On the growths of entire algebroidal functions under the treatment of p-adic analysis

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Abstract

In this paper we wish to establish some results focusing on the comparative growth properties of composition of two k-valued entire algebroidal functions in terms of their relative growth indicators from the view point of slowly changing functions in the light of p-adic analysis. The main aim of this paper is to find out the estimates of relative (p,q) -th type, relative (p,q) -th lower type and relative (p,q) -th weak type under somewhat different conditions where p and q are any two positive integers via the concept of p-adic analysis of entire algebroidal functions.

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1 Relevant definitions and notations:

Let $f \in \mathcal{A}(\mathbb{K})$ be an entire function then we denote by $|f|(r)$ the number $\sup\{|f(x)|\,|x| = r, r > 0\}$ where $|\cdot|(r)$ is a multiplicative norm on $\mathcal{A}(\mathbb{K})$. Similarly for the entire function $g \in \mathcal{A}(\mathbb{K})$, $|g|(r)$ is defined. The ratio $\frac{|f|(r)}{|g|(r)}$ as $r \to \infty$ is called the growth of f with respect to g in terms of their multiplicative norm on $\mathcal{A}(\mathbb{K})$. However, if $f \in \mathcal{A}(\mathbb{K})$ is not constant, the $|f|(r)$ is strictly increasing function of r and tends to ∞ with r therefore there exists its inverse function which is defined by $\widehat{|f|} : (|f(0)|, \infty) \to (0, \infty)$ with $\lim_{s\to\infty} \widehat{|f|}(s) = \infty$. Several authors (see [2] ,[8], [9], [16] ,[19] ,[20], [21], [22], [35]). made closed investigations on the properties of relative order of entire functions from the view point of p-adic analysis to yield many results, for example, some of which are recalled here.

Theorem [8]Let f and $g \in \mathcal{A}(\mathbb{K})$ be transcendental. If $\rho(f) \neq 0$, then $\rho(f \circ g) = +\infty$. If $\rho(f) = 0$ then

$$\rho(f \circ g) \geq \rho(g).$$

Theorem [9] Let f,$g \in \mathcal{A}(\mathbb{K})$ be any two entire functions such that $0 < \rho^{(m,n)}(g) < \lambda^{(p,q)}(f) \leq \rho^{(p,q)}(f) < \infty$,

$$\lim_{r\to\infty} \frac{\log^{[p]} |fog| (\exp^{[n+1]} r)}{\log^{[p]} |f| \left[\exp^{[q]} r^A\right]} = \infty \; if \quad q = m$$

$$and \lim_{r\to\infty} \frac{\log^{[p]} \widehat{|h|} \, |fog| (\exp^{[q+n-m+1]} r)}{\log^{[p]} |f| \left[\exp^{[q]} r^A\right]} = \infty \; if \quad q > m$$

where A>0 is a constant.

Theorem [9] Let f,g,$h \in \mathcal{A}(\mathbb{K})$ be any three entire functions such that f and g have finite *relative* (p, q)*-th* L *-order and lower order* with respect to h respectively then

$$\overline{\lim_{r\to\infty}} \frac{\log^{[p]} \widehat{|h|} \, |fog| (\exp^{[n+1]} r)}{\log^{[p]} \widehat{|h|} \left(|f| \left[\exp^{[q]} r^A\right]\right)} = \infty \; if \quad q = m$$

$$and \overline{\lim_{r\to\infty}} \frac{\log^{[p]} \widehat{|h|} \, |fog| (\exp^{[q+n-m+1]} r)}{\log^{[p]} \widehat{|h|} \left(|f| \left[\exp^{[q]} r^A\right]\right)} = \infty \; if \quad q > m$$

where A>0 is a constant.

2 Lemmas.

Lemma 1 *Let F and G be two k-valued p-adic entire algebroidal functions. Then for all sufficiently large positive numbers of r we obtain that*

$$|FoG| (r) = |F| (|G| (r)) .$$

3 Main Results.

Theorem 3.1 *Let F,G and H be three k-valued p-adic entire algebroidal functions defined by the following irreducible equations:*

$$f_k F^k + f_{k-1} F^{k-1} + f_{k-2} F^{k-2} + ... + f_0 = 0 \qquad \text{(i)}$$

$$g_k G^k + g_{k-1} G^{k-1} + g_{k-2} G^{k-2} + ... + g_0 = 0 \qquad \text{(ii)}$$

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$$h_kH^k + h_{k-1}H^{k-1} + h_{k-2}H^{k-2} + ... + h_0 = 0 \quad \text{(iii)}$$

where f_i's , g_i's and h_i's are entire functions belonging to $\mathcal{A}(\mathbb{K})$ having no common zeros $(i = 0, 1, 2, \ldots, k)$ be such that $0 < \lambda_{h_i}^{(p,q)L}(F) \le \rho_{h_i}^{(p,q)L}(F) < \infty$ and $\rho^{(m,n)L}(G) > 0$, where p, q, m, n are positive integers. Then for every positive constant A, we have

$$(a)\limsup_{r\to+\infty} \frac{\log^{[p]} \widehat{|h_i|}\, |FoG| (\exp^{[n+1]} r)}{\log^{[p]} \widehat{|h_i|} \left(|F| \left[\exp^{[q]} (rL(r))^A\right]\right)} = \infty \; if \quad q = m$$

$$(b)\limsup_{r\to+\infty} \frac{\log^{[p]} \widehat{|h_i|}\, |FoG| (\exp^{[q+n+1-m]} r)}{\log^{[p]} \widehat{|h_i|} \left(|F| \left[\exp^{[q]} (rL(r))^A\right]\right)} = \infty \; if \quad q > m.$$

$$(c)\limsup_{r\to+\infty} \frac{\log^{[p]} \widehat{|h_i|}\, |FoG| (\exp^{[n-1]} r)}{\log^{[p]} \widehat{|h_i|} \left(|F| \left[\exp^{[q]} (rL(r))^A\right]\right)} = \infty \; if \quad q \le m-1 \text{ and } \rho^{(m,n)L}(G) > A$$

Proof: For all sufficiently large values of r it is clear that $|FoG|(r) = |F|(|G|(r))$.

Now from the definition of $\rho_{h_i}^{(p,q)L}(F)$, for any arbitrary positive $\varepsilon(> 0)$ we have for all sufficiently large values of r that

$$\log^{[p]} \widehat{|h_i|} \left(|F| \left[\exp^{[q]} (rL(r))^A\right]\right) \le \left(\rho_{h_i}^{(p,q)L}(F) + \varepsilon\right) (rL(r))^A . \quad (1)$$

Also we get for a sequence of positive numbers of r tending to infinity that

$$\begin{aligned}
\log^{[m]} |G| (\exp^{[q+n+1-m]} r) &\ge \left(\rho^{(m,n)L}(G) - \epsilon\right) \log^{[n]}(\exp^{[q+n+1-m]} r) \\
\text{i.e.,} \quad \log^{[m]} |G| (\exp^{[q+n+1-m]} r) &\ge \left(\rho^{(m,n)L}(G) - \epsilon\right) \exp^{[q+1-m]} r \\
\text{i.e.,} \quad \log^{[q-m]} \log^{[m]} |G| (\exp^{[q+n+1-m]} r) &\ge \log^{[q-m]} \left(\rho^{(m,n)L}(G) - \epsilon\right) \exp^{[q+1-m]} r \\
\text{i.e.,} \quad \log^{[q]} |G| (\exp^{[q+n+1-m]} r) &\ge \exp r + O(1), \quad (2)
\end{aligned}$$

and

$$\begin{aligned}
\log^{[m]} |G| (\exp^{[n-1]} r) &\ge \left(\rho^{(m,n)L}(G) - \epsilon\right) \log^{[n]}(\exp^{[n-1]} r) \\
\log^{[m-1]} |G| (\exp^{[n-1]} r) &\ge r^{\left(\rho^{(m,n)L}(G) - \epsilon\right)}. \quad (3)
\end{aligned}$$

Since $\widehat{|h_i|}(r)$ is an increasing function of r, for all sufficiently large positive values of r we have

$$\log^{[p]} \widehat{|h_i|}\, |FoG| (r) \ge \left(\lambda_{h_i}^{(p,q)L}(F) - \varepsilon\right) \log^{[q]}(|G| r). \quad (4)$$

Case I. First let $q = m$ then it follows from (4) for a sequence of positive numbers of r tending to infinity that

$$\log^{[p]} \widehat{|h_i|}\, |FoG| (\exp^{[n+1]} r) \ge \left(\lambda_{h_i}^{(p,q)L}(F) - \varepsilon\right) \left(\rho^{(m,n)L}(G) - \epsilon\right) \exp r. \quad (5)$$

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Case II Let $q > m$. Then we get from (2) and (4) for a sequence of positive numbers of r tending to infinity that

$$\log^{[p]} \widehat{|h_i|}\, |FoG| \left(\exp^{[q+n+1-m]} r\right) \geq \left(\lambda_{h_i}^{(p,q)L}(F) - \varepsilon\right) \exp r + O(1). \quad (6)$$

Case III Again let $q \leq m-1$. Then we have from (3) and (4) for a sequence of positive numbers of r tending to infinity that

$$\begin{aligned} \log^{[p]} \widehat{|h_i|}\, |FoG| \left(\exp^{[n-1]} r\right) &\geq \left(\lambda_{h_i}^{(p,q)L}(F) - \varepsilon\right) \log^{[q]} |G| \left(\exp^{[n-1]} r\right) \\ i.e., \log^{[p]} \widehat{|h_i|}\, |FoG| \left(\exp^{[n-1]} r\right) &\geq \left(\lambda_{h_i}^{(p,q)L}(F) - \varepsilon\right) \log^{[m-1]} |G| \left(\exp^{[n-1]} r\right) \\ i.e., \log^{[p]} \widehat{|h_i|}\, |FoG| \left(\exp^{[n-1]} r\right) &\geq \left(\lambda_{h_i}^{(p,q)L}(F) - \varepsilon\right) r^{\left(\rho^{(m,n)L}(G) - \epsilon\right)}. \quad (7) \end{aligned}$$

Now combining (1) and (5) it follows for sequence of positive numbers tending to infinity that

$$\frac{\log^{[p]} \widehat{|h_i|}\, |FoG| \left(\exp^{[n+1]} r\right)}{\log^{[p]} \widehat{|h_i|} \left(|F| \left[\exp^{[q]} \left(rL(r)\right)^A\right]\right)} \geq \frac{\left(\lambda_{h_i}^{(p,q)L}(F) - \varepsilon\right) \left(\rho^{(m,n)L}(G) - \epsilon\right) \exp r}{\left(\rho_{h_i}^{(p,q)L}(F) + \varepsilon\right) \left(rL(r)\right)^A}.$$

As $\frac{\exp r}{(rL(r))^A} \to \infty$ as $r \to \infty$, then from above it follows that

$$\limsup_{r \to +\infty} \frac{\log^{[p]} \widehat{|h_i|}\, |FoG| \left(\exp^{[n+1]} r\right)}{\log^{[p]} \widehat{|h_i|} \left(|F| \left[\exp^{[q]} \left(rL(r)\right)^A\right]\right)} = \infty.$$

Hence the first part of the theorem follows.

Again combining (1) and (6) of Case II we get for a sequence of positive integers of r tending to infinity that

$$\frac{\log^{[p]} \widehat{|h_i|}\, |FoG| \left(\exp^{[q+n+1-m]} r\right)}{\log^{[p]} \widehat{|h_i|} \left(|F| \left[\exp^{[q]} \left(rL(r)\right)^A\right]\right)} \geq \frac{\left(\lambda_{h_i}^{(p,q)L}(F) - \varepsilon\right) \exp r + O(1)}{\left(\rho_{h_i}^{(p,q)L}(F) + \varepsilon\right) \left(rL(r)\right)^A}$$

$$\limsup_{r \to +\infty} \frac{\log^{[p]} \widehat{|h_i|}\, |FoG| \left(\exp^{[q+n+1-m]} r\right)}{\log^{[p]} \widehat{|h_i|} \left(|F| \left[\exp^{[q]} \left(rL(r)\right)^A\right]\right)} = \infty.$$

Thus the second part of the theorem.

Also from the Equations (1) and (7) for a sequence of positive integers of r tending to infinity that

$$\frac{\log^{[p]} \widehat{|h_i|}\, |FoG| \left(\exp^{[n-1]} r\right)}{\log^{[p]} \widehat{|h_i|} \left(|F| \left[\exp^{[q]} \left(rL(r)\right)^A\right]\right)} \geq \frac{\left(\lambda_{h_i}^{(p,q)L}(F) - \varepsilon\right) r^{\left(\rho^{(m,n)L}(G) - \epsilon\right)}}{\left(\rho_{h_i}^{(p,q)L}(F) + \varepsilon\right) \left(rL(r)\right)^A}.$$

As $\rho^{(m,n)L}(G) > A$, therefore from above we get

$$\limsup_{r \to +\infty} \frac{\log^{[p]} \widehat{|h_i|}\, |FoG| \left(\exp^{[n-1]} r\right)}{\log^{[p]} \widehat{|h_i|} \left(|F| \left[\exp^{[q]} \left(rL(r)\right)^A\right]\right)} = \infty$$

This proves the third part of the theorem.

Hence completes the proof of the theorem.

In the line of Theorem 3.1 the following theorem can be derived:

Theorem 3.2 *Let us suppose that F,G and H be three k-valued p-adic entire algebroidal functions as defined in Theorem 3.1 by the Equations (i), (ii) and (iii) where* f_i , g_i *and* $h_i \in \mathcal{A}(\mathbb{K})$ *are entire functions having no common zeros* $(i = 0, 1, 2, \ldots, k)$ *be such that* $0 < \lambda_{h_i}^{(p,q)L}(F) \le \rho_{h_i}^{(p,q)L}(F) < \infty$ *and* $\lambda^{(m,n)L}(G) > 0$. *Then for every positive constant A we obtain that*

$$(a)\lim_{r\to+\infty}\frac{\log^{[p]}\widehat{|h_i|}\,|FoG|\left(\exp^{[n+1]} r\right)}{\log^{[p]}\widehat{|h_i|}\left(|F|\left[\exp^{[q]}\left(rL(r)\right)^{A}\right]\right)} = \infty \ if \quad q = m$$

$$(b)\lim_{r\to+\infty}\frac{\log^{[p]}\widehat{|h_i|}\,|FoG|\left(\exp^{[q+n+1-m]} r\right)}{\log^{[p]}\widehat{|h_i|}\left(|F|\left[\exp^{[q]}\left(rL(r)\right)^{A}\right]\right)} = \infty \ if \quad q > m.$$

Theorem 3.3 *Let F,G and H be three k-valued p-adic entire algebroidal functions defined by the irreducible Equations (i), (ii) and (iii) as in Theorem 1, where* f_i*'s ,* g_i*'s and* h_i*'s are entire functions belonging to* $\mathcal{A}(\mathbb{K})$ *having no common zeros* $(i = 0, 1, 2, \ldots, k)$ *be such that* $0 < \lambda_{h_i}^{(p,q)L}(F) \le \rho_{h_i}^{(p,q)L}(F) < \infty$ *and* $\rho^{(m,n)L}(G) > 0$. *Also let W be another k-valued p-adic entire algebroidal functions defined by the irreducible equation:*

$$w_k W^k + w_{k-1}W^{k-1} + w_{k-2}W^{k-2} + \ldots\ldots\ldots\ldots.. + w_0 = 0$$

where w_i*'s*$(\in \mathcal{A}(\mathbb{K}))$ *are entire functions be such that* $\rho_{w_i}^{(l,n)L}(G) < \infty$, *where* p, q, m, n, l, n *are positive integers. Then for every positive constant A we obtain the followings*

$$(a)\limsup_{r\to+\infty}\frac{\log^{[p]}\widehat{|h_i|}\,|FoG|\left(\exp^{[n+1]} r\right)}{\log^{[l]}\widehat{|w_i|}\left(|G|\left[\exp^{[n]}\left(rL(r)\right)^{A}\right]\right)} = \infty \ if \quad q = m.$$

$$(b)\limsup_{r\to+\infty}\frac{\log^{[p]}\widehat{|h_i|}\,|FoG|\left(\exp^{[q+n+1-m]} r\right)}{\log^{[l]}\widehat{|w_i|}\left(|G|\left[\exp^{[n]}\left(rL(r)\right)^{A}\right]\right)} = \infty \ if \quad q > m.$$

$$(c)\limsup_{r\to+\infty}\frac{\log^{[p]}\widehat{|h_i|}\,|FoG|\left(\exp^{[n-1]} r\right)}{\log^{[l]}\widehat{|w_i|}\left(|G|\left[\exp^{[n]}\left(rL(r)\right)^{A}\right]\right)} = \infty \ if \quad q \le m-1 \text{ and } \rho^{(m,n)L}(G) > A.$$

Proof. First we consider A_0 be a positive constant be such that

$$A < A_0 < \rho^{(m,n)L}(G). \tag{8}$$

Case I. Let $q = m$. Then in view of the first part of Theorem 1, for any arbitrary positive $\varepsilon(> 0)$ we have for a sequence of positive integers of r tending to infinity and a positive integer $R > 1$

$$\log^{[p]}\widehat{|h_i|}\,|FoG|\left(\exp^{[n+1]} r\right) > R\left(\lambda_{h_i}^{(p,q)L}(F) - \varepsilon\right)\left(rL(r)\right)^{A_0}. \tag{9}$$

Case II. Let $q > m$. Then we get from the second part of Theorem 1 for a sequence of $r\,(> 0)$ tending to infinity and a positive integer $R > 1$

$$\log^{[p]}\widehat{|h_i|}\,|FoG|\left(\exp^{[q+n+1-m]} r\right) > R\left(\lambda_{h_i}^{(p,q)L}(F) - \varepsilon\right)\left(rL(r)\right)^{A_0}. \tag{10}$$

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Case III. Again let $q \leq m-1$. Then we obtain from the third part of the Theorem 1 for a sequence of positive numbers of r tending to infinity and a positive integer $R > 1$

$$\log^{[p]} \widehat{|h_i|} \, |FoG| \left(\exp^{[n-1]} r\right) > R\left(\lambda_{h_i}^{(p,q)L}(F) - \varepsilon\right) (rL(r))^{A_0} . \tag{11}$$

Now from the definition of $\rho_{h_i}^{(p,n)L}(G)$, we have for all sufficiently large values of r that

$$\log^{[l]} \widehat{|w_i|} \left(|G| \left[\exp^{[n]} (rL(r))^A\right]\right) \leq \left(\rho_{w_i}^{(l,n)L}(G) + \varepsilon\right) (rL(r))^A . \tag{12}$$

Now combining (9) and (12) it follows for a sequence of positive integers of r tending to infinity that

$$\frac{\log^{[p]} \widehat{|h_i|} \, |FoG| \left(\exp^{[n+1]} r\right)}{\log^{[l]} \widehat{|w_i|} \left(|G| \left[\exp^{[n]} (rL(r))^A\right]\right)} > \frac{R\left(\lambda_{h_i}^{(p,q)L}(F) - \varepsilon\right) (rL(r))^{A_0}}{\left(\rho_{w_i}^{(l,n)L}(G) + \varepsilon\right) (rL(r))^A}. \tag{13}$$

Since $A < A_0$, we have from (13)

$$\limsup_{r\to+\infty} \frac{\log^{[p]} \widehat{|h_i|} \, |FoG| \left(\exp^{[n+1]} r\right)}{\log^{[l]} \widehat{|w_i|} \left(|G| \left[\exp^{[n]} (rL(r))^A\right]\right)} = \infty.$$

Hence the first part of the theorem follows.

Again for $A < A_0$, it follows from the Equations (10) and (12) for all sufficiently large values of r tending to infinity that

$$\frac{\log^{[p]} \widehat{|h_i|} \, |FoG| \left(\exp^{[q+n+1-m]} r\right)}{\log^{[l]} \widehat{|w_i|} \left(|G| \left[\exp^{[n]} (rL(r))^A\right]\right)} > \frac{R\left(\lambda_{h_i}^{(p,q)L}(F) - \varepsilon\right) (rL(r))^{A_0}}{\left(\rho_{w_i}^{(l,n)L}(G) + \varepsilon\right) (rL(r))^A}$$

Thus from above we obtain

$$\limsup_{r\to+\infty} \frac{\log^{[p]} \widehat{|h_i|} \, |FoG| \left(\exp^{[q+n+1-m]} r\right)}{\log^{[l]} \widehat{|w_i|} \left(|G| \left[\exp^{[n]} (rL(r))^A\right]\right)} = \infty.$$

This implies the second part of the theorem.

By similar arguments it follows from (11) and (12) for sequence of positive values of of r tending to infinity that

$$\frac{\log^{[p]} \widehat{|h_i|} \, |FoG| \left(\exp^{[n-1]} r\right)}{\log^{[l]} \widehat{|w_i|} \left(|G| \left[\exp^{[n]} (rL(r))^A\right]\right)} > \frac{R\left(\lambda_{h_i}^{(p,q)L}(F) - \varepsilon\right) (rL(r))^{A_0}}{\left(\rho_{w_i}^{(l,n)L}(G) + \varepsilon\right) (rL(r))^A}.$$

Therefore in view of (8) and above we get that

$$\limsup_{r\to+\infty} \frac{\log^{[p]} \widehat{|h_i|} \, |FoG| \left(\exp^{[n-1]} r\right)}{\log^{[l]} \widehat{|w_i|} \left(|G| \left[\exp^{[n]} (rL(r))^A\right]\right)} = \infty.$$

This proves the third part of the theorem.

Hence the theorem follows.

Theorem 3.4 *Let F,G and H be three k-valued p-adic entire algebroidal functions defined by the irreducible Equations (i), (ii), (iii) as in Theorem 3.1 where f_i , g_i and $h_i \in \mathcal{A}(\mathbb{K})$ are entire functions having no common zeros ($i = 0, 1, 2, \ldots, k$). Also let $0 < \lambda_{h_i}^{(p,q)L}(F) \leq \rho_{h_i}^{(p,q)L}(F) < \infty$ and $\lambda^{(m,n)L}(G) > 0$ and $\rho_{k_i}^{(l,n)L}(G) < \infty$. Then for every positive constant A,*

$$(a) \lim_{r\to+\infty} \frac{\log^{[p]} \widehat{|h_i|}\, |FoG| \left(\exp^{[n+1]} r\right)}{\log^{[l]} \widehat{|h_i|} \left(|G| \left[\exp^{[n]} \left(rL(r)\right)^A\right]\right)} = \infty \; if \quad q = m$$

$$(b) \lim_{r\to+\infty} \frac{\log^{[p]} \widehat{|h_i|}\, |FoG| \left(\exp^{[q+n+1-m]} r\right)}{\log^{[l]} \widehat{|h_i|} \left(|G| \left[\exp^{[n]} \left(rL(r)\right)^A\right]\right)} = \infty \; if \quad q > m.$$

The proof of the theorem follows from Theorem 3.2 and Theorem 3.3.

Theorem 3.5 *Let F,G and H be three k-valued p-adic entire algebroidal functions defined by the irreducible Equations (i), (ii), (iii) as in Theorem 3.1 where f_i , g_i and $h_i \in \mathcal{A}(\mathbb{K})$ are entire functions having no common zeros ($i = 0, 1, 2, \ldots, k$). Also let $0 < \lambda_{h_i}^{(p,q)L}(F) \leq \rho_{h_i}^{(p,q)L}(F) < \infty$ and $0 < \lambda^{(m,n)L}(G) < \infty$. Then for a positive constant A we have*

$$(a) \limsup_{r\to+\infty} \frac{\log^{[p]} \widehat{|h_i|} \left(|F| \left[\exp^{[q]} \left(rL(r)\right)^A\right]\right)}{\log^{[p]} \widehat{|h_i|} \left(|FoG| \left(\exp^{[n]} r\right)\right)} = \infty \; if \quad q \geq m \; and \; A > 1,$$

$$(b) \limsup_{r\to+\infty} \frac{\log^{[p]} \widehat{|h_i|} \left(|F| \left[\exp^{[q]} \left(rL(r)\right)^A\right]\right)}{\log^{[p]} \widehat{|h_i|} \left(|FoG| \left(\exp^{[n-1]} r\right)\right)} = \infty$$

$$if \quad q \geq m \text{ or } q = m - 1 \text{ with } m \neq 1 \text{ and } \lambda^{(m,n)L}(G) < A,$$

$$(c) \limsup_{r\to+\infty} \frac{\log^{[p]} \widehat{|h_i|} \left(|F| \left[\exp^{[q]} \left(rL(r)\right)^A\right]\right)}{\log^{[p+m-q-1]} \widehat{|h_i|} \left(|FoG| \left(\exp^{[n-1]} r\right)\right)} = \infty \; if \quad m > q+1 \text{ and } A > \lambda^{(m,n)L}(G).$$

Proof. From the definition of $\lambda_{h_i}^{(p,q)L}(F)$, we obtain for arbitrary positive $\varepsilon\,(> 0)$ and for all sufficiently large values of r that

$$\log^{[p]} \widehat{|h_i|} \left(|F| \left[\exp^{[q]} \left(rL(r)\right)^A\right]\right) \geq \left(\lambda_{h_i}^{(p,q)L}(F) - \varepsilon\right) \left(rL(r)\right)^A \qquad (14)$$

Also we have for a sequence of positive numbers of r tending to infinity that

$$\begin{aligned} \log^{[m]} |G| \left(\exp^{[n-1]} r\right) &\leq \left(\lambda^{(m,n)L}(G) + \epsilon\right) \log r \\ \log^{[m]} |G| \left(\exp^{[n-1]} r\right) &\leq \log r^{\left(\lambda^{(m,n)L}(G)+\epsilon\right)} && (15) \\ \log^{[m-1]} |G| \left(\exp^{[n-1]} r\right) &\leq r^{\left(\lambda^{(m,n)L}(G)+\epsilon\right)}. && (16) \end{aligned}$$

Since $\widehat{|h_i|}(r)$ is an increasing function of r then for all sufficiently large values of r we obtain from Lemma 1 that

$$\log^{[p]} \widehat{|h_i|} \left(|FoG| (r)\right) \leq \left(\rho_{h_i}^{(p,q)L}(F) + \epsilon\right) \log^{[q]} \left(|G| (r)\right). \qquad (17)$$

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Case I. First let $q \geq m$. We see from the Equation (17) for a sequence of positive numbers of r tending to infinity that

$$\log^{[p]} \widehat{|h_i|} \left(|FoG| \left(\exp^{[n]} r \right) \right) \leq \left(\rho_{h_i}^{(p,q)L}(F) + \epsilon \right) \log^{[m]} \left(|G| \left(\exp^{[n]} r \right) \right)$$

$$i.e., \log^{[p]} \widehat{|h_i|} \left(|FoG| \left(\exp^{[n]} r \right) \right) \leq \left(\rho_{h_i}^{(p,q)L}(F) + \epsilon \right) \left(\lambda^{(m,n)L}(G) + \epsilon \right) \log r. \quad (18)$$

Case II Next let $q \geq m$ or $q = m - 1$ with $m \neq 1$. Then it follows from the Equations (16) and (17) for all sufficiently large values of r that

$$\log^{[p]} \widehat{|h_i|} \left(|FoG| \left(\exp^{[n-1]} r \right) \right) \leq \left(\rho_{h_i}^{(p,q)L}(F) + \epsilon \right) \log^{[m-1]} \left(|G| \left(\exp^{[n-1]} r \right) \right)$$

$$\log^{[p]} \widehat{|h_i|} \left(|FoG| \left(\exp^{[n-1]} r \right) \right) \leq \left(\rho_{h_i}^{(p,q)L}(F) + \epsilon \right) r^{\left(\lambda^{(m,n)L}(G) + \epsilon \right)}. \quad (19)$$

Case III Let $m > q + 1$. Then we get from (15) and (17) for a sequence of positive numbers of r tending to infinity that

$$\log^{[p]} \widehat{|h_i|} \left(|FoG| \left(\exp^{[n-1]} r \right) \right) \leq \left(\rho_{h_i}^{(p,q)L}(F) + \epsilon \right) \log^{[q-m]} \log^{[m]} \left(|G| \left(\exp^{[n-1]} r \right) \right)$$

$$i.e., \log^{[p]} \widehat{|h_i|} \left(|FoG| \left(\exp^{[n-1]} r \right) \right) \leq \left(\rho_{h_i}^{(p,q)L}(F) + \epsilon \right) \exp^{[m-q]} \log r^{\left(\lambda^{(m,n)L}(G) + \epsilon \right)}$$

$$i.e., \log^{[p]} \widehat{|h_i|} \left(|FoG| \left(\exp^{[n-1]} r \right) \right) \leq \left(\rho_{h_i}^{(p,q)L}(F) + \epsilon \right) \exp^{[m-q-1]} r^{\left(\lambda^{(m,n)L}(G) + \epsilon \right)}$$

$$i.e., \log^{[p]} \widehat{|h_i|} \left(|FoG| \left(\exp^{[n-1]} r \right) \right) \leq r^{\left(\lambda^{(m,n)L}(G) + \epsilon \right)} + O(1). \quad (20)$$

When $q \geq m$ and $A > 1$ we see from the Equations (14) and (18) for all sufficiently large values of r

$$\frac{\log^{[p]} \widehat{|h_i|} \left(|F| \left[\exp^{[q]} \left(rL(r) \right)^A \right] \right)}{\log^{[p]} \widehat{|h_i|} \left(|FoG| \left(\exp^{[n]} r \right) \right)} \geq \frac{\left(\lambda_{h_i}^{(p,q)L}(F) - \epsilon \right) \left(rL(r) \right)^A}{\left(\rho_{h_i}^{(p,q)L}(F) + \epsilon \right) \left(\lambda^{(m,n)L}(G) + \epsilon \right) r}$$

$$i.e., \quad \limsup_{r \to +\infty} \frac{\log^{[p]} \widehat{|h_i|} \left(|F| \left[\exp^{[q]} \left(rL(r) \right)^A \right] \right)}{\log^{[p]} \widehat{|h_i|} \left(|FoG| \left(\exp^{[n]} r \right) \right)} = \infty.$$

The first part of theorem follows.

Again if $q \geq m$ or $q = m(\neq 1) - 1$ it follows from (14) and (19) for a sequence of positive numbers of r tending to infinity that

$$\frac{\log^{[p]} \widehat{|h_i|} \left(|F| \left[\exp^{[q]} \left(rL(r) \right)^A \right] \right)}{\log^{[p]} \widehat{|h_i|} \left(|FoG| \left(\exp^{[n-1]} r \right) \right)} \geq \frac{\left(\lambda_{h_i}^{(p,q)L}(F) - \epsilon \right) \left(rL(r) \right)^A}{\left(\rho_{h_i}^{(p,q)L}(F) + \epsilon \right) r^{\left(\lambda^{(m,n)L}(G) + \epsilon \right)}}. \quad (21)$$

As $A > \lambda^{(m,n)L}(G)$ we consider an arbitrary $\epsilon > 0$ in such a manner that

$$\lambda^{(m,n)L}(G) + \epsilon < A. \quad (22)$$

Combining (21) and (22) we get that

$$\limsup_{r \to +\infty} \frac{\log^{[p]} \widehat{|h_i|} \left(|F| \left[\exp^{[q]} \left(rL(r) \right)^A \right] \right)}{\log^{[p]} \widehat{|h_i|} \left(|FoG| \left(\exp^{[n-1]} r \right) \right)} = \infty.$$

Hence the second part of the theorem.

When $m > q+1$, we obtain from (14) and (20) for all sufficiently large values of r that

$$\frac{\log^{[p]} \widehat{|h_i|}\left(|F|\left[\exp^{[q]}\left(rL(r)\right)^A\right]\right)}{\log^{[p+m-q-1]} \widehat{|h_i|}\left(|FoG|\left(\exp^{[n-1]} r\right)\right)} \geq \frac{\left(\lambda_{h_i}^{(p,q)L}(F)-\epsilon\right)\left(rL(r)\right)^A}{r^{\left(\lambda^{(m,n)L}(G)+\epsilon\right)}+O(1)}. \quad (23)$$

Combining (22) and (23) we have

$$\limsup_{r\to+\infty} \frac{\log^{[p]} \widehat{|h_i|}\left(|F|\left[\exp^{[q]}\left(rL(r)\right)^A\right]\right)}{\log^{[p+m-q-1]} \widehat{|h_i|}\left(|FoG|\left(\exp^{[n-1]} r\right)\right)} = \infty.$$

This proves the third part of the theorem.

This proves the theorem.

Theorem 3.6 *Let four k-valued p-adic entire algebroidal functions F,G, H and W where f_i , g_i , h_i and $w_i \in \mathcal{A}(\mathbb{K})$ are entire functions having no common zeros $(i=0,1,2,\ldots,k)$. Also let $\rho_{h_i}^{(p,q)L}(F)$, $\lambda^{(m,n)L}(G)$ and $\lambda_{w_i}^{(l,n)L}(G)$ are finite, where p,q,m,n,l,n are positive integers. Then for every positive constant A,*

$$(a)\limsup_{r\to+\infty} \frac{\log^{[l]} \widehat{|w_i|}\left(|G|\left[\exp^{[n]}\left(rL(r)\right)^A\right]\right)}{\log^{[p]} \widehat{|h_i|}\left(|FoG|\left(\exp^{[n]} r\right)\right)} = \infty \; if \quad q \geq m \text{ and } A > 1,$$

$$(b)\limsup_{r\to+\infty} \frac{\log^{[l]} \widehat{|w_i|}\left(|G|\left[\exp^{[n]}\left(rL(r)\right)^A\right]\right)}{\log^{[p]} \widehat{|h_i|}\left(|FoG|\left(\exp^{[n-1]} r\right)\right)} = \infty$$

$$if \quad q \geq m \text{ or } q=m-1 \text{ with } m\neq 1 \text{ and } \lambda^{(m,n)L}(G) < A,$$

$$(c)\limsup_{r\to+\infty} \frac{\log^{[l]} \widehat{|w_i|}\left(|G|\left[\exp^{[n]}\left(rL(r)\right)^A\right]\right)}{\log^{[p+m-q-1]} \widehat{|h_i|}\left(|FoG|\left(\exp^{[n-1]} r\right)\right)} = \infty \; if \quad m > q+1 \text{ and } A > \lambda^{(m,n)L}(G).$$

The proof of this theorem is omitted as it follows from Theorem 3.5 and Theorem 3.4.

Theorem 3.7 *Let us suppose that F,G and H be three k-valued p-adic entire algebroidal functions as defined in Theorem 3.1 by the Equations (i), (ii) and (iii) where f_i , g_i and $h_i \in \mathcal{A}(\mathbb{K})$ are entire functions having no common zeros $(i=0,1,2,\ldots.k)$ with finite relative (p,q)th L-order and finite relative (p,q)th L- lower order of F with respect to another entire function h_i . Also let $\rho^{(m,n)L}(G) < \infty$. Then*

$$(a)\lim_{r\to+\infty} \frac{\log^{[p]} \widehat{|h_i|}\left(|F|\left[\exp^{[q]}\left(rL(r)\right)^A\right]\right)}{\log^{[p]} \widehat{|h_i|}\left(|FoG|\left(\exp^{[n]} r\right)\right)} = \infty \; if \quad q \geq m \text{ and } A > 1,$$

$$(b)\lim_{r\to+\infty} \frac{\log^{[p]} \widehat{|h_i|}\left(|F|\left[\exp^{[q]}\left(rL(r)\right)^A\right]\right)}{\log^{[p]} \widehat{|h_i|}\left(|FoG|\left(\exp^{[n-1]} r\right)\right)} = \infty$$

$$if \quad q \geq m \text{ or } q=m-1 \text{ with } m\neq 1 \text{ and } \rho^{(m,n)L}(G) < A,$$

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$$(c)\lim_{r\to+\infty}\frac{\log^{[p]}\widehat{|h_i|}\left(|F|\left[\exp^{[q]}\left(rL(r)\right)^A\right]\right)}{\log^{[p+m-q-1]}\widehat{|h_i|}\left(|FoG|\left(\exp^{[n-1]}r\right)\right)}=\infty\, if\quad m>q{+}1 \text{ and } \rho^{(m,n)L}(G)<A.$$

Theorem 3.8 *Let us consider four k-vauled p-adic entire algebroidal functions F, G, H and W where f_i , g_i , h_i and $w_i \in \mathcal{A}(\mathbb{K})$ are entire functions having no common zeros $(i = 0, 1, 2, \ldots, k)$. Also let $\rho_{h_i}^{(p,q)L}(F)$, $\rho^{(m,n)L}(G)$ and $\lambda_{w_i}^{(l,n)L}(G)$ are finite, where p, q, m, n, l, n are positive integers. Then for every positive constant A,*

$$(a)\lim_{r\to+\infty}\frac{\log^{[l]}\widehat{|w_i|}\left(|G|\left[\exp^{[n]}\left(rL(r)\right)^A\right]\right)}{\log^{[p]}\widehat{|h_i|}\left(|FoG|\left(\exp^{[n]}r\right)\right)}=\infty\, if\quad q\geq m \text{ and } A>1,$$

$$(b)\lim_{r\to+\infty}\frac{\log^{[l]}\widehat{|w_i|}\left(|G|\left[\exp^{[n]}\left(rL(r)\right)^A\right]\right)}{\log^{[p]}\widehat{|h_i|}\left(|FoG|\left(\exp^{[n-1]}r\right)\right)}=\infty$$

$$if\quad q\geq m \text{ or } q=m-1 \text{ with } m\neq 1 \text{ and } \rho^{(m,n)L}(G)<A,$$

$$(c)\lim_{r\to+\infty}\frac{\log^{[l]}\widehat{|w_i|}\left(|G|\left[\exp^{[n]}\left(rL(r)\right)^A\right]\right)}{\log^{[p+m-q-1]}\widehat{|h_i|}\left(|FoG|\left(\exp^{[n-1]}r\right)\right)}=\infty\, if\quad m>q{+}1 \text{ and } A>\rho^{(m,n)L}(G).$$

The proof of Theorem 3.7 and Theorem 3.8 is omitted as those can be carried out in the line of Theorem 5 and Theorem 6 respectivley.

In view of Theorem 3.1 and Theorem 3.5 we have an application which is as follows.

Theorem 3.9 *Let F,G and H be three k-valued p-adic entire algebroidal functions defined by the irreducible Equations (i), (ii), (iii) as in Theorem 3.1 where f_i , g_i and $h_i \in \mathcal{A}(\mathbb{K})$ are entire functions having no common zeros $(i = 0, 1, 2, \ldots, k)$. Also let $0 < \lambda_{h_i}^{(p,q)L}(F) \leq \rho_{h_i}^{(p,q)L}(F) < \infty$ and $0 < \lambda^{(m,n)L}(G) < A < \rho^{(m,n)L}(G)$. Then for $m \neq 1$ and $q = m-1$,*

$$\liminf_{r\to+\infty}\frac{\widehat{|h_i|}\,|FoG|\left(\exp^{[n-1]}r\right)}{\widehat{|h_i|}\left(|F|\left[\exp^{[q]}\left(rL(r)\right)^A\right]\right)}\leq 1\leq\limsup_{r\to+\infty}\frac{\widehat{|h_i|}\,|FoG|\left(\exp^{[n-1]}r\right)}{\widehat{|h_i|}\left(|F|\left[\exp^{[q]}\left(rL(r)\right)^A\right]\right)}$$

Proof. From Theorem 3.1 we obtain for a sequence of positive numbers of r tending to infinity that

$$\begin{aligned}\log^{[p]}\widehat{|h_i|}\left(|FoG|\left(\exp^{[n-1]}r\right)\right) &\geq \log^{[p]}\widehat{|h_i|}\,|F|\left[\exp^{[q]}\left(rL(r)\right)^A\right]\\ i.e., \widehat{|h_i|}\left(|FoG|\left(\exp^{[n-1]}r\right)\right) &\geq \widehat{|h_i|}\,|F|\left[\exp^{[q]}\left(rL(r)\right)^A\right]\\ i.e., \limsup_{r\to+\infty}\frac{\widehat{|h_i|}\left(|FoG|\left(\exp^{[n-1]}r\right)\right)}{\widehat{|h_i|}\,|F|\left[\exp^{[q]}\left(rL(r)\right)^A\right]} &\geq 1 \qquad (24)\end{aligned}$$

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Again in view of Theorem 3.5 for all sufficiently large values of r we get

$$\begin{aligned} \log^{[p]} \widehat{|h_i|}\,|F|\left[\exp^{[q]}\left(rL(r)\right)^A\right] &\geq \log^{[p]} \widehat{|h_i|}\left(|FoG|\left(\exp^{[n-1]} r\right)\right) \\ i.e., \widehat{|h_i|}\,|F|\left[\exp^{[q]}\left(rL(r)\right)^A\right] &\geq \widehat{|h_i|}\left(|FoG|\left(\exp^{[n-1]} r\right)\right) \\ i.e., \liminf_{r\to+\infty} \frac{\widehat{|h_i|}\left(|FoG|\left(\exp^{[n-1]} r\right)\right)}{\widehat{|h_i|}\,|F|\left[\exp^{[q]}\left(rL(r)\right)^A\right]} &\leq 1. \end{aligned} \tag{25}$$

Combining (24) and (25) we get the result.

Hence completes the proof.

The next theorem can be derived in the line of Theorem 3.3 and Theorem 3.6.

Theorem 3.10 *Let us consider four k-valued p-adic entire algebroidal functions F,G, H and W where f_i , g_i , h_i and $w_i \in \mathcal{A}(\mathbb{K})$ are entire functions having no common zeros $(i = 0, 1, 2, \ldots, k)$.Also let $0 < \lambda_{h_i}^{(p,q)L}(F) \leq \rho_{h_i}^{(p,q)L}(F) < \infty$, $0 < \lambda_{w_i}^{(p,n)L}(G) \leq \rho_{k_i}^{(p,n)L}(G) < \infty$ and $0 < \lambda^{(m,n)L}(G) < A < \rho^{(m,n)L}(G)$. Then we obtain for $m \neq 1$ and $q = m - 1$ that*

$$\liminf_{r\to+\infty} \frac{\widehat{|h_i|}\,|FoG|\left(\exp^{[n-1]} r\right)}{\widehat{|w_i|}\left(|G|\left[\exp^{[n]}\left(rL(r)\right)^A\right]\right)} \leq 1 \leq \limsup_{r\to+\infty} \frac{\widehat{|h_i|}\,|FoG|\left(\exp^{[n-1]} r\right)}{\widehat{|w_i|}\left(|G|\left[\exp^{[n]}\left(rL(r)\right)^A\right]\right)}.$$

The proof of this theorem is omitted as it can be carried out by the similar arguments of Theorem 3.9.

4 Conclusion with Further Prospects.

The main aim of this chapter is actually to extend and to modify some relative growth properties of entire algebroidal functions where as the coefficients are all entire functions belonging to $\mathcal{A}(\mathbb{K})$ from the view point of p -adic analysis, where p is any positive prime. We also modify the results in the context of slowly changing function for more generalization. Further the concept of relative index-pair of an entire function belonging to $\mathcal{A}(\mathbb{K})$ have been deeply studied here with more extension. In fact, these are mainly the consequences of the theories derived as in previous chapters. we may see some more powerful results in the next chapter.

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