SOME STUDIES ON IDENTITIES WITH UNITY OR WITHOUT UNITY IN THE CENTER OF THE NON-ASSOCIATIVE RINGS.

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Abstract- In this paper, we study some properties by taking certain identities with unity or without unity in the center of Non - Associative Rings and hence we prove the commutative exist.

Keywords – Center, Non-Associative Ring, Prime ring,Alternative Ring. Char.≠*n*

I. INTRODUCTION

Introduction: "The center U of R is defined as $U = \{u \in R / [u, R] = 0\}$. A ring R is of characteristic $\neq n$ if $nx = 0$ implies $x = 0$ for all x in R and n a natural number. An alternative ring R is a ring in which $(x, x, y) = 0 = (x, y, y)$ for all x, y in R. A ring R is said to be prime if whenever A and B are ideals of R such that $AB = 0$, then either $A = 0$ or $B = 0$."

Throughout this paper R represents non - associative ring. In 1973 [10] RamAwtar proved commutativity of a prime associative ring by taking $xy^2x - yx^2y$ in the center U. In 1975 [11] Awtar has proved the commutativity of a nonassociative ring with 1 satisfying any one of the identity $(xy)^n = (yx)^n$, $(xy)^n = x^ny^n$ and $(xy)^n = y^nx^n$.

In 1988 [9] Quadri, Khan and Ashraf proved the commutativity of an associative ring R satisfying the identity $(xy)^2 = yx$ for all x, y. R.D.Giri and Modi [1] generalized this result and proved certain results on the commutativity of nonassociative rings of char. $\neq 2$ with unity satisfying $(xy)^2 \in U$ or $(xy)^2 - xy \in U$ or $((xy)z)^2 - (xy)z \in U$ or $[(xy)^2 - yx, x] = 0$ or $[(xy)^2 - yx, y] = 0$.

Yuanchun [12] proved that a semisimple ring R is commutative if and only if $(xy)^2 - xy^2x$ is central and in 2000 [4] Khan proved that commutativity of a nonassociative ring with unity satisfying $(xy)^2 = (xy^2)x$ or $(xy)^2 = (yx^2)y$.

K.Suvarna and K.Madhusudhan Reddy[5,6,7] proved commutativity in prime alternative ring by taking identities in the center and they studied commutativity without using Herstein theorem and unity. Based on the above results we have proved commutativity with unity and without unity in the non asoociative rings..

We prove some results on commutativity with unity.

Theorem1: Let R be a nonassociative ring of char. $\neq 2$ with unity satisfying $x^2 \in U$ for all x in R. Then R is commutative.

Proof: By hypothesis $x^2 \in U$ (1)

Put $x = x + 1$ in (1) and using (1), we get

$2x \in U$.

By taking char. $\neq 2$ we get $x \in U$

Therefore $xy = yx$ for all x in R.

Hence R is commutative

Theorem2: Let R be a nonassociative ring with unity satisfying $xy \in U$ for all x,y in R. Then R is commutative.

Proof: By hypothesis $xy \in U$ (2)
Now we take $y = y + 1$ in (2) and using (2), we get
$x \in U$
Therefore $xy = yx$ for all x in R.

Hence R is commutative

Theorem3: Let R be a nonassociative ring with unity satisfying $(xy)z \in U$ for all x,y and z in R. Then R is commutative.

Proof: By hypothesis$(xy)z \in U$ (3)

take z with $z + 1$ in (3) and using (3), we get

$xy \in U$

by using theorem (2), we obtain commutative .

Theorem 4 **:**Let R be an alternative ring with char. $\neq 2$ satisfying $(x \circ y) \in U$ for all x, y in R. Then R is commutative.

Proof: By hypothesis $(x \circ y) \in U$, i.e., $xy + yx \in U$. (4)

Now by replacing $x = x + 1$ in (4) and using (4), we get

$2x \in U$.

Using char. $\neq 2$ we get $x \in U$

Therefore $xy = yx$ for all x in R.

Hence R is commutative.

We prove the commutativity without unity.

First we prove the following Lemmas:

Lemma 1**:** Let R be an alternative ring with $[x, [x, y]] = 0$. Then $2[x, y]^2 = [x, [x, y^2]]$.

Proof **:** We have
$$
\begin{aligned}
[x, [x, y^2]] - 2[x, y]^2 &= [x, xy^2 - y^2x] - 2(xy - yx)^2 \\
&= x(xy^2 - y^2x) - (xy^2 - y^2x)x - 2(xy)^2 - 2(yx)^2 \\
&\quad + 2(xy)(yx) + 2(yx)(xy) \\
&= x^2y^2 - x(y^2x) - x(y^2x) + y^2x^2 - 2(xy)^2 - 2(yx)^2 \\
&\quad + 2x(y^2x) + 2y(x^2y) \\
&= x^2y^2 + y^2x^2 - 2(xy)^2 - 2(yx)^2 + 2y(x^2y) \\
&= (x^2y^2 - 2(xy)^2 + (yx^2)y) + (y^2x^2 - 2(yx)^2 + y(x^2y)) \\
&= (x^2y - 2(xy)x + yx^2)y + y(yx^2 - 2x(yx) + x^2y) \\
&= [x, [x, y]]y + y[x, [x, y]] = 0.
\end{aligned}
$$

Therefore $2[x, y]^2 = [x, [x, y^2]]$.

Lemma 2 **:** Let R be a prime alternative ring satisfying the condition $[x, y]^2 - [x^2, y^2] \in U$ for all x, y in R. Then R has no nonzero nilpotent elements.

Proof **:** By hypothesis $[x, y]^2 - [x^2, y^2] \in U$

i.e., $(xy - yx)^2 - (x^2y^2 - y^2x^2) \in U$. (5)

Now we replace x with $x + y$. Then

$((x + y)y - y(x + y))^2 - (x + y)^2y^2 + y^2(x + y)^2 \in U$

or $(xy - yx)^2 - x^2y^2 - (xy)y^2 - (yx)y^2 - y^4 + y^2x^2 + y^2(xy) + y^2(yx) + y^4 \in U$.

Using (5), we get

$y^2(xy) - (yx)y^2 + y^2(yx) - (xy)y^2 \in U$. (6)

Thus for $y \in R$, we have

$(y^2(xy) - (yx)y^2 + y^2(yx) - (xy)y^2)y = y(y^2(xy) - (yx)y^2 + y^2(yx) - (xy)y^2)$. (7)

Without loss of generality let $0 \neq x \in R$ and $x^2 = 0$.

Now by replacing y with yx in (7), we get

$((yx)^2(x(yx)) - ((yx)x)(yx)^2 + (yx)^2((yx)x) - (x(yx))(yx)^2)(yx) =$
$(yx)((yx)^2(x(yx)) - ((yx)x)(yx)^2 + (yx)^2((yx)x) - (x(yx))(yx)^2)$.

i.e., $yxyxxyxyx - yxxyxyxyx + yxyxyxxyx - xyxyxyxyx =$
$yxyxyxxyx - yxyxxyxyx + yxyxyxyxx - yxxyxyxyx$.

Now using the fact $x^2 = 0$ in the above equation, we obtain $(xy)^4x = 0$, that is $(xy)^5 = 0$ for all y in R.

By Lemma 1.1 of [3] it follows that xR is a nonzero right ideal of R in which $z^5 = 0$ where $z \in xR$. But $(xy)^5 = 0$ implies $xR = 0$, since $x^2 = 0$. Then $xRx = 0$. Hence $x = 0$, by primeness of R.

***Theorem 5*:** Let R be a prime alternative ring of char. $\neq 2$ satisfying $[x, y]^2 - [x^2, y^2] \in U$ for all x, y in R. Then R is commutative.

Proof: Refer the proof as in [7].

***Theorem 6*:** Let R be a prime alternative ring with char. $\neq 2$ satisfying (i) $[x^2y^2 + xy, z] \in U$ or (ii) $[x^2y^2 + yx, z] \in U$ for all x, y in R and for fixed z in R. Then R is commutative.

***Proof* :** (i) By hypothesis $[x^2y^2 + xy, z] \in U$. (8)

We take x with $x + y$ in (8). Then

$[(x + y)^2y^2 + (x + y)y, z] \in U$

or $[x^2y^2 + (xy)y^2 + (yx)y^2 + xy + y^4 + y^2, z] \in U$. (9)

Using (8) in (9), we get

$[(xy)y^2 + (yx)y^2 + y^4 + y^2, z] \in U$. (10)

By replacing x with y in (8), we obtain

$[y^4 + y^2, z] \in U$. (11)

Using (11)in (10), we get

$[(xy)y^2 + (yx)y^2, z] \in U$. (12)

Now put $x = x + y$ in (12), we have

$[(xy)y^2 + (yx)y^2 + 2y^4, z] \in U$.

Using (12)in above and using char. $\neq 2$, we get

$[y^4, z] \in U$. (13)

From (11) and (13), we have

$[y^2, z] \in U$. (14)

By placing $y = x + y$ in (14) and using (14), we get

$[xy + yx, z] \in U$. (15)

Now by replacing $z = yx$ in , we get

$[xy, yx] \in U$ or $(xy^2)x - (yx^2)y \in U$.

Now applying the same argument as in the Theorem 5, we conclude that R is commutative.

(ii) By hypothesis

$[x^2y^2 + yx, z] \in U$. (16)

We replace x with $x + y$ and use (16). Then

$$[(xy)y^2 + (yx)y^2 + y^4 + y^2, z] \in U.$$

Now applying the same argument as in Theorem 6(i), we conclude that R is commutative

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