Orthogonal reverse derivations and symmetric reverse biderivations of semiprime rings

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Abstract: In the present paper, we proved some results concerning orthogonal symmetric reverse biderivations such that for a two torsion free semiprime ring R. Let d_1 a reverse derivation and B_1 be a reverse biderivation. Then d_1 and B_1 are orthogonal if and only if the following conditions are equivalent

(i) $d_1 B_1 = 0$,
(ii) $B_1(x, y)d_1(x) = 0$,
(iii) $d_1 B_1$ is a biderivation,
(iv) $d_1(x)B_1(y, z) + B_1(x, y)d_1(z) = 0$, for every $x, y, z \in R$.

Keywords: semiprime ring, reverse derivation, reverse biderivation and orthogonal.

1. INTRODUCTION

Bresar and Vukman [3] have introduced the notation of orthogonality for a couple of derivations d and g of a semiprime ring, proved several necessary and sufficient conditions for d and g are to be orthogonal and they gave the related results to a classical result of Posner [6]. Daif, Tammam and Haetinger [4] have obtained the results of orthogonality conditions between derivations and biderivations in semiprime ring. Vukman [8,9] investigated some results with symmetric biderivations on prime and semiprime rings in connection with centralizing mappings. Jaya Subba Reddy and Ramoorthy Reddy [5], proved some results on orthogonal symmetric biderivations in Semiprime rings and they gave several necessary and sufficient conditions for two biderivations to be orthogonal. Abdul Rhman [1] proved orthogonality conditions for a pair of reverse derivations of a semiprime ring. And also Jaya Subba Reddy and Ramoorthy Reddy [6], obtained the conditions of orthogonality using symmetric reverse biderivations of semiprime rings. In this paper, our aim is to obtain the notation of orthogonal for reverse derivation d and symmetric reverse biderivation B on semiprime rings, and we presented several necessary and sufficient conditions for B and d to be orthogonal.

2. PRELIMINARIES

Throughout this paper R will be an associative ring. A ring R is said to be two torsion free if $2x = 0, x \in R$ implies $x = 0$. Recall that a ring R is said to be prime if the product of any two nonzero ideals of R is nonzero. Equivalently, $xRy = 0$ $x, y \in R$ implies $x = 0$ or $y = 0$. A ring R is said to be semiprime if it has no nonzero nilpotent ideals. Equivalently, $xRx = 0$ implies $x = 0$. Let us write $[x, y] = xy - yx$, for all $x, y \in R$ and the identities $[xy, z] = [x, z]y + x[y, z]$, $[x, yz] = [x, y]z + y[x, z]$. A map $d_1: R \to R$ is a derivation of a ring R if d_1 is additive, $d_1(xy) = d_1(x)y + xd_1(y)$, for all $x, y \in R$. A mapping $B_1(.,.): R \times R \to R$ is a symmetric mapping if $B_1(x, y) = B_1(y, x)$, for all $x, y \in R$. A symmetric bi additive mapping $B_1(.,.): R \times R \to R$ is called a symmetric biderivation if $B_1(xy, z) = B_1(x, z)y + xB_1(y, z)$, for all $x, y, z \in R$. Obviously, in this case also $B_1(x, yz) = B_1(x, y)z + yB_1(x, z)$, for all $x, y, z \in R$. A symmetric bi additive mapping $B_1(.,.): R \times R \to R$ is called a symmetric reverse biderivation if $B_1(xy, z) = B_1(y, z)x + yB_1(x, z)$, for all $x, y, z \in R$. Obviously, in this case also $B_1(x, yz) = B_1(x, z)y + zB_1(x, y)$, for all $x, y, z \in R$. Let R be a semiprime ring, the two derivations d_1 and g are called orthogonal if $d_1(x)Rg(y) = 0 = g(y)Rd_1(x)$, for all $x, y \in R$. Let R be a semiprime ring, symmetric biderivation B_1 and derivation d are called orthogonal if $B_1(\mathrm{x,y})\mathrm{R}d_1(\mathrm{z}) = (0) = d_1(\mathrm{x})\mathrm{R}B_1(\mathrm{y,z})$, for all $\mathrm{x,y,z} \in \mathrm{R}$. In the same manner, symmetric reverse biderivations B_1 and reverse derivation d_1 are called orthogonal if $B_1(\mathrm{x,y})\mathrm{R}d_1(\mathrm{z}) = (0) = d_1(\mathrm{x})\mathrm{R}B_1(\mathrm{y,z})$, for all $\mathrm{x,y,z} \in \mathrm{R}$.

3. MAIN TEXT

Lemma 1 ([3], lemma 1):
Let R be a two torsion free semiprime ring and a, b elements of R . Then the following conditions are equivalent.

(i) $\mathrm{axb} = 0$, for all $\mathrm{x} \in \mathrm{R}$.
(ii) $\mathrm{bxa} = 0$, for all $\mathrm{x} \in \mathrm{R}$.
(iii) $\mathrm{axb} + \mathrm{bxa} = 0$, for all $\mathrm{x} \in \mathrm{R}$.

If one of above three conditions is fulfilled, then $\mathrm{ab} = \mathrm{ba} = 0$.

Lemma 2: for a semiprime ring R. Suppose that a reverse derivation $d_1: R \to R$ and a reverse biderivation $B_1: R \times R \to R$ satisfies $d_1(x)RB_1(x,y) = 0(B_1(x,y)Rd_1(x) = 0)$, for every $x, y \in R$. Then $d_1(x)RB_1(z,y) = 0(B_1(x,y)Rd_1(z) = 0)$, for some $z \in R$.

Proof: Linearizing the expression $d_1(x)rB_1(x,y) = 0$, then $d_1(x+z)rB_1(x+z,y) = 0$
$(d_1(x) + d_1(z))r(B_1(x,y) + B_1(z,y)) = 0$
$d_1(x)rB_1(x,y) + d_1(x)rB_1(z,y) + d_1(z)rB_1(x,y) + d_1(z)rB_1(z,y) = 0.$
Then $d_1(x)rB_1(z,y) + d_1(z)rB_1(x,y) = 0$, implies that $d_1(x)rB_1(z,y) = -d_1(z)rB_1(x,y)$
Hence $d_1(x)rB_1(z,y)Rd_1(x)rB_1(z,y) = -d_1(z)rB_1(x,y)Rd_1(x)rB_1(z,y)$
$d_1(x)rB_1(z,y)Rd_1(x)rB_1(z,y) = 0$
Since R is semiprime ring then $d_1(x)rB_1(z,y) = 0$. Therefore $d_1(x)RB_1(z,y) = 0$, for some $z \in R$.

Lemma 3: For a two torsion free semiprime ring, a reverse derivation d_1 and a reverse biderivation B_1 are orthogonal if and only if $d_1(x)B_1(y,z) + B_1(x,y)d_1(z) = 0$, for every $x, y, z \in R$.

Proof: suppose B_1 is a reverse biderivation and d_1 is a reverse derivation such that
$d_1(x)B_1(y,z) + B_1(x,y)d_1(z) = 0$
Put $z = mz$, for some $m \in R$, in the above equation, we get
$d_1(x)B_1(y,mz) + B_1(x,y)d_1(mz) = 0$
$d_1(x)B_1(y,z)m + d_1(x)zB_1(y,m) + B_1(x,y)d_1(z)m + B_1(x,y)zd_1(m) = 0.$
Which implies $d_1(x)zB_1(y,m) + B_1(x,y)zd_1(m) = 0$, for some $m = x$, we obtain the above expression as in the form $d_1(x)zB_1(y,x) + B_1(x,y)zd_1(x) = 0$. Using the lemma 1, we have $d_1(x)RB_1(x,y) = 0$, which reduces to $d_1(x)RB_1(z,y) = 0$ by using the lemma 2, therefore d_1 and B_1 are orthogonal.
If d_1 and B are orthogonal reverse derivation and reverse biderivation then $d_1(x)RB_1(y,z) = B_1(x,y)Rd_1(z) = 0$
Thus for some $1 \in R$, $d_1(x)B_1(y,z) + B_1(x,y)d_1(z) = 0$.

Proposition 4: Let d_1 be a reverse derivation and B_1 a reverse biderivation of a ring R. Then the following identities holds
$d_1B_1(xy,z) = d_1\big(B_1(xy,z)\big)$
$= d_1\big(B_1(y,z)x + yB_1(x,z)\big)$
$= d_1(B_1(y,z)x) + d_1\big(yB_1(x,z)\big)$
$= d_1(x)B_1(y,z) + xd_1B_1(y,z) + B_1(x,z)d_1(y) + d_1B_1(x,z)y$
$d_1B_1(xy,z) = d_1B_1(x,z)y + d_1(x)B_1(y,z) + B_1(x,z)d_1(y) + xd_1B_1(y,z)$, for every $x, y, z \in R$

Lemma 5: For a two torsion free semiprime ring R and let $B_1: R \times R \to R$ be a Jordan reverse biderivation. Then B_1 is a reverse biderivation.

Theorem 1 : for a two torsion free semiprime ring, a reverse derivation d_1 and a reverse biderivation B_1 are orthogonal if and only if $d_1B_1 = 0$.

Proof: let a reverse derivation d_1 and a reverse biderivation B_1 such that $d_1B_1 = 0$. Use the proposition that $d_1B_1(xy,z) = d_1B_1(x,z)y + d_1(x)B_1(y,z) + B_1(x,z)d_1(y) + xd_1B_1(y,z)$, to get
$d_1(x)B_1(y,z) + B_1(x,z)d_1(y) = 0$
Hence by lemma 3, d_1 and B_1 are orthogonal.

Conversely, suppose that d_1 and B are orthogonal then $d_1(x)sB_1(y,z) = 0$, for every $x, y, z, s \in R$. Then
$d_1(d_1(x)sB_1(y,z)) = 0$
$d_1\big(sB_1(y,z)\big)d_1(x) + sB_1(y,z)d_1\big(d_1(x)\big) = 0$
$d_1B_1(y,z)sd_1(x) + B_1(y,z)d_1(s)d(x) + sB_1(y,z)d_1\big(d_1(x)\big) = 0$
In the above equation the last two terms are zero because d_1 and B_1 are orthogonal. Which implies $d_1B_1(y,z)sd_1(x) = 0$, now for $x = B_1(y,z)$, the above equation yields to $d_1B_1(y,z)sd_1B_1(y,z)$=0. Since R is semiprime , $d_1B_1(y,z) = 0$, for every $y, z \in R$.
Therefore $d_1B_1 = 0$.

Theorem 2: for a two torsion free semiprime ring, a reverse derivation d_1 and a reverse biderivation B_1 are orthogonal if and only if $B_1(x,y)d_1(x) = 0$, for every $x, y \in R$.

Proof: Assume that B_1 and d_1 are reverse biderivation and reverse derivation such that $B_1(x,y)d_1(x) = 0$, for every $x, y \in R$.
Linearizing above equation, we get
$B_1(x+z,y)d_1(x+z) = 0$, for some $z \in R$ then $\big(B_1(x,y) + B_1(z,y)\big)\big(d_1(x) + d_1(z)\big) = 0$
$B_1(x,y)d_1(x) + B_1(x,y)d_1(z) + B_1(z,y)d_1(x) + B_1(z,y)d_1(z) = 0$
$B_1(x,y)d_1(z) + B_1(z,y)d_1(x) = 0$ (1)
Taking z by zs in the above Eq.(1), we find that
$B_1(x,y)d_1(zs) + B_1(zs,y)d_1(x) = 0$
$B_1(x,y)d_1(s)z + B_1(x,y)sd_1(z) + B_1(s,y)zd_1(x) + sB_1(z,y)d_1(x) = 0$ (2)
From equation (1), $B_1(x,y)d_1(z) = -B_1(z,y)d_1(x)$, implies $B_1(x,y)d_1(s) = -B(s,y)d_1(x)$ and $B_1(z,y)d_1(x) = -B_1(x,y)d_1(z)$.
Now from equation (2), we have
$B_1(x,y)sd_1(z) - B_1(s,y)d_1(x)z + B_1(s,y)zd_1(x) - sB_1(x,y)d_1(z) = 0$.
Replacing z by $d_1(x)$ in the above equation, we get
$B_1(x,y)sd_1(d_1(x)) - B_1(s,y)d_1(x)d_1(x) + B_1(s,y)d_1(x)d_1(x) - sB_1(x,y)d_1(d_1(x)) = 0$
$[B_1(x,y),s]{d_1}^2(x) = 0$
Put s by sw in the above equation, we find that
$[B_1(x,y),sw]{d_1}^2(x) = 0$
$[B_1(x,y),s]w{d_1}^2(x) + s[B_1(x,y),w]{d_1}^2(x) = 0 = 0$
$[B_1(x,y),s]w{d_1}^2(x) = 0$.
Hence by Lemma 2, one can obtain
$[B_1(m,y),s]R{d_1}^2(x) = 0$, for every $s, m \in R$. (3)
Replacing x by xu in the above equation, we obtain
$[B_1(m,y),s]R{d_1}^2(xu) = 0$
$[B_1(m,y),s]Rd_1(d(u)x + ud_1(x)) = 0$
$[B_1(m,y),s]Rd(x)d_1(u) + [B_1(m,y),s]Rx{d_1}^2(x) + [B_1(m,y),s]R{d_1}^2(x)u + [B_1(m,y),s]Rd_1(x)d_1(u) = 0$.
Using equation (3), we have $2[B_1(m,y),s]Rd_1(x)d_1(u) = 0$, since R is two torsion free, which reduces to $[B_1(m,y),s]Rd_1(x)d_1(u) = 0$. Substituting $x = xz$ in the above equation, we find that
$[B_1(m,y),s]Rd_1(xz)d_1(u) = 0$
$[B_1(m,y),s]Rd_1(z)xd_1(u) + [B_1(m,y),s]Rzd_1(x)d_1(u) = 0$,
$[B_1(m,y),s]Rd_1(z)xd_1(u) = 0$, in particular $[B_1(m,y),s]Rd_1(z)R[B_1(m,y),s]Rd_1(z) = 0$, by the semiprimeness of R, it implied that $[B_1(m,y),s]Rd_1(z) = 0$, again in particular we can take $[B_1(m,y),s]R[B_1(m,y),s] = 0$, again by the semiprimeness of R, it implied that $[B_1(m,y),d_1(x)] = 0$, for some $s = d_1(x)$.
The above equation yields to $B_1(m,y)d_1(x) = d_1(x)B_1(m,y)$, for every $m, x, y \in R$.
Hence with the equation (2), it is clear that $B_1(x,y)d_1(z) + d_1(x)B_1(z,y) = 0$
Thus using lemma 3, we get the required result.
Conversely, for d_1 and B_1 are orthogonal, then $d_1(x)RB_1(x,y) = 0$,
Therefore $d_1(x)RB_1(x,y) = 0$, for every $x, y \in R$.

Theorem 3:For a two torsion free semiprime ring R, then a reverse derivation d_1 and a reverse biderivation B_1 are orthogonal if and only if d_1B_1 is a biderivation.

Proof:

Let d_1 be a reverse derivation and B_1 be a reverse biderivation such that d_1B_1 is a biderivation, then $d_1B_1(xy,z) = (d_1B_1)(x,z)y + xd_1B_1(y,z)$, hence the proposition 4, becomes $d_1(x)B_1(y,z) + B_1(x,z)d_1(y) = 0$, for every $x,y,z \in R$. So by lemma 3, we can clear that d_1 and B_1 are orthogonal.

Conversely, let d_1 and B_1 are orthogonal, then by lemma 3, it is clear that $d_1(x)B_1(y,z) + B_1(x,z)d_1(y) = 0$, again by proposition 4, we have $d_1B_1(xy,z) = (d_1B_1)(x,z)y + xd_1B_1(y,z)$, from this we conclude d_1B_1 is a biderivation.

Theorem 4: For a two torsion free semiprime ring R. Then a reverse biderivation B_1 and a reverse derivation d_1 are orthogonal, if and only if $d_1(x)B_1(x,y) + B_1(x,y)d_1(x) = 0$, for every $x,y \in R$.

Proof:

Let d_1 and B_1, reverse derivation and reverse biderivation satisfies $d_1(x)B_1(x,y) + B_1(x,y)d_1(x) = 0$, for every $x,y \in R$. Take $y = x$ in the proposition

$d_1B_1(xy,z) = d_1B_1(x,z)y + d_1(x)B_1(y,z) + B_1(x,z)d_1(y) + xd_1B_1(y,z)$, then we get

$d_1B_1(x^2,z) = d_1B_1(x,z)x + d_1(x)B_1(x,z) + B_1(x,z)d_1(x) + xd_1B_1(x,z)$.

Therefore $d_1B_1(x^2,z) = d_1B_1(x,z)x + xd_1B_1(x,z)$

d_1B_1 is a Jordan reverse biderivation and using lemma 5 and theorem 3, we can conclude d_1 and B_1 are orthogonal.

Conversely, let d_1 and B_1 are orthogonal. Then using lemma 3, we get $d_1(x)B_1(y,z) + B_1(x,z)d_1(y) = 0$, for each $x,y,z \in R$. Substituting $y = x$ in the above relation to get the required result.

Now we are in the position to formulate the main theorem which as shown below

Theorem 5: For a two torsion free semiprime ring R. Let d_1 a reverse derivation and B_1 be a reverse biderivation. Then d_1 and B_1 are orthogonal if and only if the following conditions are equivalent

(v) $d_1B_1 = 0$,

(vi) $B_1(x,y)d_1(x) = 0$,

(vii) d_1B_1 is a biderivation,

(viii) $d_1(x)B_1(y,z) + B_1(x,y)d_1(z) = 0$, for every $x,y,z \in R$.

Proof: proof of this theorem follows from the theorems 1,2, 3 and lemma 3.

Corollary 1: Let R be a prime ring of characteristic not equal to 2. Let B_1 and D_1 be two symmetric reverse biderivation and reverse derivation of R. If B_1 and D_1 are satisfy one of the conditions of Theorem 5, then either $B_1 = 0$ or $D_1 = 0$.

Corollary 2: Let R be a two torsion free semiprime ring and let B_1 be a symmetric reverse biderivation of R, ${B_1}^2$ is a symmetric biderivation, then $B_1 = 0$.

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