On Chromatic Zagreb Indices of Some Cycle Related Graphs

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Abstract

Real numbers that are invariant under graph isomorphism are known as topological indices. In recent literature, the concept of chromatic topological indices has been described and studied as an extended coloring version of some Zagreb indices. In this article these new indices have been created for certain types of cycle related graphs.

Keywords: Double fan graph, Fan graph, Friendship graph, Zagreb Indices.

1. Introduction

Chemical graph theory has a wide range of applications in today's world, particularly in the pharmaceutical industry. It is primarily concerned with mathematical modelling of chemical phenomena and acquiring useful insights into chemical behaviour. One of the essential conceptions of molecular descriptors in chemical graph theory is a topological index of a graph *G*, which is a real number retained under isomorphism. The chromatic topological indices of a graph *G* was recently coined in [5] to identify a novel coloring version of these indices that encompasses both proper colouring and topological indices. The vertex degrees are swapped with minimal coloring in this case, however the additional coloring conditions of proper coloring are maintained. The graphs in this work are finite, non-trivial, undirected, linked and free of loops and multiple edges. See [2,8,9,10] for notation and terminology not expressly described here.

Analogous to the definitions of Zagreb indices of graphs (see [1,3,4,7]), the notions of different chromatic Zagreb indices have been introduced in [5] as follows:

<u>Definition1.1</u>

[5] Let $C = \{c_1, c_2, ..., c_l\}$ be the proper coloring of any graph G. Since $|C| = l$, G has $l!$ minimum parameter colorings. Denote these colorings as $\phi_t(G), 1 \le t \le l!$.

Let $\phi(v_i) = c_s$, $1 \le i \le n, 1 \le s \le l$. Then for $1 \le t \le l!$,

- The *first chromatic Zagreb index* of *G* is defined as:

$$M_1^{\phi_t}(G) = \sum_{i=1}^{n} c(v_i)^2$$

$$= \sum_{j=1}^{l} \theta(c_j) \cdot j^2, \; c_j \in C$$

- The *second chromatic Zagreb index* of *G* is defined as:

$$M_2^{\phi_t}(G) = \sum_{i=1}^{n-1}\sum_{j=2}^{n}(c(v_i)\cdot c(v_j)),\ v_iv_j \in E(G)$$
$$= \sum_{1\le t,s\le l}^{t<s}(t\cdot s)\eta_{ts}$$

- The *chromatic irregularity index* of G is defined as:

$$M_3^{\phi_t}(G) = \sum_{i=1}^{n-1}\sum_{j=2}^{n}|\,c(v_i) - c(v_j)\,|,\ v_iv_j \in E(G)$$
$$= \sum_{1\le t,s\le l}^{t<s}|\,t-s\,|\,\eta_{ts}$$

- The *chromatic total irregularity index* of G is defined as:

$$M_4^{\phi_t}(G) = \frac{1}{2}\sum_{i=1}^{n-1}\sum_{j=2}^{n}|\,c(v_i) - c(v_j)\,|, v_i, v_j \in V(G)$$

Here η_{ts} is the number of edges $e(=uv)$ in G such that $c(u)=t$ and $c(v)=s$.
In account of the aforesaid considerations, the minimum and maximum chromatic Zagreb indices, as well as the related irregularity indices, are defined as follows:

$$M_i^{\phi^-}(G) = \min\left\{M_i^{\phi_t(G)} : 1\le t\le l!\right\}, 1\le i\le 4$$
$$M_i^{\phi^+}(G) = \max\left\{M_i^{\phi_t(G)} : 1\le t\le l!\right\}, 1\le i\le 4$$

2. Chromatic Zagreb Indices of Fan Graph

Definition 2.1

[6] A fan graph F_n is defined as the join of the path P_n and the graph $\widetilde{K}_1$. That is $P_n + \widetilde{K}_1$.

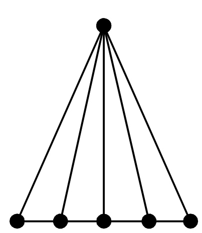

Figure 1: Fan Graph F_5.

Theorem 2.2

For any fan graph F_n, the minimum chromatic Zagreb indices and the corresponding irregularity indices are:

i. $$M_1^{\phi^-}(F_n) = \begin{cases} \dfrac{5(n+3)}{2}, & if\ n\ is\ odd. \\ \dfrac{5n+18}{2}, & if\ n\ is\ even. \end{cases}$$

ii. $$M_2^{\phi^-}(F_n) = \begin{cases} \dfrac{13n-7}{2}, & if\ n\ is\ odd. \\ \dfrac{13n-4}{2}, & if\ n\ is\ even. \end{cases}$$

iii. $$M_3^{\phi^-}(F_n) = \begin{cases} \frac{5n-1}{2}, & if\ n\ is\ odd. \\ \frac{5n-2}{2}, & if\ n\ is\ even. \end{cases}$$

iv. $$M_4^{\phi^-}(F_n) = \begin{cases} \frac{n^2+6n+1}{4}, & if\ n\ is\ odd. \\ \frac{n^2+6n}{4}, & if\ n\ is\ even. \end{cases}$$

Proof:

The chromatic number of a fan graph F_n is 3. Let $v_1, v_2, \ldots, v_n$ be the vertices of P_n and v_0 be the vertex of $\widetilde{K}_1$. To calculate the minimum Zagreb indices, we use ϕ^- coloring pattern to F_n.

Part 1: To Calculate $\boldsymbol{M_1^{\phi^-}}$ of $\boldsymbol{F_n}$, consider the following cases.

Case 1:

If $\boldsymbol{n}$ is odd, color $\boldsymbol{F_n}$ as follows: $\boldsymbol{c(v_{2k-1}) = 1;\ 1 \le k \le \frac{n+1}{2}}$, $\boldsymbol{c(v_{2k}) = 2;\ 0 \le k \le \frac{n-1}{2}}$ and $\boldsymbol{c(v_0) = 3}$. $\boldsymbol{\theta(c_1) = \frac{n+1}{2}, \theta(c_2) = \frac{n-1}{2}}$ and $\boldsymbol{\theta(c_3) = 1}$.

$$\boldsymbol{M_1^{\phi^-}(F_n) = \sum_{j=1}^{l} \theta(c_j).j^2\ ;c_j \in C}$$

$$= 1\left(\frac{n+1}{2}\right) + 4\left(\frac{n-1}{2}\right) + 9(1)$$

$$= \boldsymbol{\frac{5n+15}{2} = \frac{5(n+3)}{2}}.$$

Case 2:

If n is even, color F_n as follows: $c(v_{2k-1}) = 1;\ 1 \le k \le \frac{n}{2}$, $c(v_{2k}) = 2;\ 0 \le k \le \frac{n}{2}$ and $c(v_0) = 3$. $\theta(c_1) = \frac{n}{2}, \theta(c_2) = \frac{n}{2}$ and $\theta(c_3) = 1$.

$$\boldsymbol{M_1^{\phi^-}(F_n) = \sum_{j=1}^{l} \theta(c_j).j^2\ ;c_j \in C}$$

$$\boldsymbol{= 1\left(\frac{n}{2}\right) + 4\left(\frac{n}{2}\right) + 9(1)}$$

$$= \boldsymbol{\frac{5n+18}{2}}.$$

Part 2: To Calculate $M_2^{\phi^-}$ of F_n, consider the following cases.

Case 1:

If n is odd, $\eta_{12} = n-1, \eta_{13} = \frac{n+1}{2}, \eta_{23} = \frac{n-1}{2}$

$$M_2^{\phi^-}(F_n) = \sum_{1 \le t,s \le l}^{t<s} (t \cdot s)\eta_{ts}$$

$$= 2(n-1) + 3\left(\frac{n+1}{2}\right) + 6\left(\frac{n-1}{2}\right) = \frac{13n-7}{2}.$$

Case 2:

If n is even, $\eta_{12} = n-1, \eta_{13} = \frac{n}{2}, \eta_{23} = \frac{n}{2}$

$$\boldsymbol{M_2^{\phi^-}(F_n) = \sum_{1 \le t,s \le l}^{t<s} (t \cdot s)\eta_{ts}}$$

$$= 2(n-1) + 3\left(\frac{n}{2}\right) + 6\left(\frac{n}{2}\right)$$
$$= \frac{13n-4}{2}.$$

Part 3: To Calculate the minimum irregularity index $M_3^{\phi^-}$ of F_n, consider the following cases.

Case 1:

If n is odd, $\eta_{12} = n-1, \eta_{13} = \frac{n+1}{2}, \eta_{23} = \frac{n-1}{2}$

$$M_3^{\phi^-}(F_n) = \sum_{1\le t,s\le l}^{t<s} |t-s|\eta_{ts}$$
$$= 1(n-1) + 2\left(\frac{n+1}{2}\right) + 1\left(\frac{n-1}{2}\right) = \frac{5n-1}{2}.$$

Case 2:

If n is even, $\eta_{12} = n-1, \eta_{13} = \frac{n}{2}, \eta_{23} = \frac{n}{2}$

$$M_3^{\phi^-}(F_n) = \sum_{1\le t,s\le l}^{t<s} |t-s|\eta_{ts}$$
$$= 1(n-1) + 2\left(\frac{n}{2}\right) + 1\left(\frac{n}{2}\right)$$
$$= \frac{5n-2}{2}$$

Part 4: To compute the total irregularity indices $M_4^{\phi^-}$ of F_n, consider all the possible vertex pairs and all color combinations providing non zero distances.

Case 1:

If n is odd, $\theta(c_1) = \frac{n+1}{2}, \theta(c_2) = \frac{n-1}{2}$ and $\theta(c_3) = 1$.

$$M_4^{\phi^-}(F_n) = \frac{1}{2}\sum_{v_i,v_j\in V(F_n)} |c(v_i) - c(v_j)|$$
$$= 1\left(\frac{n+1}{2}\right)\left(\frac{n-1}{2}\right) + 2(1)\left(\frac{n+1}{2}\right) + 1(1)\left(\frac{n-1}{2}\right)$$
$$= \frac{n^2+6n+1}{4}.$$

Case 2:

If n is even, $\theta(c_1) = \frac{n}{2}, \theta(c_2) = \frac{n}{2}$ and $\theta(c_3) = 1$.

$$M_4^{\phi^-}(F_n) = \frac{1}{2}\sum_{v_i,v_j\in V(F_n)} |c(v_i) - c(v_j)|$$
$$= 1\left(\frac{n}{2}\right)\left(\frac{n}{2}\right) + 2(1)\left(\frac{n}{2}\right) + 1(1)\left(\frac{n}{2}\right) = \frac{n^2+6n}{4}.$$

Theorem 2.3

For any fan graph F_n, the maximum chromatic Zagreb indices and the corresponding irregularity indices are:

i. $$M_1^{\phi^+}(F_n) = \begin{cases} \frac{13n+7}{2}, & \text{if } n \text{ is odd.} \\ \frac{13n+2}{2}, & \text{if } n \text{ is even.} \end{cases}$$

ii. $$M_2^{\phi^+}(F_n)=\begin{cases}\frac{17n-11}{2}, & if\ n\ is\ odd.\\ \frac{17n-12}{2}, & if\ n\ is\ even.\end{cases}$$

iii. $$M_3^{\phi^+}(F_n)=\begin{cases}\frac{5n-1}{2}, & if\ n\ is\ odd.\\ \frac{5n-2}{2}, & if\ n\ is\ even.\end{cases}$$

iv. $$M_4^{\phi^+}(F_n)=\begin{cases}\frac{n^2+6n+1}{4}, & if\ n\ is\ odd.\\ \frac{n^2+6n}{4}, & if\ n\ is\ even.\end{cases}$$

Proof:

The chromatic number of a fan graph $\boldsymbol{F_n}$ is 3. Let $\boldsymbol{v_1, v_2, \dots, v_n}$ be the vertices of $\boldsymbol{P_n}$ and $\boldsymbol{v_0}$ be the vertex of $\boldsymbol{\widetilde{K}_1}$. To calculate the maximum Zagreb indices, we use $\boldsymbol{\phi^+}$ coloring pattern to $\boldsymbol{F_n}$. This results to the following cases:

Case 1: If $\boldsymbol{n}$ is odd, color $\boldsymbol{F_n}$ as follows: $\boldsymbol{c(v_0)=1,\ c(v_{2k-1})=2;\ 0\le k\le \frac{n+1}{2}}$, and $\boldsymbol{c(v_{2k})=3;\ 0\le k\le \frac{n-1}{2}}$.

$\boldsymbol{\theta(c_1)=1, \theta(c_2)=\frac{n-1}{2}}$ and $\boldsymbol{\theta(c_3)=\frac{n+1}{2}}$. Also $\boldsymbol{\eta_{12}=\frac{n-1}{2}, \eta_{13}=\frac{n+1}{2}, \eta_{23}=n-1}$.

Case 2: If $\boldsymbol{n}$ is even, color $\boldsymbol{F_n}$ as follows: $\boldsymbol{c(v_0)=1,\ c(v_{2k-1})=2;\ 0\le k\le \frac{n}{2}}$, and $\boldsymbol{c(v_{2k})=3;\ 0\le k\le \frac{n}{2}}$.

$\boldsymbol{\theta(c_1)=1, \theta(c_2)=\frac{n}{2}}$ and $\boldsymbol{\theta(c_3)=\frac{n}{2}}$. Also $\boldsymbol{\eta_{12}=\frac{n}{2}, \eta_{13}=\frac{n}{2}, \eta_{23}=n-1}$.

The remaining part of the proof is the same as in the proof of Theorem 2.2.

3. Chromatic Zagreb Indices of Double Fan Graph

<u>Definition 3.1</u>

[6] A double fan graph DF_n is defined as the join of the path P_n and the graph $\widetilde{K}_2$. That is $P_n+\widetilde{K}_2$.

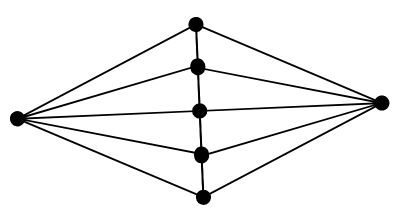

Figure 2: Double Fan Graph

<u>Theorem 3.2</u>

For any fan graph DF_n, the minimum chromatic Zagreb indices and the corresponding irregularity indices are:

i. $$M_1^{\phi^-}(DF_n)=\begin{cases}\frac{5n+33}{2}, & if\ n\ is\ odd.\\ \frac{5n+36}{2}, & if\ n\ is\ even.\end{cases}$$

ii. $M_2^{\phi^-}(DF_n) = 11n - 5$

iii. $M_3^{\phi^-}(DF_n) = 4n$

iv. $M_4^{\phi^-}(DF_n) = \begin{cases} \frac{n^2+12n+3}{4}, & \text{if } n \text{ is odd.} \\ \frac{n^2+6n}{4}, & \text{if } n \text{ is even.} \end{cases}$

Proof:

The chromatic number of a double fan graph DF_n is 3. Let $v_1, v_2, \ldots, v_n$ be the vertices of P_n and v_0 and v_0' be the vertices of $\widetilde{K}_2$. To calculate the minimum Zagreb indices, we use ϕ^- coloring pattern to DF_n.

Part 1: To Calculate $M_1^{\phi^-}$ of DF_n, consider the following cases.

Case 1:

If n is odd, color DF_n as follows: $c(v_{2k+1}) = 1;\ 0 \le k \le \frac{n-1}{2}$, $c(v_{2k}) = 2;\ 0 \le k \le \frac{n}{2}$ and $c(v_0) = c(v_0') = 3$.

$\theta(c_1) = \frac{n+1}{2}, \theta(c_2) = \frac{n-1}{2}$ and $\theta(c_3) = 2$.

$$M_1^{\phi^-}(DF_n) = \sum_{j=1}^{l} \theta(c_j).j^2\ ; c_j \in C$$

$$= 1\left(\frac{n+1}{2}\right) + 4\left(\frac{n-1}{2}\right) + 9(2) \quad = \frac{5n+33}{2}.$$

Case 2:

If n is even, color DF_n as follows: $c(v_{2k+1}) = 1;\ 0 \le k \le \frac{n}{2}$, $c(v_{2k}) = 2;\ 0 \le k \le \frac{n}{2}$ and $c(v_0) = c(v_0') = 3$.

$\theta(c_1) = \frac{n}{2}, \theta(c_2) = \frac{n}{2}$ and $\theta(c_3) = 2$.

$$M_1^{\phi^-}(DF_n) = \sum_{j=1}^{l} \theta(c_j).j^2\ ; c_j \in C$$

$$= 1\left(\frac{n}{2}\right) + 4\left(\frac{n}{2}\right) + 9(2)$$

$$= \frac{5n+36}{2}.$$

Part 2: To Calculate $\boldsymbol{M_2^{\phi^-}}$ of $\boldsymbol{DF_n}$,

$$\boldsymbol{\eta_{12} = n - 1, \eta_{13} = n + 1, \eta_{23} = n - 1}$$

$$\boldsymbol{M_2^{\phi^-}(DF_n)} = \sum_{\substack{t<s \\ 1\le t,s\le l}} \boldsymbol{(t \cdot s)\eta_{ts}}$$

$$\boldsymbol{= 2(n-1) + 3(n+1) + 6(n-1) = 11n - 5}.$$

Part 3: To Calculate the minimum irregularity index $\boldsymbol{M_3^{\phi^-}}$ of $\boldsymbol{DF_n}$,

$$\eta_{12} = n - 1, \eta_{13} = n + 1, \eta_{23} = n - 1$$

$$M_3^{\phi^-}(DF_n) = \sum_{\substack{t<s \\ 1\le t,s\le l}} |t - s|\eta_{ts}$$

$$= 1(n-1) + 2(n+1) + 1(n-1) = 4n.$$

Part 4:

To compute the total irregularity indices $M_4^{\phi^-}$ of DF_n, consider all the possible vertex pairs and all color combinations providing non zero distances.

Case 1:

If n is odd, $\theta(c_1) = \frac{n+1}{2}, \theta(c_2) = \frac{n-1}{2}$ and $\theta(c_3) = 2$.

$$M_4^{\phi^-}(DF_n) = \frac{1}{2}\sum_{v_i,v_j\in V(F_n)} |c(v_i) - c(v_j)|$$

$$= 1\left(\frac{n+1}{2}\right)\left(\frac{n-1}{2}\right) + 2(2)\left(\frac{n+1}{2}\right) + 1(2)\left(\frac{n-1}{2}\right)$$

$$= \frac{n^2 + 12n + 3}{4}.$$

Case 2:

If n is even, $\theta(c_1) = \frac{n}{2}, \theta(c_2) = \frac{n}{2}$ and $\theta(c_3) = 2$.

$$M_4^{\phi^-}(DF_n) = \frac{1}{2}\sum_{v_i,v_j\in V(F_n)} |c(v_i) - c(v_j)|$$

$$= 1\left(\frac{n}{2}\right)\left(\frac{n}{2}\right) + 2(2)\left(\frac{n}{2}\right) + 1(2)\left(\frac{n}{2}\right) = \frac{n^2 + 12n}{4}.$$

Theorem 3.3

For any fan double graph DF_n, the maximum chromatic Zagreb indices and the corresponding irregularity indices are:

i. $$M_1^{\phi^+}(DF_n) = \begin{cases} \frac{13n+1}{2}, & \text{if } n \text{ is odd.} \\ \frac{13n+4}{2}, & \text{if } n \text{ is even.} \end{cases}$$

ii. $M_2^{\phi^+}(DF_n) = 11n - 5$

iii. $M_3^{\phi^+}(DF_n) = 4n$

iv. $$M_4^{\phi^+}(DF_n) = \begin{cases} \dfrac{n^2+12n+3}{4}, & \text{if } n \text{ is odd.} \\ \dfrac{n^2+6n}{4}, & \text{if } n \text{ is even.} \end{cases}$$

Proof:

The chromatic number of a double fan graph $\boldsymbol{DF_n}$ is 3. Let $\boldsymbol{v_1, v_2, \ldots, v_n}$ be the vertices of $\boldsymbol{P_n}$ and $\boldsymbol{v_0}$ and $\boldsymbol{v_0'}$ be the vertices of $\widetilde{\boldsymbol{K}}_{\boldsymbol{2}}$. To calculate the maximum Zagreb indices, we use $\boldsymbol{\phi^+}$ coloring pattern to $\boldsymbol{DF_n}$ This results to the following cases:

<u>*Case 1:*</u> If $\boldsymbol{n}$ is odd, color $\boldsymbol{DF_n}$ as follows: $\boldsymbol{c(v_0) = c(v_0') = 1}$, $\boldsymbol{c(v_{2k-1}) = 2;\ 1 \le k \le \frac{n+1}{2}}$ and $\boldsymbol{c(v_{2k}) = 3;\ 0 \le k \le \frac{n-1}{2}}$.

$\boldsymbol{\theta(c_1) = 1, \theta(c_2) = \frac{n-1}{2}}$ and $\boldsymbol{\theta(c_3) = \frac{n+1}{2}}$.

<u>*Case 2:*</u> If $\boldsymbol{n}$ is even, $\boldsymbol{c(v_0) = c(v_0') = 1}$, $\boldsymbol{c(v_{2k-1}) = 2;\ 1 \le k \le \frac{n}{2}}$ and $\boldsymbol{c(v_{2k}) = 3;\ 0 \le k \le \frac{n}{2}}$.

$\boldsymbol{\theta(c_1) = 1, \theta(c_2) = \frac{n}{2}}$ and $\boldsymbol{\theta(c_3) = \frac{n}{2}}$.

Also in both the cases $\boldsymbol{\eta_{12} = n-1\ \eta_{13} = n+1, \eta_{23} = n-1}$.

The remaining part of the proof is the same as in the proof of Theorem 3.2.

4. Chromatic Zagreb Indices of Friendship Graph

<u>Definition 4.1</u>

[6] A friendship graph Fr_n^k is a planar graph with $kn+1$ vertices and $n(k+1)$ edges constructed by joining n copies of the cycle C_{k+1} with a common vertex.

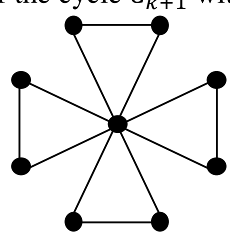

Figure 3: Friendship Graph $\boldsymbol{Fr_4^2}$.

<u>Theorem 4.2</u>

For any friendship graph Fr_n^k, the minimum chromatic Zagreb indices and the corresponding irregularity indices are:

i. $$M_1^{\phi^-}(\mathrm{Fr}_n^k) = \begin{cases} \dfrac{5nk-3n+18}{2}, & \text{if } n \text{ is odd.} \\ \dfrac{5nk+18}{2}, & \text{if } n \text{ is even.} \end{cases}$$

ii. $$M_2^{\phi^-}(\mathrm{Fr}_n^k) = \begin{cases} 2nk+4n, & \text{if } n \text{ is odd.} \\ 2nk+7n, & \text{if } n \text{ is even.} \end{cases}$$

iii. $$M_3^{\phi^-}(\mathrm{Fr}_n^k) = \begin{cases} (k+3)n, & \text{if } n \text{ is odd.} \\ (k+2)n, & \text{if } n \text{ is even.} \end{cases}$$

iv. $$M_4^{\phi^-}(\mathrm{Fr}_n^k) = \begin{cases} \dfrac{n^2(k^2-1)+6kn+2n}{4}, & \text{if } n \text{ is odd.} \\ \dfrac{nk(nk+6)}{2}, & \text{if } n \text{ is even.} \end{cases}$$

Proof:

The chromatic number of a friendship graph Fr_n^k is 3. Let $\{v_1{}^1, v_2{}^1, \dots, v_k{}^1\}$, $\{v_1{}^2, v_2{}^2, \dots, v_k{}^2\}$, …, $\{v_1{}^n, v_2{}^n, \dots, v_k{}^n\}$ be the vertices of n copies of P_n and v_0 be the central vertex. To calculate the minimum Zagreb indices, we use ϕ^- coloring pattern to Fr_n^k.

Part 1: To Calculate $M_1^{\phi^-}$ of Fr_n^k, consider the following cases.

Case 1:

If n is odd, color Fr_n^k as follows, $c(v_{2k+1}{}^i) = 1;\ 1 \le k \le \frac{n+1}{2}$, $c(v_{2k}{}^i) = 2;\ 1 \le k \le \frac{n-1}{2}$, $1 \le i \le n$ and $c(v_0) = 3$.

$\theta(c_1) = n\left(\frac{k+1}{2}\right), \theta(c_2) = n\left(\frac{k-1}{2}\right)$ and $\theta(c_3) = 1$.

$$M_1^{\phi^-}(Fr_n^k) = \sum_{j=1}^{l} \theta(c_j).j^2 \ ; c_j \in C$$

$$= 1(n)\left(\frac{k+1}{2}\right) + 4(n)\left(\frac{k-1}{2}\right) + 9(1) \quad = \frac{5nk - 3n + 18}{2}.$$

Case 2:

If n is even, color Fr_n^k as follows, $c(v_{2k+1}{}^i) = 1$, $c(v_{2k}{}^i) = 2;\ 0 \le k \le \frac{n}{2}$, $1 \le i \le n$ and $c(v_0) = 3$.

$\theta(c_1) = \frac{nk}{2}, \theta(c_2) = \frac{nk}{2}$ and $\theta(c_3) = 1$.

$$M_1^{\phi^-}(Fr_n^k) = \sum_{j=1}^{l} \theta(c_j).j^2 \ ; c_j \in C$$

$$= 1\left(\frac{nk}{2}\right) + 4\left(\frac{nk}{2}\right) + 9(1)$$

$$= \frac{5nk + 18}{2}.$$

Part 2: To Calculate $\boldsymbol{M_2^{\phi^-}}$ of $\boldsymbol{Fr_n^k}$, consider the following cases.

Case 1:

$\eta_{12} = k(n-1), \eta_{13} = 2n_{,} \eta_{23} = 0$.

$$\boldsymbol{M_2^{\phi^-}(Fr_n^k) = \sum_{1 \le t,s \le l}^{t<s} (t \cdot s)\eta_{ts}}$$

$$\boldsymbol{= 2n(k-1) + 3(2n) + 6(0) = 2nk + 4n}.$$

Case 2:

If $\boldsymbol{n}$ is even

$$\eta_{12} = n(k-1), \eta_{13} = n, \eta_{23} = n$$

$$M_3^{\phi^-}(Fr_n^k) = \sum_{1 \le t,s \le l}^{t<s} |t-s|\eta_{ts}$$

$$= 2n(k-1) + 3n + 6n = \boldsymbol{2nk + 7n}.$$

Part 3: To Calculate the minimum irregularity index $\boldsymbol{M_3^{\phi^-}}$ of $\boldsymbol{Fr_n^k}$,

Case 1:

If $\boldsymbol{n}$ is odd

$\boldsymbol{\eta_{12} = k(n-1), \eta_{13} = 2n_{,} \eta_{23} = 0}$.

$$M_3^{\phi^-}(Fr_n^k) = \sum_{1 \le t,s \le l}^{t<s} |t-s|\eta_{ts} \quad = n(k-1) + 2(2n) + 1(0)$$

$$= nk + 3n = (k+3)n.$$

Case 2:
If $\boldsymbol{n}$ is even

$$\eta_{12} = n(k-1), \eta_{13} = n, \eta_{23} = n$$

$$M_3^{\phi^-}(Fr_n^k) = \sum_{1 \le t,s \le l}^{t<s} |t-s|\eta_{ts} = 1n(k-1) + 2(n) + 1(n) = nk + 2n = (k+2)n.$$

Part 4: To compute the total irregularity indices $M_4^{\phi^-}$ of Fr_n^k, consider all the possible vertex pairs and all color combinations providing non zero distances.

Case 1:

If n is odd, $\theta(c_1) = n\left(\frac{k+1}{2}\right), \theta(c_2) = n\left(\frac{k-1}{2}\right)$ and $\theta(c_3) = 1$.

$$M_4^{\phi^-}(Fr_n^k) = \frac{1}{2}\sum_{v_i, v_j \in V(F_n)} |c(v_i) - c(v_j)|$$

$$= 1\left(n\left(\frac{k+1}{2}\right)\right)\left(n\left(\frac{k-1}{2}\right)\right) + 2(1)\left(n\left(\frac{k+1}{2}\right)\right) + 1(1)\left(n\left(\frac{k-1}{2}\right)\right)$$

$$= \frac{n^2(k^2-1) + 6kn + 2n}{4}.$$

Case 2:

If $\boldsymbol{n}$ is even, $\boldsymbol{\theta(c_1) = \frac{nk}{2}, \theta(c_2) = \frac{nk}{2}}$ and $\boldsymbol{\theta(c_3) = 1}$.

$$\boldsymbol{M_4^{\phi^-}(Fr_n^k) = \frac{1}{2}\sum_{v_i, v_j \in V(F_n)} |c(v_i) - c(v_j)|}$$

$$\boldsymbol{= 1\left(\frac{nk}{2}\right)\left(\frac{nk}{2}\right) + 2(1)\left(\frac{nk}{2}\right) + 1(1)\left(\frac{nk}{2}\right) = \frac{n^2k^2 + 6nk}{4} = \frac{nk(nk+6)}{4}}.$$

Theorem 4.3

For any friendship graph Fr_n^k, the maximum chromatic Zagreb indices and the corresponding irregularity indices are:

i. $$M_1^{\phi^+}(\mathrm{Fr}_n^k) = \begin{cases} \frac{13nk+5n+2}{2}, & \text{if } n \text{ is odd.} \\ \frac{13nk+2}{2}, & \text{if } n \text{ is even.} \end{cases}$$

ii. $$M_2^{\phi^+}(\mathrm{Fr}_n^k) = \begin{cases} 6nk, & \text{if } n \text{ is odd.} \\ 6nk - n, & \text{if } n \text{ is even.} \end{cases}$$

iii. $$M_3^{\phi^+}(\mathrm{Fr}_n^k) = \begin{cases} (k+3)n, & \text{if } n \text{ is odd.} \\ (k+2)n, & \text{if } n \text{ is even.} \end{cases}$$

iv. $$M_4^{\phi^+}(\mathrm{Fr}_n^k) = \begin{cases} \frac{n^2(k^2-1)+6kn+2n}{4}, & \text{if } n \text{ is odd.} \\ \frac{nk(nk+6)}{2}, & \text{if } n \text{ is even.} \end{cases}$$

Proof:

The chromatic number of a double fan graph Fr_n^k is 3. Let $\{v_1{}^1, v_2{}^1, \dots, v_k{}^1\}$, $\{v_1{}^2, v_2{}^2, \dots, v_k{}^2\}$, …, $\{v_1{}^n, v_2{}^n, \dots, v_k{}^n\}$ be the vertices of n copies of P_n and v_0 be the central vertex. To calculate the maximum Zagreb indices, we use ϕ^+ coloring pattern to Fr_n^k. This results to the following cases:

Case 1:

If n is odd, color Fr_n^k as follows, $c(v_0) = 1, c({v_{2k+1}}^i) = 2;\ 1 \le k \le \frac{n+1}{2}$ and $c({v_{2k}}^i) = 2;\ 1 \le k \le \frac{n-1}{2}$, $1 \le i \le n$.

$\theta(c_1) = 1, \theta(c_2) = n\left(\frac{k-1}{2}\right)$and $\theta(c_3) = n\left(\frac{k+1}{2}\right)$.

$$\eta_{12} = 0, \quad \eta_{13} = 2n, \quad \eta_{23} = n(k-1).$$

Case 2: If n is even, $c(v_0) = 1, ({v_{2k+1}}^i) = 2, c({v_{2k}}^i) = 3;\ 0 \le k \le \frac{n}{2}, 1 \le i \le n$

$\theta(c_1) = 1, \theta(c_2) = \frac{nk}{2}$ and $\theta(c_3) = \frac{nk}{2}$.

$$\eta_{12} = n, \quad \eta_{13} = n, \quad \eta_{23} = n(k-1).$$

The remaining part of the proof is the same as in the proof of Theorem 4.2.

5. Chromatic Zagreb Indices of Complement Friendship Graph

Definition 5.1

[6] A complement friendship graph $\overline{\overline{Fr_n^k}}$, is a planar graph with $kn+1$ vertices and $2nk-1$ edges constructed by joining n copies of the fan graph F_k with a common vertex.

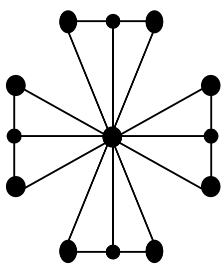

Figure 3: Complement Friendship Graph $\overline{\overline{Fr_4^3}}$.

Theorem 5.2

For any complement friendship graph $\overline{\overline{Fr_n^k}}$, the minimum chromatic Zagreb indices and the corresponding irregularity indices are:

i. $$M_1^{\varphi^-}\left(\overline{\overline{Fr_n^k}}\right) = \begin{cases} \frac{5nk-3n+18}{2}, & \text{if } n \text{ is odd.} \\ \frac{5nk+18}{2}, & \text{if } n \text{ is even.} \end{cases}$$

ii. $$M_2^{\varphi^-}\left(\overline{\overline{Fr_n^k}}\right) = \begin{cases} \frac{13nk-7n}{2}, & \text{if } n \text{ is odd.} \\ \frac{13nk-4n}{2}, & \text{if } n \text{ is even.} \end{cases}$$

iii. $$M_3^{\varphi^-}\left(\overline{\overline{Fr_n^k}}\right) = \begin{cases} \frac{5nk-n}{2}, & \text{if } n \text{ is odd.} \\ \frac{5nk-2n}{2}, & \text{if } n \text{ is even.} \end{cases}$$

iv. $$M_4^{\varphi^-}\left(\overline{\overline{Fr_n^k}}\right) = \begin{cases} \frac{n^2(k^2-1)+6kn+2n}{4}, & \text{if } n \text{ is odd.} \\ \frac{nk(nk+6)}{2}, & \text{if } n \text{ is even.} \end{cases}$$

Proof:

The chromatic number of a complement friendship graph $\overline{\overline{Fr_n^k}}$ is 3. Let $\{v_1{}^1, v_2{}^1, \ldots, v_k{}^1\}$, $\{v_1{}^2, v_2{}^2, \ldots, v_k{}^2\}$, …, $\{v_1{}^n, v_2{}^n, \ldots, v_k{}^n\}$ be the vertices of n copies of P_n and v_0 be the central vertex. To calculate the minimum Zagreb indices, we use ϕ^- coloring pattern to $\overline{\overline{Fr_n^k}}$.

Part 1: To Calculate $M_1^{\phi^-}$ of $\overline{\overline{Fr_n^k}}$ consider the following cases.

Case 1:

If n is odd, color $\overline{\overline{Fr_n^k}}$ as follows, $c(v_{2k+1}{}^i) = 1;\ 1 \le k \le \frac{n+1}{2}$, $c(v_{2k}{}^i) = 2;\ 1 \le k \le \frac{n-1}{2}$, $1 \le i \le n$ and $c(v_0) = 3$.

$\theta(c_1) = n\left(\frac{k+1}{2}\right), \theta(c_2) = n\left(\frac{k-1}{2}\right)$ and $\theta(c_3) = 1$.

$$M_1^{\phi^-}(Fr_n^k) = \sum_{j=1}^{l} \theta(c_j).j^2 \ ; c_j \in C$$
$$= 1(n)\left(\frac{k+1}{2}\right) + 4(n)\left(\frac{k-1}{2}\right) + 9(1) = \frac{5nk - 3n + 18}{2}.$$

Case 2:

If n is even, color $\overline{\overline{Fr_n^k}}$ as follows, $c(v_{2k+1}{}^i) = 1$, $c(v_{2k}{}^i) = 2;\ 0 \le k \le \frac{n}{2}$, $1 \le i \le n$ and $c(v_0) = 3$.

$\theta(c_1) = \frac{nk}{2}, \theta(c_2) = \frac{nk}{2}$ and $\theta(c_3) = 1$.

$$M_1^{\phi^-}(Fr_n^k) = \sum_{j=1}^{l} \theta(c_j).j^2 \ ; c_j \in C$$
$$= 1\left(\frac{nk}{2}\right) + 4\left(\frac{nk}{2}\right) + 9(1)$$
$$= \frac{5nk + 18}{2}.$$

Part 2: To Calculate $\boldsymbol{M_2^{\phi^-}}$ of $\overline{\overline{\boldsymbol{Fr_n^k}}}$, consider the following cases.

Case 1:

$\eta_{12} = k(n-1), \eta_{13} = n\left(\frac{k+1}{2}\right), \eta_{23} = n\left(\frac{k-1}{2}\right)$.

$$\boldsymbol{M_2^{\phi^-}\left(\overline{\overline{Fr_n^k}}\right) = \sum_{1 \le t,s \le l}^{t<s} (t \cdot s)\eta_{ts}}$$
$$\boldsymbol{= 2n(k-1) + 3\left(n\left(\frac{k+1}{2}\right)\right) + 6\left(n\left(\frac{k-1}{2}\right)\right) = \frac{13nk - 7n}{2}}.$$

Case 2:

If $\boldsymbol{n}$ is even

$$\eta_{12} = n(k-1), \eta_{13} = \frac{nk}{2}, \eta_{23} = \frac{nk}{2}$$
$$\boldsymbol{M_2^{\phi^-}\left(\overline{\overline{Fr_n^k}}\right) = \sum_{1 \le t,s \le l}^{t<s} (t \cdot s)\eta_{ts}}$$
$$= 2n(k-1) + 3\left(\frac{nk}{2}\right) + 6\left(\frac{nk}{2}\right) = \frac{\boldsymbol{13nk} - 4\boldsymbol{n}}{\boldsymbol{2}}.$$

Part 3: To Calculate the minimum irregularity index $M_3^{\phi^-}$ of $\overline{\overline{Fr_n^k}}$,

Case 1:

$\eta_{12} = k(n-1), \eta_{13} = n\left(\frac{k+1}{2}\right), \eta_{23} = n\left(\frac{k-1}{2}\right)$.

$$M_3^{\phi^-}\left(\overline{\overline{Fr}}_n^k\right) = \sum_{1\le t,s\le l}^{t<s} |t-s|\eta_{ts} = n(k-1) + 2\left(n\left(\frac{k+1}{2}\right)\right) + 1\left(n\left(\frac{k-1}{2}\right)\right)$$
$$= \frac{5nk-n}{2}.$$

Case 2:
If n is even

$$\eta_{12} = n(k-1), \eta_{13} = \frac{nk}{2}, \eta_{23} = \frac{nk}{2}$$
$$M_3^{\phi^-}\left(\overline{\overline{Fr}}_n^k\right) = \sum_{1\le t,s\le l}^{t<s} |t-s|\eta_{ts}$$
$$= n(k-1) + 2\left(\frac{nk}{2}\right) + 1\left(\frac{nk}{2}\right) = \frac{5nk-2n}{2}.$$

Part 4: To compute the total irregularity indices $M_4^{\phi^-}$ of $\overline{\overline{Fr}}_n^k$, consider all the possible vertex pairs and all color combinations providing non zero distances.

Case 1:
If n is odd, $\theta(c_1) = n\left(\frac{k+1}{2}\right), \theta(c_2) = n\left(\frac{k-1}{2}\right)$ and $\theta(c_3) = 1$.

$$M_4^{\phi^-}\left(\overline{\overline{Fr}}_n^k\right) = \frac{1}{2}\sum_{v_i,v_j\in V(F_n)} |c(v_i) - c(v_j)|$$
$$= 1\left(n\left(\frac{k+1}{2}\right)\right)\left(n\left(\frac{k-1}{2}\right)\right) + 2(1)\left(n\left(\frac{k+1}{2}\right)\right) + 1(1)\left(n\left(\frac{k-1}{2}\right)\right)$$
$$= \frac{n^2(k^2-1) + 6kn + 2n}{4}.$$

Case 2:
If n is even, $\theta(c_1) = \frac{nk}{2}, \theta(c_2) = \frac{nk}{2}$ and $\theta(c_3) = 1$.

$$M_4^{\phi^-}\left(\overline{\overline{Fr}}_n^k\right) = \frac{1}{2}\sum_{v_i,v_j\in V(F_n)} |c(v_i) - c(v_j)|$$
$$= 1\left(\frac{nk}{2}\right)\left(\frac{nk}{2}\right) + 2(1)\left(\frac{nk}{2}\right) + 1(1)\left(\frac{nk}{2}\right)$$
$$= \frac{n^2k^2 + 6nk}{4} = \frac{nk(nk+6)}{4}.$$

Theorem 5.3

For any complement friendship graph $\overline{\overline{Fr}}_n^k$ the maximum chromatic Zagreb indices and the corresponding irregularity indices are:

i. $$M_1^{\phi^+}\left(\overline{\overline{\mathrm{Fr}}}_n^k\right) = \begin{cases} \dfrac{13nk+5n+2}{2}, & \text{if } n \text{ is odd.} \\ \dfrac{13nk+2}{2}, & \text{if } n \text{ is even.} \end{cases}$$

ii. $$M_2^{\phi^+}\left(\overline{\overline{\mathrm{Fr}}}_n^k\right) = \begin{cases} \dfrac{17nk-11n}{2}, & \text{if } n \text{ is odd.} \\ \dfrac{11nk-6n}{2}, & \text{if } n \text{ is even.} \end{cases}$$

iii. $$M_3^{\phi^+}\left(\overline{\overline{\mathrm{Fr}}}_n^k\right)=\begin{cases}\dfrac{5kn-n}{2}, & if\ n\,is\,odd.\\ \dfrac{5kn-2n}{2}, & if\ n\,is\,even.\end{cases}$$

iv. $$M_4^{\phi^+}\left(\overline{\overline{\mathrm{Fr}}}_n^k\right)=\begin{cases}\dfrac{n^2(k^2-1)+6kn+2n}{4}, & if\ n\,is\,odd.\\ \dfrac{nk(nk+6)}{2}, & if\ n\,is\,even.\end{cases}$$

Proof:

The chromatic number of a double fan graph $\overline{\overline{Fr}}_n^k$ is 3. Let $\{v_1{}^1, v_2{}^1, \dots, v_k{}^1\}$, $\{v_1{}^2, v_2{}^2, \dots, v_k{}^2\}$, …, $\{v_1{}^n, v_2{}^n, \dots, v_k{}^n\}$ be the vertices of n copies of P_n and v_0 be the central vertex. To calculate the minimum Zagreb indices, we use ϕ^+ coloring pattern to $\overline{\overline{Fr}}_n^k$. This results to the following cases:

Case 1:

If n is odd, color $\overline{\overline{Fr}}_n^k$ as follows, $c(v_0) = 1, c\left(v_{2k+1}{}^i\right) = 2;\ 1 \le k \le \frac{n+1}{2}$ and $c\left(v_{2k}{}^i\right) = 2;\ 1 \le k \le \frac{n-1}{2}$, $1 \le i \le n$.

$\theta(c_1) = 1, \theta(c_2) = n\left(\frac{k-1}{2}\right)$and $\theta(c_3) = n\left(\frac{k+1}{2}\right)$.

$\eta_{12} = \frac{n(k-1)}{2},\ \ \eta_{13} = \frac{n(k+1)}{2},\ \eta_{23} = n(k-1)$.

Case 2: If n is even, $c(v_0) = 1, \left(v_{2k+1}{}^i\right) = 2, c\left(v_{2k}{}^i\right) = 3;\ 0 \le k \le \frac{n}{2}, 1 \le i \le n$

$\theta(c_1) = 1, \theta(c_2) = \frac{nk}{2}$ and $\theta(c_3) = \frac{nk}{2}$.

$$\eta_{12} = \frac{nk}{2}, \qquad \eta_{13} = \frac{nk}{2}, \qquad \eta_{23} = n(k-1).$$

The remaining part of the proof is the same as in the proof of Theorem 5.2.

Conclusion

In chemical graph theory and distribution theory, the concept presented in this work has several applications. This document gives an overview of chromatic Zagreb indices for a few cycle related graphs. The study appears to hold promise for future research because these indices can be generated for a wide range of graph classes and derived graph classes. For graph operations, graph products, and graph powers, chromatic topological indices can be calculated. The research on the subject in relation to various forms of graph colorings also seems to be quite promising. The approach can be used to edge colorings and map colorings as well. The subject of this study has a wide range of applications in the chemical industry. If $\boldsymbol{c(v_i)}$ assumes values such as energy, valency, bond strength, and so on, some interesting research using the above-mentioned principles are conceivable in Chemistry. Similar research can be done in a variety of other domains. All these facts show that there is a lot of room for more research in this field. Even the chromatic version of other topological indices opens new study avenues with wide-ranging applicability.

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