On polynomials over valued fields

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Abstract. Polynomials are well known for their ability to improve their properties and for their applicability in the interdisciplinary fields of engineering and science. Many problems arising in engineering, physics, chemistry and other disciplines of science are mathematically constructed by them. Polynomials are originally algebraic structures which are investigated in many branches of mathematics. The investigation of factorization, irreducibility and roots of polynomials are from the most important aspects of study of polynomials. This paper is devoted to explore polynomials from an algebraic view point in valuation theory. The special feature is the focus on application of valuation theory to explore some properties of polynomials. More precisely, we present the newest irreducibility criteria, and some results in relation to the roots of polynomials with coefficients in certain valued fields. In this study, it is used some applicable tools of valuation theory such as lifting of polynomials.

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I.Introduction

Polynomials are one of the significant concepts of mathematics which appear in many areas of science. They are used to form polynomial equations, which encode a wide range of problems from elementary word problems to complicated scientific problems; they are used to define polynomials functions, which appear in settings ranging from basic physics, chemistry and engineering to economics and social science. For example, a roller coaster would use polynomials to model the curves, while a civil engineer would use polynomials to design roads, buildings and other structures. Specially in mathematics, they are used in calculus and numerical analysis to approximate other functions or to construct polynomial rings and algebraic varieties, which are central concepts in algebra and valuation theory. Consequently, the comprehension of polynomials is important throughout multiple tasks because it holds a large position in various academic subjects.
It is worthwhile mentioning that polynomials are originally algebraic structures investigated in many branches of mathematics, specially in valuation theory. Valuation theory is a branch of math, which forms a solid link between algebra, number theory and analysis. This paper is devoted to examine the newest research on polynomials with coefficients in valued fields.
One of the most important aspects of studying polynomials is to determine whether a polynomial is irreducible or not. It is noted that the property of irreducibility depends on the nature of the coefficients that are accepted for the possible factors, that is, the field or ring to which the coefficients of the polynomial and its possible factors are supposed to belong. Irreducible polynomials appear naturally in the study of polynomial factorization and algebraic field extensions. In this paper, we present the latest irreducibility criteria, which generalize the classical and known ones in valuation theory.
Moreover, one of the other subjects which holds a prominent position in investigating polynomials is to discuss the roots of polynomials. Finding roots of a polynomial is an extremely important work in math because many problems need to be solved by determining the roots of the polynomials. In fact, factorization of a polynomial and determination

of its roots are closely related to each other. This paper also deals with this object from an algebraic point of view over valued fields. More precisely, we study some results concerned with finding the roots of polynomials over valued fields under certain conditions. In this direction, it is used applicable tools of valuation theory such as minimal pairs, complete distinguished chains and lifting of polynomials, which are beneficial to obtain new results about polynomials.

II.Preliminaries

Valuation theory has become important through its applications in many fields of mathematics. It arose in the early part of the twentieth century in connection with number theory and has many important applications to math; specially to algebra, geometry and analysis: the classical application to the study of algebraic curves and to Dedekind and Prufer domains; the close connection to the famous resolution of the singularities problem; the study of the absolute Galois group of a field; the connection between ordering, valuations, and quadratic forms over a formally real field; the application to real algebraic geometry; the study of non-commutative rings; etc. Valuation theory has not only produced new methods which could be profitably used in number theoretical research, but it has also led to a change of viewpoint. The special feature of this paper is its focus on application of valuation theory to explore properties of polynomials. Throughout, we assume the reader to be familiar with the elementary notions of algebra and valuation theory (see for example [1-3]).
Firstly, the basic notion that must be defined is the concept of a valued field.

Definition 2.1 *Let K be a field, G be a totally ordered abelian group and ∞ be a symbol that satisfies, for all $g \in G$, in the axioms: $\infty = \infty + \infty = g + \infty = \infty + g$, and $\infty \geq g$.*
A valuation v of K is a surjective map

$$v: K \to G \cup \{\infty\}$$

which satisfies the following properties for all $x, y \in K$:
1. $v(x) = \infty \leftrightarrow x = 0$,
2. $v(xy) = v(x) + v(y)$,
3. $v(x + y) \geq \min\{v(x), v(y)\}$.

(K, v) is called a valued field. For a valued field (K, v), G is said to be the value group of v and denoted by $G(K)$. The set $R_v = \{x \in K | v(x) \geq 0\}$ is said to be the valuation ring of v that has the unique maximal ideal $M_v = \{x \in K | v(x) > 0\}$. R_v/M_v is called the residue field of v and denoted by $R(K)$.

Let K_2/K_1 be an extension of fields. A valued field (K_2, v_2) is called an extension of a valued field (K_1, v_1) if $v_2|_{K_1} = v_1$. This statement is denoted by $(K_2, v_2)/(K_1, v_1)$ or briefly by K_2/K_1. A valued field (K, v) is called henselian if v has a unique extension to every algebraic extension of K, and discrete if its value group is the ring of integers $\mathbb{Z}$. When v is henselian, $\bar{v}$ is denoted the unique extension of v to a fixed algebraic closure $\overline{K}$ of K. For an overfield K' of K contained in $\overline{K}$, we shall denote by $G(K')$ and $R(K')$ respectively the value group and the residue field of the valuation v' of K' obtained by restricting $\bar{v}$ to K'. For any $\alpha \in R_{v'}$ and polynomial $f(x) \in R_{v'}[x]$, we let $\bar{\alpha}$ and $\bar{f}(x)$ denote the canonical image of α and $f(x)$ in $R(K')$ and $R(K')[x]$, respectively. By the degree of an element $\alpha \in \overline{K}$, we shall mean the degree of the extension $K(\alpha)/K$ and denote it by deg α.
After mentioning these necessary notions, let us recall some well-known and classical irreducibility criteria. We begin with Eisenstein's criterion that gives a sufficient condition for a polynomial with integer coefficients to be irreducible over the rational numbers that is, for it to not be factorizable into the product of non-constant polynomials with rational coefficients. Both Eisenstein' statement and proof are virtually identical to how we would formulate them today [4]. Eisenstein was actually concerned with the lemniscate, where the relevant question was irreducibility of polynomials with coefficients in the Gaussian integers, rather than in the ordinary integers, but, as he observed, the statement and proof are identical in either case.

Theorem 2.2 (Eisenstein irreducibility criterion) *Let $f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n$ be a polynomial with coefficient in the ring $\mathbb{Z}$ of integers. Suppose that there exists a prime number p such that*
1. a_0 is not divisible by p ($p \nmid a_0$),
2. a_i is divisible by p for $1 \leq i \leq n$ ($p | a_i$),
3. a_n is not divisible by p^2 ($p^2 \nmid a_n$).

$f(x)$ is irreducible over the field $\mathbb{Q}$ of rational numbers.
A polynomial satisfying above conditions is called an Eisenstein polynomial.

Example: Consider the $p$the cyclotomic polynomial

$$x^{p-1}+x^{p-2}+\cdots+x+1=\frac{x^p-1}{x-1}$$

On changing x to $x+1$ it becomes

$$\frac{(x+1)^p-1}{(x+1-1)}=x^{p-1}+\binom{p}{1}x^{p-2}+\cdots+\binom{p}{p-1}$$

and hence irreducible over $\mathbb{Q}$.

The Dumas criterion generalized the the Eisenstein's criterion [5].

Theorem 2.3 (Dumas irreducibility criterion) *Let $f(x)=a_0x^n+a_1x^{n-1}+\cdots+a_n$ be a polynomial with coefficients in the ring $\mathbb{Z}$. Suppose there exists a prime p whose exact power p^{r_i} dividing a_i (where $r_i=\infty$ if $a_i=0$) satisfy*

1. $r_0=0$,
2. $(r_i/i)>(r_n/n)$ for $1\le i\le n-1$,
3. $\gcd(r_n,n)$ equals 1.

Then $f(x)$ is irreducible over $\mathbb{Q}$.
A polynomial satisfying above conditions is called an Eisenstein-Dumas polynomial. For example, for polynomial x^3+3x^2+9x+9, we can not use Eisenstein irreducibility criterion, but with choosing $p=3, r_1=1, r_2=2, r_3=2$, it is irreducible over $\mathbb{Q}$ by Dumas criterion. However, there are many polynomials where are not satisfied in the conditions of Dumas criterion. Here valuation theory can be used to extend the area where tests can be satisfied. A special kind of valuations defined below, which are from the most applicable valuations and used extensively in number theory and algebraic geometry, can be employed to generalize Dumas criterion as follows.

Definition 2.4 *For a given prime number p, let v_p stand for the mapping $v_p:\mathbb{Q}^*\to\mathbb{Z}$ defined as follows. Write any non zero rational number $x=p^r\frac{a}{b}$, $p\nmid ab$. Set $v_p(x)=r$. Then*

1. $v_p(0)=\infty$,
2. $v_p(xy)=v_p(x)+v_p(y)$,
3. $v_p(x+y)\ge\min\{v_p(x),v_p(y)\}$.

v_p is called the p-adic valuation of $\mathbb{Q}$.

Kurschak presented Dumas criterion for fields with p-adic valuations [6].

Theorem 2.5 (Dumas criterion with p-adic valuations) *Let $f(x)=a_0x^n+a_1x^{n-1}+\cdots+a_n$ be a polynomial with coefficient in $\mathbb{Z}$. Suppose there exists a prime p such that*

1. $v_p(a_0)=0$,
2. $v_p(a_i)/i>v_p(a_n)/n$ for $1\le i\le n-1$, and
3. $v_p(a_n)$ is coprime to n.

Then $f(x)$ is irreducible over $\mathbb{Q}$.

Theorem 2.6 (classical Schönemann irreducibility criterion) *If a polynomial $f(x)$ belonging to $\mathbb{Z}[x]$ has the form $f(x)=\phi(x)^s+pm(x)$ where p is a prime number,*

1. $\phi(x)$ belonging to $\mathbb{Z}[x]$ is a monic polynomial which is irreducible modulo p,
2. $\phi(x)$ is coprime to $m(x)$ modulo p, and
3. the degree of $m(x)$ is less than the degree of $f(x)$.

Then $f(x)$ is irreducible over $\mathbb{Q}$.
Eisenstein's criterion is easily seen to be a particular case of Schönemann criterion [7] by setting $\phi(x)=x$.

III. New generalizations of irreducibility criteria and roots of polynomials with coefficients in valued fields

Recently, some of researchers in valuation theory have worked on irredusility criteria. They try to generalize them in order that a wider range of polynomials can be examined. From almost the last three decades, a new series of papers have been published in math journals deal with this object.

In 1995, Popescu and Zaharescu gave an irreducibility criterion for polynomials over a complete discrete rank one valued field, also named a local field (see [2] for the definition of such valued fields) which generalizes the usual Eisenstein irreducibility criterion as follows [8].

Theorem 3.1 *Let (K, v) be a complete discrete rank one valued field. Let*

$$g(x) = x^n + a_1 x^{n-1} + \cdots + a_n$$

be a polynomial of $K[x]$. Assume that $v(a_n) = s$ is a positive integer relatively prime to n. Then g is an irreducible polynomial if and only if $v(a_i) > \frac{is}{n},\ 1 \le i \le n-1$.

More generally, Popescu and Zaharescu examined the structure of irreducible polynomials over local fields. They defined a system of invariant factors for each monic irreducible polynomial over a local field such that these invariant are characteristic. i.e., by using invariants we may describe the set of irreducible polynomials over a local field. There some important invariants associated to algebraic elements were defined. one of the most important of such invariants is the invariant $\delta_K(\theta)$ referred to as the main invariant of an algebraic element θ over K. They defined the invariant $\delta_K(\theta)$ for algebraic elements $\theta \in \overline{K} \backslash K$ when (K, v) is a complete discrete rank one valued field. By the main invariant of an algebraic element θ is defined the supremum of the set $M(\theta, K)$ defined by

$$M(\theta, K) = \{\bar{v}(\theta - \xi) |\ \xi \in \overline{K},\ \deg\xi < deg\theta\}.$$

For a complete discrete rank one valued field (K, v), they also introduced in [8] the notions of distinguished pairs, complete distinguished chains and lifting of polynomials. After that, many notions and results of [8] have been generalized to henselian valued fields or arbitrary valued fields (see for example, [9-11]). For completeness, we give a concrete example of a valued field of rank greater than one.

Example: Let v_x denote the x-adic valuation (see [3] for the definition of x-adic valuation) of the field $\mathbb{Q}(x)$ of rational functions in x trivial on $\mathbb{Q}$ with $v_x(\sum a_i x^i) = min\{v(a_i)\}$ and v_p denote the p-adic valuation of $\mathbb{Q}$. For any non-zero polynomial $f(x)$ belonging to $\mathbb{Q}(x)$, we shall denote by f^* the constant term of the polynomial $f(x)/x^{v_x(f(x))}$. Let v be the mapping from non-zero elements of $\mathbb{Q}(x)$ to $\mathbb{Z} \times \mathbb{Z}$ (lexicographically ordered) defined on $\mathbb{Q}[x]$ by

$$v(f(x)) = (v_x f(x), v_p(f^*)).$$

Then v gives a valuation on $\mathbb{Q}(x)$.

Let us remark some of the most important notions which have recently been useful tools of valuation theory (see for example [12-18]).

A pair (θ, α) of elements of $\overline{K}$ with $\deg\theta > deg\alpha$ is called a distinguished pair (more precisely a (K, v)-distinguished pair) if α is an element of smallest degree over K such that $\bar{v}(\theta - \alpha) = \delta_K(\theta)$. Distinguished pairs give rise to distinguished chains in a natural manner. A chain $\theta = \theta_0, \theta_1, \ldots, \theta_s$ of elements of $\overline{K}$ is called a complete (often called saturated) distinguished chain for θ (with respect to (K, v)) if (θ_i, θ_{i+1}) is a (K, v)-distinguished pair for $0 \le i \le s-1$ and $\theta_s \in K$. The concept of lifting of a polynomial is another important tool for investigating the properties of irreducible polynomials with coefficients in valued fields (see, [16], [17] and [19] for example). We briefly recall a survey of it.

If $f(x)$ is a fixed nonzero polynomial in $K[x]$, then using the Euclidean algorithm, each $F(x) \in K[x]$ can be uniquely represented as a finite sum $\sum_{i \ge 0} F_i(x) f(x)^i$, where for any i, the polynomial $F_i(x)$ is either 0 or has degree less than that of $f(x)$. This representation will be referred to as the f-expansion of $F(x)$.

For a pair $(\alpha, \delta) \in \overline{K} \times G(\overline{K})$, the valuation $\overline{w}_{\alpha,\delta}$ of $\overline{K}(x)$ defined on $\overline{K}[x]$ by

$$\overline{w}_{\alpha,\delta}(\textstyle\sum_i c_i (x - \alpha)^i) = \min_i \{\bar{v}(c_i) + i\delta\},\ c_i \in \overline{K},$$

will be referred to as the valuation defined by the pair (α, δ). The description of $\overline{w}_{\alpha,\delta}$ on $K(x)$ is given by the already known theorem stated below (see [20]).

Theorem 3.2 *Let $\overline{w}_{\alpha,\delta}$ be the valuation of $\overline{K}(x)$ defined by a minimal pair (α, δ) (a pair $(\alpha, \delta) \in \overline{K} \times G(\overline{K})$ is* *said to be minimal if whenever* $\beta \in \overline{K}$ *satisfies* $\bar{v}(\alpha-\beta) \geq \delta$, *then* $deg\alpha \leq deg\beta$*) and* $w_{\alpha,\delta}$ *be the valuation of* $K(x)$ *obtained by restricting* $\overline{w}_{\alpha,\delta}$. *If* $f(x)$ *is the minimal polynomial of* α *over* K *of degree* n *and* λ *is an element of* $G(\overline{K})$ *such that* $w_{\alpha,\delta}(f(x)) = \lambda$, *then the following hold.*

1. For any $F(x)$ belonging to $K[x]$ with f-expansion $\sum_i F_i(x)f(x)^i$, we have $w_{\alpha,\delta}(F(x)) = \min_i\{\overline{v}(F_i(\alpha)) + i\lambda\}$.
2. Let e be the smallest positive integer such that $e\lambda \in G(K(\alpha))$ and $h(x)$ belonging to $K[x]$ be a polynomial of degree less than n with $\overline{v}(h(\alpha)) = e\lambda$, then the $w_{\alpha,\delta}$-residue $\left(\overline{f(x)^e/h(x)}\right)$ of $(f(x)^e/h(x))$ is transcendental over $R(K(\alpha))$ and the residue field of $w_{\alpha,\delta}$ is canonically isomorphic to $R(K(\alpha))\left(\left(\overline{f(x)^e/h(x)}\right)\right)$.

Using the canonical homomorphism from the valuation ring of v onto its residue field, one can lift any monic irreducible polynomial having coefficients in $R(K)$ to yield a monic irreducible polynomial with coefficients in K. The description of the residue field of $w_{\alpha,\delta}$ given in Theorem 3.2(2) led Popescu and Zaharescu to generalize the notation of usual lifting (see [8]). In fact, they introduced the notation of lifting of a polynomial belonging to $R(K(\alpha))[Y]$ (Y an indeterminate) with respect to a minimal pair (α,δ) as follows.
For a (K,v)-minimal pair (α,δ), let $f(x)$, n, λ, and e be as in Theorem 3.2. As in [8], a monic polynomial $F(x)$ belonging to $K[x]$ is said to be a lifting of a monic polynomial $Q(Y)$ belonging to $R(K(\alpha))[Y]$ having degree $m \geq 1$ with respect to (α,δ) if there exists $h(x) \in K[x]$ of degree less than n such that

1. $\deg F(x) = emn$,
2. $w_{\alpha,\delta}(F(x)) = mw_{\alpha,\delta}(h(x)) = em\lambda$,
3. the $w_{\alpha,\delta}$-residue of $F(x)/h(x)^m$ is $Q((\overline{f^e/h}))$.

Now, we have necessary tools of valuation theory, let us present the latest irreducibility criteria, which generalize the classical and known ones mentioned in the previous section.
In 1997, Khanduja and Saha by usable tools introduced above generalized the Eisenstein-Dumas-Kurschak criterion as follows [21].

Theorem 3.3 *Let* v *be a valuation of a field K with value group* G *and* $f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n$ *be a polynomial over* K. *If*

1. $v(a_0) = 0$,
2. $v(a_i)/i \geq v(a_n)/n$ for $1 \leq i \leq n$,
3. there does not exit any integer $d > 1$ dividing n such that $v(a_n)/n \in G$.

Then $f(x)$ is irreducible over K.

The following example intelligibly shows that how Theorem 3.3 generalizes the former criteria.

Example: Let $F(x,y) = g(y)x^s + h(y)$ be a polynomial over a field L in independent variables x, y . If $g(y), h(y)$ have no common factors and $\deg g(y) - \deg h(y)$ is coprime to s, then $F(x,y)$ is irreducible over L. For the verification, it is sufficient to regard $F(x,y)/g(y)$ as a polynomial in x with coefficients over the field $K = L(y)$ with valuation on K defined by $v(a(y)/b(y)) = \deg b(y) - \deg a(y)$ and apply Theorem 3.3.

The following theorem proved by Bhatia and Khanduja presented a different criterion for irreducibility [22].

Theorem 3.4 *Let* v *be a valuation of a field* K *with value group the set of integers. Let* $g(x) = x^m + a_1x^{m-1} + \cdots + a_m$ *be a polynomial with coefficients in* K *such that* $v(a_i)/i \geq v(a_m)/m$ *for* $1 \leq i \leq m-1$. *Let* r *denote* $gcd(v(a_m), m)$ *and* b *be an element of* K *with* $v(b) = v(a_m)/r$. *Suppose that the polynomial* $z^r + \overline{(a_m/b^r)}$ *in the indeterminate* z *is irreducible over the residue field of* v. *Then* $g(x)$ *is irreducible over* K.

It is remarked that a polynomial $f_1(x_1) + \cdots + f_n(x_n)$ with complex coefficients is irreducible provided the degrees of $f_1(x_1), \ldots, f_n(x_n)$ have greatest common divisor one [23]. The following theorem has its roots in this statement by using valuations.

Theorem 3.5 *Let f(x) and g(y) be nonconstant polynomials with coefficients in a field* K. *Let* c *and* c_0 *denote respectively the leading coefficients of* $f(x)$ *and* $g(y)$ *and* n, m *their degrees. If* $gcd(m,n) = r$ *and if* $z^r -$

(c_0/c) is irreducible over K, then so is $f(x)-g(y)$.

In this context, a question can arise is that "when is a translate $g(x+a)$ of a given polynomial $g(x)$ with coefficients in a valued fleld (K,v) an Eisenstein-Dumas polynomial with respect to v?" In 2010, such polynomials have been characterized using distinguished pairs [14].

Theorem 3.6 *Let v be a henselian valuation of a field K. Let $g(x)$ belonging to $R_v[x]$ be a monic polynomial of degree e having a root θ. Then for an element a of K, $g(x+a)$ is an Eisenstein-Dumas polynomial with respect to v if and only if (θ,a) is a distinguished pair and $K(\theta)/K$ is a totally ramified extension of degree e.*

Theorem 3.6 has an interesting corollary as follows.

Corollary 3.7 *Let $g(x)=\sum_{i=0}^{e} a_i x^i$ be a monic polynomial with coefficients in a henselian valued field (K,v). Suppose that the characteristic of the residue field of v does not divide e. If there exists an element b belonging to K such that $g(x+b)$ is an Eisenstein-Dumas polynomial with respect to v, then so is $g(x-\frac{a_{e-1}}{e})$.*

In [21], Khanduja and Saha gave a generalization of classical Schönemann irreducibility criterion using the theory of extensions of a valuation defined on K to a simple transcendental extension of K which was initiated by MacLane [24] and developed further by Popescu at el. In 2008, Brown [25] has given a different proof of this result and obtained some results about roots of polynomials. In fact, he considered the following sense for generalized Schönemann polynomials over a valued field (K,v).

Definition 3.8 *A polynomial $h(x)\in R_v[x]$ is called a generalized Schönemann polynomial over (K,v) if it can be written in the form $k(x)=p(x)^e+th(x)$ where $e\geq 1$; $p(x)\in R_v[x]$ is monic with $\bar{p}(x)$ irreducible over $R(K)$; $h(x)\in R_v[x]$ has degree less than $edegp(x)$; $\bar{p}(x)$ does not divide $\bar{h}(x)$; and, finally, $t\in R_v$ is nonzero and $v(t)\not\in sG(K)$ for any divisor $s>1$ of e.*

If a generalized Schönemann polynomial f is tame, i.e., a root of f generates a tamely ramified extension of K, Brown gave a best-possible criterion for when the existence in a henselian extension field K of an approximate root of f guarantees the existence of an exact root of f in the extension field of K [25]. More precisely, he established the following result about the existence of roots of generalized Schönemann polynomials over valued fields.

Theorem 3.9 *Suppose $k(x)=p(x)^e+th(x)$ is a generalized Schönemann polynomial over (K,v) with $\bar{p}(x)$ separable over $R(K)$ and e not divisible by the characteristic of K. If a henselian extension (K',v') of (K,v) has an element α with $v'(k(\alpha))>v(t)$, then $k(x)$ has a root in K'.*

In [26], Brown introduced a class of irreducible polynomials p and their invariants (see [26, Sec. 1]) and obtained some results about roots of polynomials in this class. Actually, he gave a class p of monic irreducible polynomials over K such that to each $g(x)$ belonging to p, there corresponds a smallest constant γ_g belonging to $G(\overline{K})$ with the property that whenever $\bar{v}(g(\beta))$ is more than γ_g with $K(\beta)$ a tamely ramified extension of (K,v), then $K(\beta)$ contains a root of $g(x)$.

Theorem 3.10 *Suppose that (K,v) is henselian and $h(x)\in p$. Suppose that α is an element of a tamely ramified finite degree extension (K',v') of (K,v) with $v'(h(\alpha))>\gamma_h$ (γ_h is the invariant of h). Then there is a root of $h(x)$ in K'.*

In 2010, it was given new results about roots of irreducible polynomials (see [13]), then using the previous results, Khanduja and Saha extended the generalized Schönemann-Eisenstein irreducibility citerion in [27].

Theorem 3.11 *Let v be a discrete valuation of K with value group $\mathbb{Z}$ and π be an element of K with $v(\pi)=1$. Let $f(x)$ belonging to $R_v[x]$ be a monic polynomial of degree m such that $\bar{f}(x)$ is irreducible over $R(K)$. Let $F(x)$ belonging to $R_v[x]$ be a monic polynomial having $f(x)$-expansion $\sum_{i=0}^{n} A_i(x)f(x)^i$. Assume that there exists $s\leq n$ such that π does not divide the content of $A_s(x)$, π divides the content of each $A_i(x), 0\leq i\leq s-1$ and π^2 does not divide the content of $A_0(x)$.Then $F(x)$ has an irreducible factor of degree sm over the completion $(\widetilde{K},\tilde{v})$*

of (K, v) which is a Schönemann polynomial with respect to $\tilde{v}$ and $f(x)$.

Theorem 3.12 *Let (K, v), π be as above and $F(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$ be a polynomial over R_v satisfying the following conditions for an index $s \leq n - 1$.*
1. $\pi | a_i$ for $0 \leq i \leq s - 1$, $\pi^2 \nmid a_0$, $\pi \nmid a_s$.
2. The polynomial $x^{n-s} + \bar{a}_{n-1}x^{n-s-1} + \cdots + \bar{a}_s$ is irreducible over the residue field of v.
3. $\bar{d} \neq \bar{a}_s$ for any divisor d of a_0 in R_v.

Then $F(x)$ is irreducible over K.

On the other hand, it was established a generalization of the classical Eisenstein irreducibility criterion by providing a bound on the degrees of factors of a polynomial with integer coefficients [28] as follows.

Theorem 3.13 *Let $f(x) = a_n x^n + \cdots + a_0 \in \mathbb{Z}[x]$ be a polynomial and suppose there is a prime p such that p does not divide a_n, p divides a_i for $i = 0, \ldots, n - 1$, and for some k with $0 \leq k \leq n - 1$, p^2 does not divide a_k. Let k_0 be the smallest such value of k. If $f(x) = g(x)h(x)$, a factorization in $\mathbb{Z}[x]$, then $min(deg(g(x)), deg(h(x))) \leq k_0$. In particular, for a primitive polynomial $f(x)$, if $k_0 = 0$, then $f(x)$ is irreducible, and if $k_0 = 1$ and $f(x)$ does not have a root in $\mathbb{Q}$, then f(x) is irreducible.*

This has an interesting corollary about solvability of polynomials by radicals.

Corollary 3.14 *Let $p \geq 5$ be prime and let $f_0(x) = x^p - p^p x + p$ and $f_1(x) = x^p - p2^p x + p^2$. Then neither $f_0(x)$ nor $f_1(x)$ is solvable by radicals.*

In 2016, a family of tests for irreducibility of polynomials with coefficients in $\mathbb{Z}$ was classified [29]. Before, we need the following definition.

Definition 3.15 *Let q be a prime. Then $f(x) \in \mathbb{Z}[x]$ satisfies condition (q, l) if its (mod q) reduction $\overline{f}(x)$ is not divisible by any irreducible polynomial of degree l in $\mathbb{F}_q[x]$.*

Theorem 3.16 *Let $f(x) = a_n x^n + \cdots + a_0 \in \mathbb{Z}[x]$ be a polynomial and suppose there is a prime p such that p does not divide a_n, p divides a_i for $i = 0, \ldots, n - 1$, and for some k with $1 \leq k \leq n - 1$, p^2 does not divide a_k. Let k_0 be the smallest such value of k. Suppose furthermore that for some prime $q \neq p$, $f(x)$ satisfies condition (q, l) for $1 \leq l \leq k_0$. Then $f(x)$ is irreducible in $\mathbb{Z}[x]$.*

In 2018, Jakhar with defining the following notion extended Theorem 3.16 as follows [30].

Definition 3.17 *Let q be a prime number. Then for a polynomial $f(x) \in \mathbb{Z}[x]$ let $\bar{f}(x)$ denote the polynomial obtained by reducing its coefficients modulo q. We say that $f(x)$ satisfies condition (q, l) if $\bar{f}(x)$ is not divisible by any irreducible polynomial of degree l in $\mathbb{F}_q[x]$.*

Theorem 3.18 *Let $f(x) = A_s(x)\phi(x)^s + A_{s-1}(x)\phi(x)^{s-1} + \cdots + A_1(x)\phi(x) + A_0(x)$ in $\mathbb{Z}[x]$ be any primitive polynomial, where $deg(A_i(x)) < deg(\phi(x))$ for all i and $\phi(x)$ is a monic polynomial. Suppose there is a prime number p such that $\phi(x)$ is irreducible modulo p, $p \nmid A_s(x)$, and p divides $A_i(x)$ for $i = 0, 1, \ldots, s - 1$. Assume that not all of the polynomials $A_0(x), \ldots, A_{s-1}(x)$ are divisible by p^2, and let $k < s$ be the smallest integer such that $p^2 \nmid A_k(x)$. Suppose either $deg(A_s(x)) + k(deg(\phi(x)) = 0$, or for some prime $q \neq p$, the leading coefficient of $f(x)$ is not divisible by q and $f(x)$ satisfies condition (q, l) for $1 \leq l \leq deg(A_s(x)) + k(deg(\phi(x)))$. Then $f(x)$ is irreducible in $\mathbb{Z}[x]$.*

See the following example by using the above theorem.

Example: Let a_0, a_1 be odd integers divisible by the prime $p = 5$ and suppose 5^2 does not divide a_1 but divides a_0. Let $f_n(x) = (x^2 + 2)^n + a_1(x^2 + 2) + a_0$, where $n \geq 2$ and $n \neq 2$ (mod 3). Observe that $k = 1$ with respect to p and $\phi(x) = x^2 + 2$. One can see that $\bar{f}(x)$ is not divisible by any irreducible polynomial in $\mathbb{F}_2[x]$ of degree $l, 1 \leq l \leq 2$. $f_n(x)$ is irreducible.

By using some tools of valuation theory Theorem 3.13 can be extended with a much weaker hypothesis in a more general setup for polynomials having coefficients from the valuation ring of an arbitrary valued field [31]. The following theorem also extends the Dumas irreducibility criterion.

Theorem 3.19 *Let p be a prime number and let $f(x) = a_n x^n + \cdots + a_0$ be a polynomial with integer coefficients. Suppose that p does not divide a_s for some $s \leq n$ and that $a_j \neq 0$ for some j with $0 \leq j < s$. For $0 \leq i < s$ let r_i be the largest positive integer such that p^{r_i} divides a_i (where $r_i = \infty$ if $a_i = 0$). Let $k(< s)$ be the smallest non-negative integer such that*

$$\min_{0 \leq i < s} \frac{r_i}{s-i} \geq \frac{r_k}{s-k}.$$

Suppose further that r_k and $(s-k)$ are relatively prime. Then $f(x)$ has an irreducible factor of degree at least $(s-k)$ over $\mathbb{Q}$. Moreover, $f(x)$ has an irreducible factor $g(x)$ over the field $\mathbb{Q}_p$ of p-adic numbers with $s-k \leq \deg g(x) \leq s$.

Theorem 3.20 *Let v be a valuation of a field K with value group $G(K)$ and valuation ring R_v having maximal ideal M_v. Let $\phi(x)$ belonging to $R_v[x]$ be a monic polynomial of degree m which is irreducible modulo M_v. Let $f(x)$ belonging to $R_v[x]$ be a polynomial having $\phi(x)$-expansion $f(x) = \sum_{i=0}^{n} a_i(x)\phi(x)^i$, where $v_x(a_0(x)) \neq 0$ and for some j with $j \leq n$, $v_x(a_j(x)) = 0$. Let s be the smallest such value of j. Assume that there exists some i with $i < s$ such that $a_i(x) \neq 0$. Let $k(< s)$ be the smallest non-negative integer such that*

$$\min_{0 \leq i \leq s-1} \left\{ \frac{v_x(a_i(x))}{s-i} \mid 0 \leq i \leq s-1 \right\} = \frac{v_x a_k(x)}{s-k}.$$

Suppose that $v_x(a_k(x)) \notin dG(K)$ for any number $d > 1$ dividing $s-k$. Then the following hold:

1. $f(x)$ has an irreducible factor of degree at least $(s-k)m$ over K.
2. If v is a henselian valuation of K and $v_x(f(x)) = 0$, then $f(x)$ has an irreducible factor $g(x)$ over K such that $(s-k)m \leq \deg g(x) \leq sm$.

Moreover, the $\phi(x)$-expansion of $g(x) = b_t(x)\phi(x)^t + \cdots + b_0(x)$ with $s-k \leq t \leq s$ satisfies $v_x(b_t(x)) = 0$, and there exists $u, 0 \leq u < t$ such that $v_x(a_k(x)) = v_x(b_u(x)), s-k = t-u$, and

$$\min_{0 \leq i \leq t-1} \left\{ \frac{v_x(b_i(x))}{t-i} \mid 0 \leq i \leq t-1 \right\} = \frac{v_x(b_u(x))}{t-u}.$$

Corollary 3.21 *Assume that all the hypotheses above are satisfied. Then for any factorization $f_1(x)f_2(x)$ of $f(x)$ over K, we have*

$$\min\{\deg f_1(x), \deg f_2(x)\} \leq (n-s+k)m + \deg a_n(x).$$

It may be pointed out that the second assertion of Theorem 3.20 and its corollary extend many previous results proved in [25], [27], [32-35].

Finally, let us see some concrete examples.

Example: Let $b \in \mathbb{Z}, b \neq 0, \pm 1$; q prime and $q | b$; r be the highest power of q such that $q^r | b$; $a \in \mathbb{Z}$, $q \nmid a$; $m \in \mathbb{Z}$, $(m, r) = 1$; $n \in \mathbb{Z}$, $n > m$. Then by Theorem 3.19, $f(x) = x^n + ax^m + b$ has an irreducible factor of degree at least m. Moreover,

- In case $n = m+1$, either $f(x)$ has a linear factor or it is irreducible over $\mathbb{Q}$.
- Moreover, in this case, $f(x)$ is a product of two irreducible polynomials having degrees $n-1$ and 1 over $\mathbb{Q}_p$.
- Furthermore, in this case, if $b = q^r c$, any monic linear factor of $f(x)$ must be $x-d$ for some d dividing c.
- In particular, if $b = \pm q^r$ and $a \neq -1-b, 1+(-1)^n b$, then $f(x)$ is irreducible over $\mathbb{Q}$.

Example: Take

$$f(x) = x(x^2+3)^3 + 5(x^2+3)^2 + 25(x^2+3) + b$$

with $b = \pm 5, \pm 25$. Suppose $K = \mathbb{Q}$ the field and $\phi(x) = x^2 + 3$ with the 5-adic valuation $v_5(5) = 1$. Then by Corollary 3.21, $f(x)$ is irreducible over $\mathbb{Q}$.

IV. Conclusion

Polynomials which are the oldest subjects in mathematics are still researched intensively; for example, Eisenstein's irreducibility criterion proved in 1850 has still been generalized to wider domains. The remarkable point of this paper is investigating some of the newest results of polynomials in valuation theory. These results show that valuation theory has usable tools to obtain new consequences for polynomials.

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