Survival Distribution In Cancer Research : The Case Of Beta Exponentiated Modified Weibull Distribution

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Abstract

Weibull distribution is a prominent distribution in statistics. In specific fields they are used in modified forms especially in problems related to cancer survival. A more generalized Weibull distribution called Beta Exponentiated Modified Weibull(BEMW) distribution which consolidate various forms of distribution shapes into one density with additional parameters. This distribution is claimed to contain 19 lifetime distributions providing feasible selection of distributions in cancer related studies. The hazard function provides usable forms which will make it feasible in reliability analysis effectively. Parameters can be estimated using Maximum Likelihood method and Least Square method. This distribution was introduced as a life time distribution and have been used to model and analyze the lifetime data of numerous applied fields. A lot of works and studies are there to model the expected lifetime of cancer patients using various models like Weibull distribution(WD), Modified Weibull distribution(MWD), Exponentiated Modified Weibull distribution (EMWD)etc. Here to model the expected life time of patients with gastric cancer , the more generalized form called BEMW distribution is applied.

Keywords:

Beta Modified Weibull Disrtibution ,Beta Generalized Distribution, Exponentiated Weibull distribution, , Hazard Function, Survival Distribution.

1. Introduction

Gastric cancer is a type of malignant tumor which invades the surrounding tissues and is a common cause of cancer related deaths in recent years. Caner related studies are very relevant in present situation. Gastric cancer ranks the top five most common cancers in India. It is the second most common cause of cancer related deaths among Indian men and women in the age between 15 and 44. The expected lifetime of patients infected by malignant tumor cells are modeled by cox model, weibull model and many other life time models . Zhu[3] explained the application of Weibull model for survival of patients with gastric cancer, and compares the Cox model with Weibull model and concluded that weibull model can elicit more precise results as an alternative to Cox. BEMW, is a new six parameter (α,β,γ,δ,λ,θ) generalized distribution which is expected to provide a generalized platform for studies related to weibull distribution. Many distributions become the special case of proposed distribution such as EMW distribution, the EW distribution, the generalized Rayleigh distribution , the beta EW distribution , the Beta Weibull distribution, the Weibull distribution, the generalized linear failure rate distribution, the exponentiated exponential distribution, the exponential distribution, the Rayleigh distribution.

Mirza et.al.,[4] (2019) was introduced the concept of Beta Generalized(Beta-G) and BEMW distributions and their confidence is BEMW will be superior to all areas where Weibull Distribution is applicable. These distributions can be detailed by using Beta- normal distribution and its applications, Eugene et al.,[2](2002). Diana and Igor [1] (2012), explained a Weibull Model Extension to Estimate Cancer Latency. Pandiyan and Koventhan [6] (2018) estimated the Survival time of Cancer Patients and Pandiyan et.al.,[7] **(2018)** estimated the survival time of cancer drinker patients using stochastic model.

2. Model description

The Beta –G distribution is defined using cdf

$$F(y) = I_{D(y)}(\alpha,\beta) = \frac{B_{D(y)(\alpha,\beta)}}{B(\alpha,\beta)}$$

$$F(y)= \frac{1}{B(\alpha,\beta)} \int_0^{D(y)} z^{\alpha-1} (1-z)^{\beta-1} dz \, ; for\ \alpha,\beta > 0 \qquad (1)$$

Where D(y) denotes the cdf of parent distribution , α,β are the shape parameters, and its role is to increase skewness and tail weight. The $B_{D(y)}(\alpha,\beta)$and $I_{D(y)}(\alpha,\beta)$ denote the incomplete beta function and incomplete beta ratio respectively.

Let d(y) be the pdf of the parent distribution. Then the pdf of the new Beta –G distribution can be obtained as

$$f(y)=\frac{1}{B(\alpha,\beta)} [D(y)]^{\alpha-1}[1-D(y)]^{\beta-1} d(y) \qquad (2)$$

There are many other studies that have applied the methodology of the Beta –G distribution and obtained more flexible distributions than their base distribution. Here we use the BEMW distribution using the generalizations given above taking the EMW distribution as the base distribution. For the EMW distribution with four parameters (γ,δ,λ,θ), the cdf and pdf are given as follows

D(y) = $\left[1-e^{-(\delta y+(\theta y)^{\lambda})}\right]^{\gamma}$ (3)

d(y)= $\gamma\left[\delta+\lambda\theta^{\lambda}y^{\lambda-1}\right]e^{-(\delta y+(\theta y)^{\lambda})}\left[1-e^{-(\delta y+(\theta y)^{\lambda})}\right]^{\gamma-1}$ (4)

where δ and θare scale ,λ and γ are shape parameters, and y,γ,λ,θ,δ > 0. The cdf of BEMW distribution is obtained by taking D(y) given in (3) of EMW distribution in (1) in the following form

F(y) = $I_{\left[1-e^{-(\delta y+(\theta y)^{\lambda})}\right]^{\gamma}}$ (α,β)

F(y)= $\frac{1}{B(\alpha,\beta)}\int_0^{\left[1-e^{-\left(\delta y+(\theta y)^{\lambda}\right)}\right]^{\gamma}} z^{\alpha-1}(1-z)^{\beta-1}dz\,;\, for\ y,\alpha,\beta,\gamma,\delta,\theta>0$ (5)

The corresponding density function of BEMW distribution can be obtained using (2) to (4) as

f(y) = $\frac{\gamma\left[\delta+\lambda\theta^{\lambda}y^{\lambda-1}\right]}{B(\alpha,\beta)}e^{-(\delta y+(\theta y)^{\lambda})}\left[1-e^{-(\delta y+(\theta y)^{\lambda})}\right]^{\alpha\gamma-1}\left\{1-\left[1-e^{-(\delta y+(\theta y)^{\lambda})}\right]^{\gamma}\right\}^{\beta-1}$ (6)

It is observed that the pdf of BEMW distribution assumed the exponential skewed to symmetric shape , depending on the values of the parameters.

3. Special cases of BEMW Distribution

This distribution includes 19 lifetime distributions as special cases. This specialty itself shows its importance in reliability theory and survival analysis. The most popularly used distribution in survival analysis is two parameter Weibull distribution(WD), Wallodi Weibull [9] (1951) for α=β=γ=1and δ =0 which is a special case of this new distribution. Beta Exponentiated Weibull (BEW)distribution (α,β,γ,λ,θ) defined by Singla et al.[8](2012) is a special case of this new distribution for its δ = 0 . For α=β=1and δ =0 its density reduces its form to Exponentiated Weibull (EW) Distribution developed by Pal et al., [5](2006). The Beta Modified Weibull Distribution (BMW), Beta Exponentiated Exponential Distribution(BEE) and 14 other distributions of this family are special cases of this six parameter distribution.

4.**Reliability analysis**

Reliability analysis is a procedure to quantitatively assess the mature product at every stage of its life cycle. In reliability analysis the survival function and hazard rate function are very important.

5.Survival function

The survival function provides the probability of a product surviving for a specific time. The survival function is

$\bar{F}$(y) = P(Y>y) = $\int_{y}^{\infty} f(y)dy$ = 1- F(y) = $I_{[1-D(y)]}(\beta,\alpha)$

The cdf of BEMW distribution be substituted to obtain the survival function of the distribution as

$$\bar{F}(y) = \frac{1}{B(\beta,\alpha)} \sum_{j=0}^{\alpha-1} \sum_{k=0}^{\beta+j} \frac{(-1)^{j+k}\Gamma(\alpha)}{\Gamma(\alpha-j)\Gamma(j+1)(\beta+j)} \binom{\beta+j}{k} D_{\gamma k,\delta,\lambda,\theta}(y)$$

6.Hazard function

The Hazard function of BEMW distribution is given by the expression

$$h(y) = \frac{f(y)}{\bar{F}(y)}$$

$$h(y) = \frac{\gamma[\delta+\lambda\theta^{\lambda}y^{\lambda-1}]}{B(\alpha,\beta)I_{[1-D(y)]}(\beta,\alpha)} e^{-(\delta y+(\theta y)^{\lambda})} \left[1-e^{-(\delta y+(\theta y)^{\lambda})}\right]^{\alpha\gamma-1} \left\{1-\left[1-e^{-(\delta y+(\theta y)^{\lambda})}\right]^{\gamma}\right\}^{\beta-1}$$

As the BEMW model is most flexible it can accommodate possible forms of hazard function such as

Decreasing ii) bathtub(decreasing-stable-increasing) iii) upside-down bathtub(increasing-decreasing) iv) stable and increasing and v) unimodal

7.Parameter estimation- maximum likely hood estimation

Let Y_1, Y_2......... Y_n be independent random variables following a BEMW distribution of size n with parameters (α,β,γ,δ,λ,θ).

The log likelihood function is obtained in the following form

$$\log L = \log\gamma + \log(\delta+\lambda\theta^{\lambda}y^{\lambda-1}) - (\delta y+(\theta y)^{\lambda}) - \log B(\alpha,\beta) + (\alpha\gamma-1)\log\left[1-e^{-(\delta y+(\theta y)^{\lambda}}\right] + (\beta-1)\log\left[1-(1-e^{-(\delta y+(\theta y)^{\lambda}})^{\gamma}\right]$$

The ML estimators for the parameters (α,β,γ,δ,λ,θ) are obtained as follows

$$\frac{\partial \log L}{\partial\alpha} = \psi(\alpha+\beta) - \psi(\alpha) + \gamma\log\left[1-e^{-(\delta y+(\theta y)^{\lambda}}\right]$$

$$\frac{\partial \log L}{\partial\beta} = \psi(\alpha+\beta) - \psi(\beta) + \log\left[1-(1-e^{-(\delta y+(\theta y)^{\lambda}})^{\gamma}\right]$$

$$\frac{\partial \log L}{\partial\gamma} = \frac{1}{\gamma} + \alpha\log\left[1-e^{-(\delta y+(\theta y)^{\lambda}}\right] - \frac{(\beta-1)(1-e^{-(\delta y+(\theta y)^{\lambda}})^{\gamma}\log\left[1-e^{-(\delta y+(\theta y)^{\lambda}}\right]}{\left[1-(1-e^{-(\delta y+(\theta y)^{\lambda}})^{\lambda}\right]}$$

$$\frac{\partial \log L}{\partial \lambda}=\frac{\theta^{\lambda}y^{\lambda-1}\left[1+\lambda\log(\theta)+\lambda\log(y)\right]}{\left[\delta+\lambda\theta^{y}y^{\lambda-1}\right]}-(\theta y)^{\lambda}\log(\theta y)+\frac{(\alpha\gamma-1)(\theta y)^{\lambda}\log(\theta y)e^{-(\delta y+(\theta y)^{\lambda})}}{\left[1-e^{-(\delta y+(\theta y)^{\lambda})}\right]}$$

$$\frac{\partial \log L}{\partial \delta}=\frac{1}{\left[\delta+\lambda\theta^{\lambda}y^{\lambda-1}\right]}-y+\frac{(\alpha\gamma-1)ye^{-(\delta y+(\theta y)^{\lambda})}}{\left[1-e^{-(\delta y+(\theta y)^{\lambda})}\right]}-\frac{(\beta-1)\gamma ye^{-(\delta y+(\theta y)^{\lambda})}\left[1-e^{-(\delta y+(\theta y)^{\lambda})}\right]^{\gamma-1}}{\left[1-(1-e^{-(\delta y+(\theta y)^{\lambda})})^{\gamma}\right]}-\frac{\gamma(\beta-1)y(\theta y)^{\lambda}\log(\theta y)e^{-(\delta y+(\theta y)^{\lambda})}\left[1-e^{-(\delta y+(\theta y)^{\lambda})}\right]^{\gamma-1}}{\left[1-(1-e^{-(\delta y+(\theta y)^{\lambda})})^{\gamma}\right]}$$

$$\frac{\partial \log L}{\partial \theta}=\frac{\lambda^{2}y^{\lambda-1}\theta^{\lambda-1}}{\left[\delta+\lambda\theta^{\lambda}y^{\lambda-1}\right]}-\lambda\theta^{\lambda}y^{\lambda-1}+\frac{(\alpha\gamma-1)e^{-(\delta y+(\theta y)^{\lambda})}\lambda\theta^{\lambda}y^{\lambda-1}}{\left[1-e^{-(\delta y+(\theta y)^{\lambda})}\right]}-\frac{\gamma(\beta-1)\lambda\theta^{\lambda}y^{\lambda-1}e^{-(\delta y+(\theta y)^{\lambda})}\left[1-e^{-(\delta y+(\theta y)^{\lambda})}\right]^{\gamma-1}}{\left[1-(1-e^{-(\delta y+(\theta y)^{\lambda})})^{\gamma}\right]}$$

8. Data analysis

The pdf of BMW distribution for varying parameters are shown below (table 1)

Table 1 The pdf of BMW distribution for varying parameters

y	First	Second	Third	Fourth
0.1	2.61E-10	2.61E-10	3.65E-06	1.502701
0.2	8.1E-08	8.29E-08	0.000264	0.23705
0.3	2.07E-06	2.24E-06	0.00258	0.046951
0.4	1.98E-05	2.39E-05	0.011142	0.010287
0.5	0.000118	0.000164	0.030935	0.002384
0.6	0.000565	0.000902	0.065163	0.000572
0.7	0.002502	0.004418	0.11378	0.00014
0.8	0.01099	0.019788	0.173604	3.5E-05
0.9	0.048053	0.079669	0.239461	8.82E-06
1	0.197785	0.275007	0.305597	2.24E-06
1.1	0.689253	0.763668	0.366818	5.72E-07
1.2	1.770531	1.594528	0.419179	1.47E-07
1.3	2.90154	2.358158	0.460225	3.77E-08
1.4	2.691693	2.367239	0.488914	9.72E-09
1.5	1.314821	1.580196	0.505358	2.51E-09
1.6	0.328572	0.700727	0.510505	6.48E-10
1.7	0.041076	0.208087	0.505819	1.68E-10
1.8	0.002413	0.041471	0.493016	4.34E-11
1.9	5.91E-05	0.005463	0.473863	1.12E-11

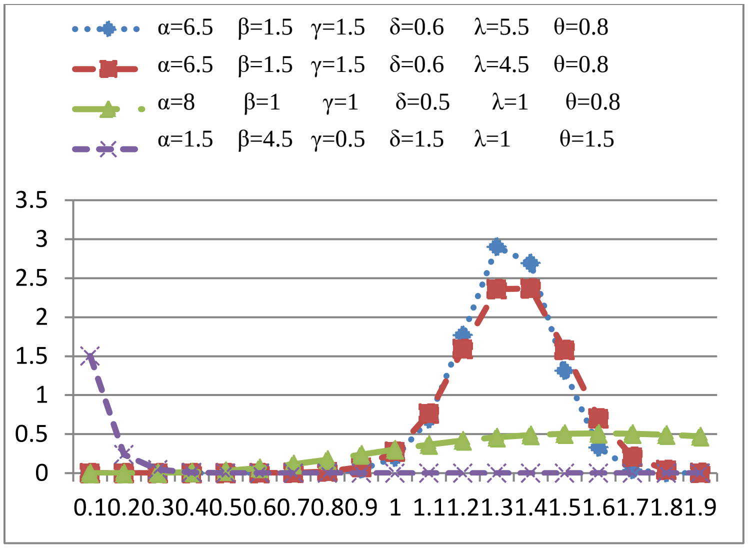

Figure 1 The pdf of BMW distribution for varying parameters

The possible shapes of hazard function of various parametric values of BEMW distribution are sketched below:

Table 2. Hazard function of various parametric values of BEMW distribution

y	1 st	2nd	3 rd
0.5	0.105713	0.36809	4.6E-05
2	0.029935	0.138046	0.002199
4	0.015903	0.086259	0.010559
6	0.010977	0.066063	0.02354
8	0.008435	0.054918	0.039628
10	0.006877	0.047724	0.057924
25	0.002965	0.027718	0.069235
30	0.002567	0.025073	0.070974
35	0.00232	0.023092	0.071054
40	0.002178	0.021544	0.070174
50	0.002092	0.019272	0.055838
65	0.00219	0.017038	0.036268
70	0.00223	0.016483	0.03557
75	0.002245	0.015994	0.034887
80	0.002223	0.01556	0.034222
85	0.002156	0.015171	0.033581

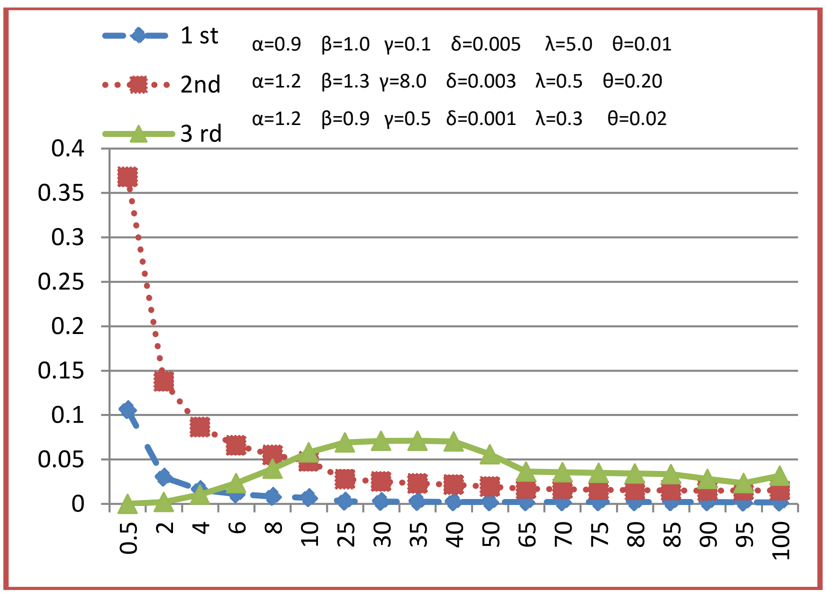

Figure 2. Hazard function of various parametric values of BEMW distribution

9. Conclusion

Beta Exponentiated Modified Weibull distribution is a more generalized form to model lifetime distributions especially cancer related studies. The proposed distribution is more flexible than the four parameter weibull distribution due to the additional two parameters. This distribution can assume a wide variety of shapes such as exponential, skewed and symmetric. The ML estimators can be obtained through Newton Raphson numerical solution algorithm by solving the six nonlinear equations simultaneously. The Hazard function also includes different forms such as decreasing, bathtub, unimodal. It can be concluded that the proposed distribution suggest a better model to the expected life time and survival time of cancer related studies.

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