a generalized (σ, τ)-derivation of R, if there exists a (σ, τ)-derivation $d: R \to R$ such that $F(xy) = F(x)\sigma(y) + \tau(x)d(y)$ holds for all $x, y \in R$.

Throughout this paper, we shall make use of the basic commutator identities:

$[x, yz] = y[x, z] + [x, y]z,$

$[xy, z] = [x, z]y + x[y, z],$

$$(xo(yz)) = (xoy)z - y[x, z] = y(xoz) + [x, y]z,$$

$[xy, z]_{\sigma,\tau} = x[y, z]_{\sigma,\tau} + [x, \tau(z)]y = x[y, \sigma(z)] + [x, z]_{\sigma,\tau}y,$

$[x, yz]_{\sigma,\tau} = \tau(y)[x, z]_{\sigma,\tau} + [x, y]_{\sigma,\tau}\sigma(z), (xo(yz))_{\sigma,\tau} = (xoy)_{\sigma,\tau}\sigma(z) - \tau(y)[x, z]_{\sigma,\tau} = \tau(y)(xoz)_{\sigma,\tau} + [x, y]_{\sigma,\tau}\sigma(z),$

$$(xy)oz)_{\sigma,\tau} = x(yoz)_{\sigma,\tau} - [x, \tau(z)]y = (xoz)_{\sigma,\tau}y + x[y, \sigma(z)].$$

3. MAIN TEXT

Lemma 1: [Bergen et al. (1981), Lemma 3] Let R be a 2-torsion free prime ring and U be a Lie ideal of R. If $U \not\subseteq Z(R)$, then $C_R(U) = Z(R)$.

Lemma 2: [Bergen et al. (1981), Lemma 4] If $U \not\subseteq Z(R)$ is a Lie ideal of a 2-torsion free prime ring $R$and $a, b \in R$ are such that $a \cup b = (0)$, then $a = 0$ or $b = 0$.

Lemma 3: [Rehman (2002), Lemma 2.6] Let R be a 2-torsion free prime ring and U be a Lie ideal of R. If U is a commutative Lie ideal of R, then $U \not\subseteq Z(R)$.

Theorem 1: Let R be a 2-torsion free prime ring and U be a non zero square closed Lie ideal of R. If R admits a generalized (σ, τ)-derivatiom $F: R \to R$ associated with the (σ, τ)-derivation the map $g: R \to R$ such that $F(uv) \pm u\sigma(v) = 0$, for all $u, v\epsilon U$, then $g(U) = (o)$ and $F(U) = \mp U$ for all$u\epsilon U$.

Proof: Assume first that $F(uv) - u\sigma(v) = 0$, for all $u, v\epsilon U$ (1)

We replace v by$2vw$ in (1), we get

$$F(2uvw) - u\sigma(2vw) = 0$$

$$F(2uv)\sigma(w) + \tau(2uv)g(w) - u\sigma(2v)\sigma(w) = 0$$

$$2\big(F(uv) - u\sigma(v)\sigma(w) + \tau(uv)g(w)\big) = 0$$

Since R is 2-torsion free prime ring, we get

$$\big(F(uv) - u\sigma(v)\sigma(w) + \tau(uv)g(w)\big) = 0$$

Using equation (1), in the above equation we get

$\tau(uv)g(w) = 0$, for all $u, v, w\epsilon U$ (2)

We replace u by $[u, r], r \in R$ in equation(2), we get

$$\tau([u, r]v)g(w) = 0$$

$$\tau(\mathrm{urv} - \mathrm{ruv})\mathrm{g(w)} = 0$$

Using equation (2), the above equation become

$$\tau(urv)g(w) = 0$$

$$\tau(u)\tau(r)\tau(v)g(w) = 0$$

$$\tau(u)R\tau(v)g(w) = 0$$

Since R is prime ring and U is a non zero lie ideal of R, we get

$\tau(v)g(w) = 0,$ for all $u, v \epsilon U$ (3)

We replace v by $[v, r], r \in R$ in equation(3), we get

$$\tau([v, r])g(w) = 0$$

$$\tau(\mathrm{vr} - \mathrm{rv})\mathrm{g}(\mathrm{w}) = 0$$

$$\tau(\mathrm{vr})\mathrm{g}(\mathrm{w}) - \tau(\mathrm{rv})\mathrm{g}(\mathrm{w}) = 0$$

$$\tau(\mathrm{v})\tau(\mathrm{r})\mathrm{g}(\mathrm{w}) - \tau(\mathrm{r})\tau(\mathrm{v})\mathrm{g}(\mathrm{w}) = 0$$

Using equation (3), the above equation becomes

$$\tau(\mathrm{v})\tau(\mathrm{r})\mathrm{g}(\mathrm{w}) = 0$$

$$\tau(\mathrm{v})\mathrm{R}\mathrm{g}(\mathrm{w}) = 0$$

Since R is prime ring and U is non zero Lie ideal of R,we have

$g(U) = 0$ (4)

$$F(uv) = F(u)\sigma(v) + \tau(u)g(v)$$

From equation (4), we get

$F(uv) = F(u)\sigma(v)$,for all $u, v \in U$ (5)

Substitute equation (5) in equation (1), we get

$$F(u)\sigma(v) - u\sigma(v) = 0$$

$(F(u) - u)\sigma(v) = 0$, for all $u, v \in U$ (6)

We replace v by $[v, r], r \in R$ in equation (6), we get

$$(F(u) - u)\sigma([v, r]) = 0$$

$$(F(u) - u)\big(\sigma(vr) - \sigma(rv)\big) = 0$$

$$(F(u) - u)\sigma(v)\sigma(r) - (F(u) - u)\sigma(r)\sigma(v) = 0$$

Using equation (6) in the above equation, we get

$$(F(u) - u)\sigma(r)\sigma(v) = 0$$

$(F(u) - u)R\sigma(v) = 0$,for all $u, v \in U$

Using prime ness of Rwe conclude that

$F(u) = u,$ for all $u \in U$

In similar manner, we can prove the result for the cause $F(uv) + u\sigma(v) = 0$, for all $u, v \in U$.

There by the proof of the theorem is completed.

Theorem2: Let$R$ be a 2-torsion free prime ring and U be a non zero square closed Lie ideal of R. If R admits a generalized (σ, τ)- derivation$F: R \to R$ associated with the (σ, τ)-derivation the map $g: R \to R$ such that $F(uv) \pm \sigma(vu) = 0$, for all $u, v \epsilon U$, then $U \subseteq Z(R), g(U) = (o)$ and $F(U) = \mp\sigma(U)$, for all$u\epsilon U$.

Proof: suppose $U \nsubseteq Z(R)$

By the assumption, we have

$F(uv) - \sigma(uv) = 0$,for all $u, v \in U$ (7)

We replace v by $2vw$ in (7), we get

$$F(2uvw) - \sigma(2uvw) = 0$$

$$F(2uv)\sigma(w) + \tau(2uv)g(w) - \sigma(2uvw) = 0$$

Using 2-torsion free in (7) and using (6), we get.

$$\sigma(vu)\sigma(w) + \tau(uv)g(w) - \sigma(uvw) = 0$$

$$\sigma(vuw - vwu) + \tau(uv)g(w) = 0$$

$\sigma(v)\sigma[u, w] + \tau(uv)g(w) = 0$, for all $u, v, w \in U$ (8)

We replace$w$ by u in (8), we get

$\tau(uv)g(u) = 0$,for all $u, v \in U$ (9)

We replace v by rv in equation (9), we get

$$\tau(urv)g(u) = 0$$

$$\tau(u)\tau(r)\tau(v)g(u) = 0$$

$$\tau(u)R\tau\,(v)g(u) = 0$$

$R$is prime ring and U is non zero lie ideal of R

$\tau\,(v)g(u) = 0$, for all $u, v \in U$ (10)

Again we replace $v$by vr, $r \in R$ in equation (10), we get

$$\tau\,(vr)g(u) = 0$$

$$\tau\,(v)\tau\,(r)g(u) = 0$$

$$\tau\,(v)Rg(u) = 0$$

Since R is prime and U is non zero lie ideal of R

$g(U) = 0$, for all $u\epsilon U$ (11)

We replace w by v and using equation (11) in equation (8), we get

$\sigma(v)\sigma[u,v] = 0$, for all $u, v \in U$ (12)

We replace u by $2wu$ in equation (12), we get

$$\sigma(v)\sigma[2wu,v] = 0$$

$$\sigma(v[2wu,v]) = 0$$

Using 2-torsaion free ness in above equation

$$\sigma(v[wu,v]) = 0$$

$$\sigma(vw[u,v] + v[w,v]u) = 0$$

$$\sigma(vw)\sigma([u,v] + \sigma(v)\sigma([w,v])\sigma(u) = 0$$

Using (12), the above equation becomes

$$\sigma(v)\sigma(w)\sigma([u,v] = 0$$

$$\sigma(v)R\sigma[u,v] = 0$$

Since R is prime ring and U is non zero lie ideal of R, we get

$\sigma[u,v] = 0$, since σ is an automorphism

$$[u,v] = 0$$

Using lemma 3,we get $U \subseteq Z(R)$, a contradiction

Therefore, we must have $U \subseteq Z(R)$

$$F(uv) = F(u)\sigma(v) + \tau(u)g(v) = F(u)\sigma(v)$$

Given that $F(uv) - \sigma(vu) = 0$

$$F(u)\sigma(v) - \sigma(uv) = 0$$

$$\big(F(u) - \sigma(u)\big)\sigma(v) = 0$$

We replace v by rv in the above equation, we get

$$\big(F(u) - \sigma(u)\big)\sigma(rv) = 0$$

$$\big(F(u) - \sigma(u)\big)\sigma(r)\sigma(v) = 0$$

$\big(F(u) - \sigma(u)\big)R\sigma(v) = 0$. Since R is primering and σ is an automorphism

$F(u) = \sigma(u)$in the similar manner, we can prove our conclusions when $F(uv) + \sigma(vu) = 0$, for all $u, v \in U$, there by the proof of the theorem is completed.

Theorem 3: Let R be a 2-torsion free prime ring and U be a non zero square closed Lie ideal of R. If Radmits a generalized (σ,τ)-derivatiom$F: R \to R$ associated with the (σ,τ)-derivation the map $g: R \to R$ such that $F(u)F(v) \pm \sigma(uv) = 0$, for all$u, v \epsilon U$, then $g(U) = (0)$and $U \subseteq Z(R)$,and $[F(u),\sigma(u)] = 0$, for all $u \epsilon U$.

Proof:$F(u)F(v) - \sigma(uv) = 0$, for all $u, v\epsilon U$ (13)

We replace v by $2vw$ in equation (13), we get

$$F(u)F(2vw) - \sigma(2uvw) = 0$$

$F(u)F(2v)\sigma(w) - \sigma(2uvw) + 2F(u)\tau(v)g(w) = 0$, for all $u, v, w\epsilon U$

Using 2-torsion freeness the above equation becomes

$$\big(F(u)F(v) - \sigma(uv)\big)\sigma(w) + F(u)\tau\{v\}g(w) = 0$$

Using (13) in the above equation becomes

$F(u)\tau\{v\}g(w) = 0$, for all $u, v, w\epsilon U$ ` (14)

Left multiply equation (14) by $F(t)$, we get

$F(t)F(u)\tau\{v\}g(w) = 0$,for all $u, v, w\epsilon U$ (15)

Using (13) in (15), we get

$\sigma(tu)\tau(v)g(w) = 0$, for all $u, v, w\epsilon U$ (16)

We replace t by $[t, r], r\epsilon R$ in the equation (16), we get

$$\sigma([t, r]u)\tau(v)g(w) = 0$$

$$\big(\sigma(tru) - \sigma(rtu)\big)\tau(v)g(w) = 0$$

Using (16), the above equation becomes

$$\sigma(t)\sigma(r)\sigma(u)\tau(v)g(w) = 0$$

$$\sigma(t)R\sigma(u)\tau(v)g(w) = 0$$

Since R is prime ring and U is a non zero lie ideal of R we find that

$\sigma(u)\tau(v)g(w) = 0$,for all $u, v, w\epsilon U$ (17)

Following the same technique twice we finally we get

$g(U) = (0)$,for all $u, v, \epsilon U$ (18)

Now $F(uv) = F(u)\sigma(v) + \tau(u)g(v)$

Using (18), in the above equation, we get

$F(uv) = F(u)\sigma(v)$, for all $u, v\epsilon U$ (19)

We replace u by $2uv$ in (13), we get

$$F(u2v)F(v) - \sigma(2uv^2) = 0$$

$$F(u)\sigma(2v)F(v) - \sigma(2uv^2) = 0$$

Since R is 2-torsion free ring, we obtain

$F(u)\sigma(v)F(v) - \sigma(uv^2) = 0$, for all $u, v\epsilon U$ (20)

Right multiply by $\sigma(v)$ to equation (13), we get

$F(u)F(v)\sigma(v) - \sigma(uv^2) = 0$, for all $u, v\ \epsilon U$ (21)

Subtracting equation (20) with equation (21), we get

$F(u[F(v), \sigma(v)] = 0$, for all $u, v \epsilon U$ (22)

Using u by $2uw$ and using (19), we get

$$F(u2w[F(v), \sigma(v)] = 0$$

$2F(u)\sigma(w)[F(v), \sigma(v)] = 0$

Using 2-torsion free ringof R, we have

$$F(u)\sigma(w)[F(u), \sigma(v)] = 0$$

It follows that $[F(u), \sigma(u) \cup [F(u), \sigma(u)]] = 0$

Lemma 2 gives $[F(u), \sigma(u)] = 0$

And the same condition is obtain if $U \subseteq Z(R)$

In similar manner we can prove the same conclusion holds for $F(u)F(v) + \sigma(uv) = 0$, for all$u, v\epsilon$U.

Theorem 4: Let R be a 2-torsion free prime ring and U be a non zero square closed Lie ideal of R. If R admits a generalized (σ, τ)-derivatiom $F: R \to R$ associated with the (σ, τ)-derivation the map $g: R \to R$ such that $F(u)F(v) \pm \sigma(v\, u) = 0$, for all$u, v\ \epsilon U$, then $U \subseteq Z(R)$,and$g(U) = (0)$, for all $u\ \epsilon\ U$.

Proof: suppose on contrary$U \nsubseteq Z(R)$

We assume that

$F(u)F(v) - \sigma(uv) = 0$, for all $u, v\epsilon U$ (23)

We replace v by $2vu$ in equation (23), we get

$$F(u)F(2vu) - \sigma(2vu^2) = 0$$

$$F(u)F(2v)\sigma(u) - 2\sigma(vu^2) + 2F(u)\tau(v)g(u) = 0$$

$\left(F(u)F(2v) - \sigma(2vw)\right)\sigma(u) + 2F(u)\tau(v)g(u) = 0$, for all $u, v, w\epsilon U$ (24)

Using (23) in (24), we get

$$2F(u)\tau(v)g(u) = 0$$

Using 2-torsion free ring,we get

$F(u)\tau(v)g(u) = 0$, for all $u, v, w\epsilon U$ (25)

Left multiplicative equation (25) by $F(w)$ and using

$$F(w)F(u)\tau(v)g(u) = 0$$

$$\sigma(uw)\tau(v)g(u) = 0$$

$$\sigma(u)\sigma(w)\tau(v)g(u) = 0$$
$$\sigma(u)\sigma(w)\tau(v)g(u) = 0$$

$$\sigma(u)U(v)g(u) = 0$$

By lemma2 we have for each $u\epsilon U$ either $\sigma(u) = 0$ ie $U = 0$ or $\tau(v)g(u) = 0$

We replace v by vr in the above equation

$$\tau(v)\tau(r)g(u) = 0$$

$$\tau(v)Rg(u) = 0$$

Since R is prime, we get

$g(U) = 0$, for all $u \in U$

Now replacing$v$ by v^2 and using the fact$g(U) = 0$, we get

$F(u)F(v)\sigma(v) - \sigma(v^2u) = 0$, for all $u, v \in U$ (26)

Right multiplying equation (23) by $\sigma(v)$ and substracting from (26), we get

$\sigma(v)\sigma[v, u] = 0$then by the same argument as given in the proof the theorem 2, we have

$U \subseteq Z(R)$.

$F(u)F(v) - \sigma(uv) = 0$, for all $u, v \in U$

This is view of theorem 3, we get $g(U) = (0)$

In the similar manner,we can prove that the same conclusion holds for $F(u)F(v) + \sigma(vu) = 0$ for all $u, v\epsilon U$.

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