STOCHASTIC MODEL OF THREE PRODUCTS INVENTORY SYSTEM ASSUMING SEASONAL DEMAND FOLLOWING HYPER EXPONENTIAL DISTRIBUTION

D.Kanagajothi[1] B.Rameshkumar[2]
[1,2]Department of Mathematics
[1,2]Vel Tech Rangarajan Dr. Sagunthala R & D Institute of Science and Technology, Chennai, India

Abstract- Many companies are using an inventory system for their success and growth. In real life inventory models, many different products are required to manufacture one unit for sale. Marketing and sales of them begin when certain numbers of them are produced for economic reasons. In this paper three different products A, B and C are assumed to constitute a unit. This work deals with the production and sales of three different products A, B and C with maximum production size k sets of them. The sales of the sets begin when k sets are produced or when the season for the products starts whichever occurs first. The production times of the products are general and the season stars in a hyper exponential distribution time. The double Laplace Transform of production and sale time and their means are obtained. The results are illustrated numerically are presented with Matlab Tool.

Keywords – Inventory Systems, Production, Products, Hyper exponential.

I. Introduction

Inventory management is an art. It connects different aspects in a business like supply chain management, logistics, etc. These functions are interlinked. Inventory is the backbone of the business. Hence, in an organized inventory should be maintained at optimum level. If it is below optimum level, it will affect the product and it is above optimum level, it will not yield the desired profit. Hence the dynamic valuation, internal and external factors have to be controlled and this has to be done through careful planning and review. Anderson et al. (2009) established the stationary distribution of the inventory level (stock on hand) in a continuous-review inventory system with compound Poisson demand, Erlang as well as hyper exponentially distributed lead times, and lost sales. Ramanarayanan and Jacob (1998) analyzed of (S, s) inventory systems with random lead time and bulk demands. Tijms (1986) considered a detailed analysis of the inventory system under (s, S) policy. Parvathi et al. (2013) worked two products inventory system with seasonal production and sales. Bagchi and Hayya (1984) assumed demand as normal and lead time as Erlang and they obtained an expression for demand during lead time from this expression they predicted protection level and potential lost sales. For expansion items these assumptions are valued for uncertain demand this is not valid. David and Robert (1985) reported a stochastic (Q, R) inventory system. In this case demands assume Poisson distribution lead times are assumed as Erlang distribution. The sums of random variables are of interest in many areas of the sciences. In tele traffic analysis, the sum of hyper exponential distribution is used as a model for the holding time distribution. Many authors examined this model and discussed its probability density function. Kadri and Smaili (2015) described hyper exponential distribution for holding time and they discoursed the possible cases. Kalpakam and Arivarignan (1985) considered a system which is subjected to random failure, they have assumed Erlangian life time under renewal demands.

II. Proposed Algorithm

2.1 Assumptions:–

The following are presented below

(i) The company produces three different products $A, B \text{ and } C$ and at a time only one is produced. The production of the products $A, B \text{ and } C$ are done in order. Only when one set of $A, B \text{ and } C$ is produced, we say that one unit is produced. Only after the production of one unit, the production of the next unit starts. The production times of $A, B \text{ and } C$ are random variables $X, Y \text{ and } Z$ respectively. The production time X of product A has c.d.f $H_X(.)$ and p.d.f $\square_X(.)$; the production time Y of product B has c.d.f $H_Y(.)$ and p.d.f $\square_Y(.)$ and the production time Z of product C has c.d.f $H_Z(.)$ and p.d.f $\square_Z(.)$. The production time $X+Y+Z$ of one unit of triplet has c.d.f $H(.)$.

(ii) The season demand for the units starts in hyper exponential time whose parameter is λ_i with probability $p_i > 0$ for $1 \leq i \leq m$ and $\sum_{i=1}^{m} p_i = 1$.

(iii) When k number of units of triplet products are produced time of sales begin. Sales time also begins when demand for the products arises.

(iv) The units are sold one by one and the selling time of a unit is a random variable with c.d.f $S(.)$ with p.d.f $s(.)$. The c.d.f of selling times of individual products $A, B \text{ and } C$ are $S_A(.)$, $S_B(.)$ and $S_C(.)$with p.d.f $s_A(.)$, $s_B(.)$ and $s_C(.)$respectively.

2.2 ANALYSIS:

We note that the probability of n units produced in the interval $(0, t)$ is $H_n(t) - H_{n+1}(t)$ for $n \geq 0$ where H_n(t) is n fold c.d.f convolution of $H(t)$ and $H_n(t)$ is the c.d.f of $\sum_{i=1}^{n}(X_i + Y_i + Z_i)$. When the n^{th} triplet is completed at time $x < t$, then during the period (x, t) there are possibilities of completion or incompletion of products $A, B, \text{and } C$ as follows. During the period (x, t)

a) The production time for product A is not over or

b) The product A is produced but the product B is not produced or

c) The products $A \text{ and } B$ are produced but the product C is not produced.

Their respective probabilities are given below. Let $\square_n$(x) be n fold convolution of $\square(x)$ with itself where $\square(x)$ is the p.d.f of $X + Y + Z$ and $\square_n$(x) is the pdf of $\sum_{i=1}^{n}(X_i+Y_i+Z_i)$. Then the probability of n triplet units are produced in $(0,t)$ and the $(n+1)^{th}$ production of A is not completed before t is

$$P\left[\sum_{i=1}^{n}(X_i+Y_i+Z_i) < t < \sum_{i=1}^{n}(X_i+Y_i+Z_i)+X_{n+1}\right] = \int_0^t \square_n(x)\bar{H}_X(t-x)dx,$$

where $\bar{H}$(x) = 1- H(x).The probability of n triplet units are produced and one production of A is over before t, but production for B is not over before t is

$$P\left[\sum_{i=1}^{n}(X_i+Y_i+Z_i) + X_{n+1} < t < \sum_{i=1}^{n}(X_i+Y_i+Z_i)+X_{n+1}+Y_{n+1}\right]$$

$$= \int_0^t \square_n(x)\int_0^{t-x}\square_X(u)\bar{H}_Y(t-x-u)dudx.$$

The probability of n triplet units are produced and one production of A and B are over before t, but production for C is not over before t is

$$P\left[\sum_{i=1}^{n}(X_i+Y_i+Z_i) + X_{n+1}+Y_{n+1} < t < \sum_{i=1}^{n}(X_i+Y_i+Z_i)+X_{n+1}+Y_{n+1}+Z_{n+1}\right]$$

$$= \int_0^t \square_n(x)\int_0^{t-x}\int_0^{v}\square_X(u)\square_Y(v-u)du\bar{H}_Z(t-x-v)dvdx.$$

The selling time of the three products is based on either the season starts or when k triplets are produced T= min (time to produce k triplets, the time from which the season starts).

The hyper exponential random variable X has parameter λ_i with probability p_i and its probability density function is

$$f(x) = \sum_{i=1}^{m} p_i\lambda_i e^{-\lambda_i x}$$

For all 1 ≤ i ≤ m, $p_i, \lambda_i > 0$ and
$\sum_{i=1}^{m} p_i$ =1.

$$f_T(t) = h_k(t)\sum_{i=1}^{m} p_i e^{-\lambda_i t} + \sum_{i=1}^{m} p_i\lambda_i e^{-\lambda_i t}\sum_{i=0}^{k-1}\int_0^t h_i(x)\bar{H}_X(t-x)dx$$
$$+\sum_{i=1}^{m} p_i\lambda_i e^{-\lambda_i t}\sum_{i=0}^{k-1}\int_0^t h_i(x)\int_0^{t-x} h_X(u)\bar{H}_Y(t-x-u)dudx$$
$$+\sum_{i=1}^{m} p_i\lambda_i e^{-\lambda_i t}\sum_{i=0}^{k-1}\int_0^t h_i(x)\int_0^{t-x}\int_0^{v} h_X(u)h_Y(v-u)\bar{H}_Z(t-x-v)dudvdx \quad (1.1)$$

An explanation of the four terms in the right hand side of Equation 1.1 will be in order. As regards the first term, it provides the probability distribution function for the time t required for the production of k number of triplets while the season remains non-started up to time t. In the case of the second term, it indicates the probability distribution for the start of the season at time t, the time x needed for the manufacture of i number of the combined units of A, B and C, while the $(i+1)^{th}$ product A is not ready during the time $t-x$. Concerning the term term, it referes to the probability distribution for the start of the season at time t, the time x to manufacture i units of the combination of A, B and C, the $(i+1)^{th}$ A product is manufactured at time $x+u$ but $(i+1)^{th}$ A product is not manufactured during the time t-x-u with the restriction on i provided by $0 \le i \le k-1$. Finally, the fourth term denotes the probability distribution for the start of the season at time t, the time x to manufacture i number of the combinations of A, B and C, the $(i+1)^{th}$ product A is manufactured at time $x+u$, the $(i+1)^{th}$ B product is manufactured at time $x+v$ but the product C is not manufactured during the time t-x-v with the constraint $0 \le i \le k-1$.

This gives the joint p.d.f of time to start sales T and total sales time S as follows considering the sales time of the triplet units in the first and second term; the sales time of triplet units and the sale time of the product A in the third term; and the sales time of triplet units and the sales time of the products A and B in the fourth term.

$$f(x,y) = h_k(x)\sum_{i=1}^{m} p_i e^{-\lambda_i(x)} s_k(y) + \sum_{i=1}^{m} p_i \lambda_i e^{-\lambda_i(x)}$$
$$\sum_{i=0}^{k-1}\int_0^x h_i(u)\bar{H}_X(x-u)du s_i(y)$$
$$+\sum_{i=1}^{m} p_i \lambda_i e^{-\lambda_i(x)}\sum_{i=0}^{k-1}\int_0^x h_i(u)\int_0^{x-u} h_X(v)\bar{H}_Y(x-u-v)dvdu$$
$$\int_0^y s_i(p)s_A(y-p)dp$$
$$+\sum_{i=1}^{m} p_i \lambda_i e^{-\lambda_i(x)}\sum_{i=0}^{k-1}\int_0^x h_i(u)\int_0^v h_X(w)h_Y(v-w)dw\bar{H}_Z(x-u-w)dvdu$$
$$\times\int_0^y s_i(p)\int_0^{y-p} s_A(q)s_B(y-p-q)dpdq \qquad (1.2)$$

The double Laplace Transform of p.d.f

$$f^*_{T,S}(\varepsilon,\eta) = \int_0^\infty\int_0^\infty e^{-\varepsilon x}e^{-\eta y} f_{T,S}(x,y)dxdy$$
$$= \int_0^\infty\int_0^\infty e^{-\varepsilon x}e^{-\eta y} h_k(x)\sum_{i=1}^{m} p_i e^{-\lambda_i x} s_k(y)dxdy$$
$$+\int_0^\infty\int_0^\infty e^{-\varepsilon x}e^{-\eta y}\sum_{i=1}^{m} p_i\lambda_i e^{-\lambda_i x}\sum_{i=0}^{k-1}\int_0^x h_i(u)\bar{H}_X(x-u)s_i(y)dudxdy$$
$$+\int_0^\infty\int_0^\infty e^{-\varepsilon x}e^{-\eta y}\sum_{i=1}^{m} p_i\lambda_i e^{-\lambda_i x}\sum_{i=0}^{k-1}\int_0^x h_i(u)\int_0^{x-u} h_X(v)\bar{H}_Y(x-u-v)dvdu$$

is

$$\int_0^y s_i(p)s_A(y-q)dpdxdy$$
$$+\int_0^\infty\int_0^\infty e^{-\varepsilon x}e^{-\eta y}\sum_{i=1}^{m} p_i\lambda_i e^{-\lambda_i x}\sum_{i=0}^{k-1}\int_0^x h_i(u)\int_0^{x-u}\int_0^v h_X(w)h_Y(v-w)dw\bar{H}_Z(x-u-v)dvdu$$
$$\times\int_0^y s_i(p)\int_0^{y-p} s_A(q)s_B(y-p-q)dpdqdxdy \qquad (7.3)$$

$$f^{*}_{T,S}(\varepsilon,\eta)=\sum_{i=1}^{m}p_i h^{*k}(\varepsilon+\lambda_i)s^{*k}(\eta)+\sum_{i=1}^{m}p_i\lambda_i\sum_{i=0}^{k-1}h^{*i}(\varepsilon+\lambda_i)\bar{H}^{*}_{X}(\varepsilon+\lambda_i)s^{*i}(\eta)$$

$$+\sum_{i=1}^{m}p_i\lambda_i\sum_{i=0}^{k-1}h^{*i}(\varepsilon+\lambda_i)h^{*}_{X}(\varepsilon+\lambda_i)\bar{H}^{*}_{Y}(\varepsilon+\lambda_i)s^{*i}(\eta)s_A(\eta)$$

$$+\sum_{i=1}^{m}p_i\lambda_i\sum_{i=0}^{k-1}h^{*i}(\varepsilon+\lambda_i)h^{*}_{X}(\varepsilon+\lambda_i)h^{*}_{Y}(\varepsilon+\lambda_i)\bar{H}^{*}_{Y}(\varepsilon+\lambda_i)$$

$$s^{*i}(\eta)s_A(\eta)s_B(\eta) \qquad (1.4)$$

$$f^{*}(\varepsilon,\eta)=\sum_{i=1}^{m}p_i h^{*k}(\varepsilon+\lambda_i)s^{*k}(\eta)+\sum_{i=1}^{m}p_i\lambda_i\left[\frac{1-(h^{*}(\varepsilon+\lambda_i)s^{*}(\eta))^{k}}{1-(h^{*}(\varepsilon+\lambda_i)s^{*}(\eta))}\right]\bar{H}^{*}_{X}(\varepsilon+\lambda_i)$$

$$+\sum_{i=1}^{m}p_i\lambda_i\left[\frac{1-(h^{*}(\varepsilon+\lambda_i)s^{*}(\eta))^{k}}{1-(h^{*}(\varepsilon+\lambda_i)s^{*}(\eta))}\right]h^{*}_{X}(\varepsilon+\lambda_i)\bar{H}^{*}_{Y}(\varepsilon+\lambda_i)s^{*}_{A}(\eta)$$

$$+\sum_{i=1}^{m}p_i\lambda_i\left[\frac{1-(h^{*}(\varepsilon+\lambda_i)s^{*}(\eta))^{k}}{1-(h^{*}(\varepsilon+\lambda_i)s^{*}(\eta))}\right]h^{*}_{X}(\varepsilon+\lambda_i)h^{*}_{Y}(\varepsilon+\lambda_i)\bar{H}^{*}_{Z}(\varepsilon+\lambda_i)$$

$$s^{*}_{A}(\eta)s^{*}_{B}(\eta)$$

On differentiation of Equation 1.4,

$$\frac{\partial}{\partial \varepsilon} f^{*}(\varepsilon,0)=\sum_{i=1}^{m} p_i\lambda_i k h^{*k-1}(\varepsilon+\lambda_i)h^{*'}(\varepsilon+\lambda_i)$$

$$+\sum_{i=1}^{m} p_i\lambda_i\left[\frac{-kh^{*k-1}(\varepsilon+\lambda_i)h^{*'}(\varepsilon+\lambda_i)}{1-h^{*}(\varepsilon+\lambda_i)}\right]\left[\begin{array}{l}\bar{H}^{*}_{X}(\varepsilon+\lambda_i)+h^{*}_{X}(\varepsilon+\lambda_i)\bar{H}^{*}_{Y}(\varepsilon+\lambda_i)\\+h^{*}_{X}(\varepsilon+\lambda_i)h^{*}_{Y}(\varepsilon+\lambda_i)\bar{H}^{*}_{Z}(\varepsilon+\lambda_i)\end{array}\right]$$

$$+\sum_{i=1}^{m} p_i\lambda_i\left[\frac{1-(h^{*}(\varepsilon+\lambda_i))^{k}}{(1-h^{*}(\varepsilon+\lambda_i))^{2}}\right]h^{*'}(\varepsilon+\lambda_i)\left[\begin{array}{l}\bar{H}^{*}_{X}(\varepsilon+\lambda_i)+h^{*}_{X}(\varepsilon+\lambda_i)\bar{H}^{*}_{Y}(\varepsilon+\lambda_i)\\+h^{*}_{X}(\varepsilon+\lambda_i)h^{*}_{Y}(\varepsilon+\lambda_i)\bar{H}^{*}_{Z}(\varepsilon+\lambda_i)\end{array}\right]$$

$$+\sum_{i=1}^{m} p_i\lambda_i\left[\frac{1-(h^{*}(\varepsilon+\lambda_i))^{k}}{(1-h^{*}(\varepsilon+\lambda_i))}\right]\left[\begin{array}{l}\bar{H}^{*'}_{X}(\varepsilon+\lambda_i)+h^{*'}_{X}(\varepsilon+\lambda_i)\bar{H}^{*}_{Y}(\varepsilon+\lambda_i)\\+h^{*}_{X}(\varepsilon+\lambda_i)\bar{H}^{*'}_{Y}(\varepsilon+\lambda_i)\\+h^{*'}_{X}(\varepsilon+\lambda_i)h^{*}_{Y}(\varepsilon+\lambda_i)\bar{H}^{*}_{Y}(\varepsilon+\lambda_i)\\+h^{*}_{X}(\varepsilon+\lambda_i)h^{*'}_{Y}(\varepsilon+\lambda_i)\bar{H}^{*}_{Z}(\varepsilon+\lambda_i)\\+h^{*}_{X}(\varepsilon+\lambda_i)h^{*}_{Y}(\varepsilon+\lambda_i)\bar{H}^{*'}_{Z}(\varepsilon+\lambda_i)\end{array}\right]$$

$$\frac{\partial}{\partial\varepsilon} f^{*}(0,0) = \sum_{i=1}^{m} p_i\lambda_i k h^{*k-1}(\varepsilon+\lambda_i) h^{*'}(\varepsilon+\lambda_i)$$

$$+\sum_{i=1}^{m} p_i\lambda_i \left[\frac{-kh^{*k-1}(\varepsilon+\lambda_i)h^{*'}(\varepsilon+\lambda_i)}{1-h^{*}(\varepsilon+\lambda_i)}\right]\left[\begin{array}{l}\bar{H}^{*}_{X}(\varepsilon+\lambda_i)+h^{*}_{X}(\varepsilon+\lambda_i)\bar{H}^{*}_{Y}(\varepsilon+\lambda_i)\\ +h^{*}_{X}(\varepsilon+\lambda_i)h^{*}_{Y}(\varepsilon+\lambda_i)\bar{H}^{*}_{Z}(\varepsilon+\lambda_i)\end{array}\right]$$

$$+\sum_{i=1}^{m} p_i\lambda_i \left[\frac{1-(h^{*}(\varepsilon+\lambda_i))^{k}}{(1-h^{*}(\varepsilon+\lambda_i))^{2}}\right] h^{*'}(\varepsilon+\lambda_i)\left[\begin{array}{l}\bar{H}^{*}_{X}(\varepsilon+\lambda_i)+h^{*}_{X}(\varepsilon+\lambda_i)\bar{H}^{*}_{Y}(\varepsilon+\lambda_i)\\ +h^{*}_{X}(\varepsilon+\lambda_i)h^{*}_{Y}(\varepsilon+\lambda_i)\bar{H}^{*}_{Z}(\varepsilon+\lambda_i)\end{array}\right]$$

$$+\sum_{i=1}^{m} p_i\lambda_i \left[\frac{1-(h^{*}(\varepsilon+\lambda_i))^{k}}{(1-h^{*}(\varepsilon+\lambda_i))}\right]\left[\begin{array}{l}\bar{H}^{*'}_{X}(\varepsilon+\lambda_i)+h^{*'}_{X}(\varepsilon+\lambda_i)\bar{H}^{*}_{Y}(\varepsilon+\lambda_i)\\ +h^{*}_{X}(\varepsilon+\lambda_i)\bar{H}^{*'}_{Y}(\varepsilon+\lambda_i)\\ +h^{*'}_{X}(\varepsilon+\lambda_i)h^{*}_{Y}(\varepsilon+\lambda_i)\bar{H}^{*}_{Y}(\varepsilon+\lambda_i)\\ +h^{*}_{X}(\varepsilon+\lambda_i)h^{*'}_{Y}(\varepsilon+\lambda_i)\bar{H}^{*}_{Z}(\varepsilon+\lambda_i)\\ +h^{*}_{X}(\varepsilon+\lambda_i)h^{*}_{Y}(\varepsilon+\lambda_i)\bar{H}^{*'}_{Z}(\varepsilon+\lambda_i)\end{array}\right]$$

$$\frac{\partial}{\partial\varepsilon} f^{*}(0,0) = -E(T)$$

$$
\begin{aligned}
E(T) = &-\sum_{i=1}^{m} p_i\lambda_i k h^{*k-1}(\varepsilon+\lambda_i) h^{*'}(\varepsilon+\lambda_i) \\
&+\sum_{i=1}^{m} p_i\lambda_i \left[\frac{-kh^{*k-1}(\varepsilon+\lambda_i)h^{*'}(\varepsilon+\lambda_i)}{1-h^{*}(\varepsilon+\lambda_i)}\right]\begin{bmatrix}\bar{H}^{*}_{X}(\varepsilon+\lambda_i)+h^{*}{}_{X}(\varepsilon+\lambda_i)\bar{H}^{*}_{Y}(\varepsilon+\lambda_i)\\ +h^{*}{}_{X}(\varepsilon+\lambda_i)h^{*}{}_{Y}(\varepsilon+\lambda_i)\bar{H}^{*}_{Z}(\varepsilon+\lambda_i)\end{bmatrix} \\
&-\sum_{i=1}^{m} p_i\lambda_i \left[\frac{1-(h^{*}(\varepsilon+\lambda_i))^{k}}{(1-h^{*}(\varepsilon+\lambda_i))^{2}}\right] h^{*'}(\varepsilon+\lambda_i)\begin{bmatrix}\bar{H}^{*}_{X}(\varepsilon+\lambda_i)+h^{*}{}_{X}(\varepsilon+\lambda_i)\bar{H}^{*}_{Y}(\varepsilon+\lambda_i)\\ +h^{*}{}_{X}(\varepsilon+\lambda_i)h^{*}{}_{Y}(\varepsilon+\lambda_i)\bar{H}^{*}_{Z}(\varepsilon+\lambda_i)\end{bmatrix} \\
&-\sum_{i=1}^{m} p_i\lambda_i \left[\frac{1-(h^{*}(\varepsilon+\lambda_i))^{k}}{(1-h^{*}(\varepsilon+\lambda_i))}\right]\begin{bmatrix}\bar{H}^{*'}_{X}(\varepsilon+\lambda_i)+h^{*'}{}_{X}(\varepsilon+\lambda_i)\bar{H}^{*}_{Y}(\varepsilon+\lambda_i)\\ +h^{*}{}_{X}(\varepsilon+\lambda_i)\bar{H}^{*'}_{Y}(\varepsilon+\lambda_i)\\ +h^{*'}{}_{X}(\varepsilon+\lambda_i)h^{*}{}_{Y}(\varepsilon+\lambda_i)\bar{H}^{*}_{Y}(\varepsilon+\lambda_i)\\ +h^{*}{}_{X}(\varepsilon+\lambda_i)h^{*'}{}_{Y}(\varepsilon+\lambda_i)\bar{H}^{*}_{Z}(\varepsilon+\lambda_i)\\ +h^{*}{}_{X}(\varepsilon+\lambda_i)h^{*}{}_{Y}(\varepsilon+\lambda_i)\bar{H}^{*'}_{Z}(\varepsilon+\lambda_i)\end{bmatrix}
\end{aligned}
\tag{1.5}
$$

It can be seen that it reduces to

$$
E(T)=\sum_{i=1}^{m} p_i\lambda_i\left[\frac{1-(h(\lambda_i))^{k}}{\lambda_i}\right]
$$

$$
\begin{aligned}
E(T) = &\; p_1\lambda_1\left[\frac{1-(h(\lambda_1))^{k}}{\lambda_1}\right]+p_2\lambda_2\left[\frac{1-(h(\lambda_2))^{k}}{\lambda_2}\right] \\
&+p_3\lambda_3\left[\frac{1-(h(\lambda_3))^{k}}{\lambda_3}\right]+p_4\lambda_4\left[\frac{1-(h(\lambda_4))^{k}}{\lambda_4}\right]
\end{aligned}
\tag{1.6}
$$

The Laplace transform of total sales time S is

$$f^{*}(0,\eta)=\sum_{i=1}^{m}p_i h^{*k}(\lambda_i)s^{*k}(\eta)+\sum_{i=1}^{m}p_i\lambda_i\left[\frac{1-(h^{*}(\lambda_i)s^{*}(\eta))^{k}}{1-(h^{*}(\lambda_i)s^{*}(\eta))}\right]\bar{H}^{*}_{X}(\lambda_i)$$
$$+\sum_{i=1}^{m}p_i\lambda_i\left[\frac{1-(h^{*}(\lambda_i)s^{*}(\eta))^{k}}{1-(h^{*}(\lambda_i)s^{*}(\eta))}\right]h^{*}_{X}(\lambda_i)\bar{H}^{*}_{Y}(\lambda_i)s^{*}_{A}(\eta)$$
$$+\sum_{i=1}^{m}p_i\lambda_i\left[\frac{1-(h^{*}(\lambda_i)s^{*}(\eta))^{k}}{1-(h^{*}(\lambda_i)s^{*}(\eta))}\right]h^{*}_{X}(\lambda_i)h^{*}_{Y}(\lambda_i)\bar{H}^{*}_{Z}(\lambda_i)s^{*}_{A}(\eta)s^{*}_{B}(\eta)$$

(1.7)

Similarly $\frac{\partial}{\partial\eta}f^{*}(0,0)=-E(S)$

$$E(S)=-\sum_{i=1}^{m}p_i h^{*k}(\lambda_i)k(-E(S))$$
$$-\sum_{i=1}^{m}p_i\lambda_i\left[\frac{kh^{*k}(\lambda_i)E(s_1)}{1-h^{*k}(\lambda_i)}\right]\left[\begin{array}{l}\bar{H}^{*}_{X}(\lambda)+h^{*}_{X}(\lambda)\bar{H}^{*}_{Y}(\lambda)\\+h^{*}_{X}(\lambda)h^{*}_{Y}(\lambda)\bar{H}^{*}_{Z}(\lambda)\end{array}\right]$$
$$-\sum_{i=1}^{m}p_i\lambda_i\left[\frac{1-h^{*k}(\lambda_i)}{(1-h^{*}(\lambda_i))^{2}}\right]h^{*}(\lambda)(-E(s_1))\left[\begin{array}{l}\bar{H}^{*}_{X}(\lambda)+h^{*}_{X}(\lambda)\bar{H}^{*}_{Y}(\lambda)\\+h^{*}_{X}(\lambda)h^{*}_{Y}(\lambda)\bar{H}^{*}_{Z}(\lambda)\end{array}\right]$$
$$-\lambda\left[\frac{1-h^{*k}(\lambda)}{(1-h^{*}(\lambda))}\right]\left[\begin{array}{l}\left\{h^{*}_{X}(\lambda)\bar{H}^{*}_{Y}(\lambda)\right\}(-E(s_A))\\+\left\{h^{*}_{X}(\lambda)h^{*}_{Y}(\lambda)\bar{H}^{*}_{Z}(\lambda)\right\}(-E(s_A))\\+\left\{h^{*}_{X}(\lambda)h^{*}_{Y}(\lambda)\bar{H}^{*}_{Z}(\lambda)\right\}(-E(s_B))\end{array}\right]$$

(1.8)

$$E(S)=\sum_{i=1}^{m} p_i\lambda_i\left[\frac{1-h^{*k}(\lambda_i)}{1-h^{*}(\lambda_i)}\right]\left\{h^{*}(\lambda_i)\left[\begin{array}{l}E(s_1)-E(s_A)-E(s_B)+E(s_A)h_X^{*}(\lambda_i)\\+E(s_B)h_X^{*}(\lambda_i)h_Y^{*}(\lambda_i)\end{array}\right]\right\} \quad (1.9)$$

Take i=1 and 2

$$E(S)=p_1\lambda_1\left[\frac{1-h^{*k}(\lambda_1)}{1-h^{*}(\lambda_1)}\right]\left\{h^{*}(\lambda_1)\left[\begin{array}{l}E(s_1)-E(s_A)-E(s_B)+E(s_A)h_X^{*}(\lambda_1)\\+E(s_B)h_X^{*}(\lambda_1)h_Y^{*}(\lambda_1)\end{array}\right]\right\}$$
$$+p_2\lambda_2\left[\frac{1-h^{*k}(\lambda_2)}{1-h^{*}(\lambda_2)}\right]\left\{h^{*}(\lambda_2)\left[\begin{array}{l}E(s_1)-E(s_A)-E(s_B)+E(s_A)h_X^{*}(\lambda_2)\\+E(s_B)h_X^{*}(\lambda_2)h_Y^{*}(\lambda_2)\end{array}\right]\right\} \quad (1.10)$$

III. Some Special Cases

Most of stochastic models, the c.d.f are exponential. In the Equation 1.8, k is constant and sales for the system are considered when it completes the production of k triplets. As a more general case, k may be treated as a random variable taking positive integral values such that

$$P(k=i)=p_i>0,\ i>0 \text{ and } \sum_{i=1}^{n} p_i=1.$$

The unique case in which X and Y are exponential random variableswith parameters l, m and n respectively is considered below which states

$$h^{*}{}_X(\lambda)=\frac{l}{l+\lambda};\ h^{*\prime}{}_X(\lambda)=\frac{-l}{(l+\lambda)^2}$$

$$h^{*}{}_Y(\lambda)=\frac{m}{m+\lambda};\ h^{*\prime}{}_Y(\lambda)=\frac{-m}{(m+\lambda)^2}$$

$$h^{*}{}_Z(\lambda)=\frac{n}{n+\lambda};\ h^{*\prime}{}_Z(\lambda)=\frac{-n}{(n+\lambda)^2}$$

$$\bar{H}^{*}{}_X(\lambda)=\frac{1}{l+\lambda};\ \bar{H}^{*\prime}{}_X(\lambda)=\frac{-1}{(l+\lambda)^2}$$

$$\bar{H}^{*}{}_{Y}(\lambda) = \frac{1}{m+\lambda};\ \bar{H}^{*'}{}_{Y}(\lambda) = \frac{-1}{(m+\lambda)^2}$$

$$\bar{H}^{*}{}_{Z}(\lambda) = \frac{1}{n+\lambda};\ \bar{H}^{*'}{}_{Z}(\lambda) = \frac{-1}{(n+\lambda)^2} \quad (1.11)$$

E (T) and E (S) are found when X and Y are exponentials with parameter l ,m and n as follows.

$$E(T) = \sum_{i=1}^{m} p_i \left(1 - \left(\frac{lmn}{(l+\lambda_i)(m+\lambda_i)(n+\lambda_i)}\right)^k\right)\frac{1}{\lambda_i} \quad (1.12)$$

and

$$E(S) = \sum_{i=1}^{m} p_i \left\{\frac{1-\left(\frac{lmn}{(l+\lambda_i)(m+\lambda_i)(n+\lambda_i)}\right)^k}{1-\left(\frac{lmn}{(l+\lambda_i)(m+\lambda_i)(n+\lambda_i)}\right)}\right\}\left[\left(\frac{lmn}{(l+\lambda_i)(m+\lambda_i)(n+\lambda_i)}\right)\right]$$

$$\left(E(s_1) - E(s_A) - E(s_B) + E(s_A)\frac{l}{l+\lambda_i} + E(s_B)\frac{l}{l+\lambda_i}\frac{m}{m+\lambda_i}\right) \quad (1.13)$$

Take i=1, 2, 3 and 4

$$E(T) = p_1\left(1-\left(\frac{lmn}{(l+\lambda_1)(m+\lambda_1)(n+\lambda_1)}\right)^k\right)\frac{1}{\lambda_1}$$

$$+ p_2\left(1-\left(\frac{lmn}{(l+\lambda_2)(m+\lambda_2)(n+\lambda_2)}\right)^k\right)\frac{1}{\lambda_2}$$

$$+ p_3\left(1-\left(\frac{lmn}{(l+\lambda_3)(m+\lambda_3)(n+\lambda_3)}\right)^k\right)\frac{1}{\lambda_3}$$

$$+ p_4\left(1-\left(\frac{lmn}{(l+\lambda_4)(m+\lambda_4)(n+\lambda_4)}\right)^k\right)\frac{1}{\lambda_4} \quad (1.14)$$

Take i=1 and 2

$$E(S) = p_1 \left\{ \frac{1-\left(\frac{lmn}{(l+\lambda_1)(m+\lambda_1)(n+\lambda_1)}\right)^k}{1-\left(\frac{lmn}{(l+\lambda_1)(m+\lambda_1)(n+\lambda_1)}\right)} \right\} \left[\left(\frac{lmn}{(l+\lambda_1)(m+\lambda_1)(n+\lambda_1)}\right)\right]$$
$$\left(E(s_1) - E(s_A) - E(s_B) + E(s_A)\frac{l}{l+\lambda_1} + E(s_B)\frac{l}{l+\lambda_1}\frac{m}{m+\lambda_1}\right)$$
$$+ p_2 \left\{ \frac{1-\left(\frac{lmn}{(l+\lambda_2)(m+\lambda_2)(n+\lambda_2)}\right)^k}{1-\left(\frac{lmn}{(l+\lambda_2)(m+\lambda_2)(n+\lambda_2)}\right)} \right\} \left[\left(\frac{lmn}{(l+\lambda_2)(m+\lambda_2)(n+\lambda_2)}\right)\right]$$
$$\left(E(s_1) - E(s_A) - E(s_B) + E(s_A)\frac{l}{l+\lambda_2} + E(s_B)\frac{l}{l+\lambda_2}\frac{m}{m+\lambda_2}\right) \quad (1.15)$$

IV. NUMERICAL EXAMPLE

To illustrate the applications of the above result different values for l, m, n, λ and k are given and E (T), E (S) are obtained in the following table.

Table: 1.1 Numerical tabulation for obtaining of E (T) and E (S) value

l	M	N	λ	p	E (T)	E (S)
0.01	0.02	0.03	0.04	0.25	24.2857	2.539
0.01	0.02	0.03	0.04	0.25	24.9796	2.582
0.01	0.02	0.03	0.04	0.25	24.9994	2.5829
0.01	0.02	0.03	0.04	0.25	25	2.5830
0.01	0.02	0.03	0.04	0.25	25	2.5832
0.01	0.02	0.03	0.04	0.25	25	2.5832

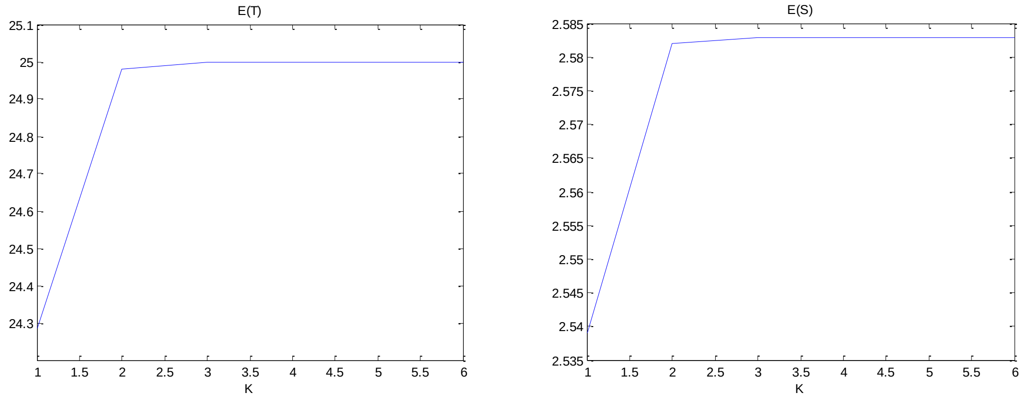


Figure 1.1 Graphs of E (T) and E (S)

Inference

In the above table, when the value of λ increases, E (T) and E (S) also increases.

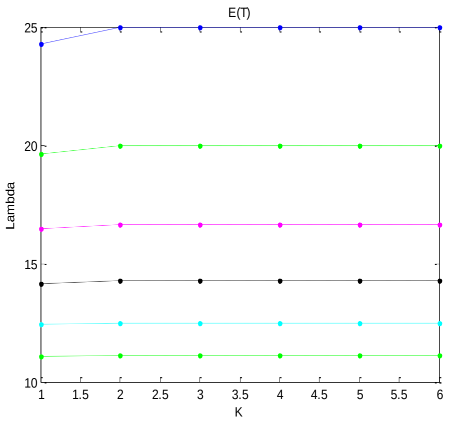
Taking l=1, λ=1 to 6 E (S) =20 and k=1 to 6, 3D Graphs of E (T) and E (S) are drawn.

Table: 1.2 Numerical tabulation for obtaining of E (T) values

	E (T)					
K	$\lambda_1 = \lambda_2$=0.04	$\lambda_1 = \lambda_2$=0.05	$\lambda_1 = \lambda_2$=0.06	$\lambda_1 = \lambda_2$=0.07	$\lambda_1 = \lambda_2$=0.08	$\lambda_1 = \lambda_2$=0.09
1	24.2857	19.6429	16.4683	14.1667	12.4242	11.0606
2	24.9796	19.9936	16.6643	14.2847	12.4995	11.1109
3	24.9994	19.9999	16.6666	14.2857	12.5	11.1111
4	25	20	16.6667	14.2857	12.5	11.1111
5	25	20	16.6667	14.2857	12.5	11.1111
6	25	20	16.6667	14.2857	12.5	11.1111

Table: 1.3 Numerical tabulation for obtaining of E (S) value

	E (S)					
K	$\lambda_1 = \lambda_2$=0.04	$\lambda_1 = \lambda_2$=0.05	$\lambda_1 = \lambda_2$=0.06	$\lambda_1 = \lambda_2$=0.07	$\lambda_1 = \lambda_2$=0.08	$\lambda_1 = \lambda_2$=0.09
1	1.678	1.938	2.1977	2.4661	2.746	3.0388
2	1.8458	2.0349	2.2605	2.5101	2.7787	3.0642
3	1.8626	2.0397	2.2623	2.5109	2.7791	3.0644
4	1.8643	2.04	2.2624	2.5109	2.7791	3.0644
5	1.8644	2.04	2.2624	2.5109	2.7791	3.0644
6	1.8644	2.04	2.2624	2.5109	2.7791	3.0644



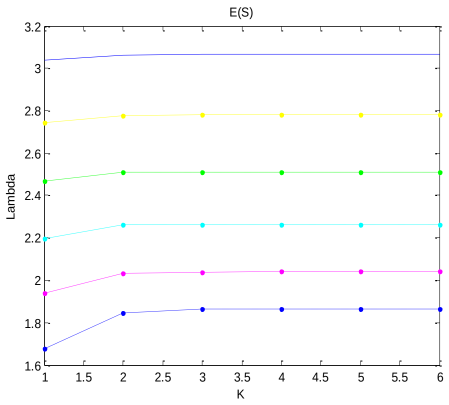


Figure 1.2: Graphs of E (T) and E (S)

Inference

In the above table, as k increases and the parameter λ is fixed, the expected values of E (T) and E (S) increase.

As k is fixed and the parameter λ increases, the expected values of E (T) and E (S) decrease.

V.CONCLUSION

When k, the number of products increases, both the expected time to sales E (T) and sales time E (S) increase.

When the exponential parameter λ increases, the expected time to sales E (T) decreases and sales time E (S) increases.

As k increases and the parameter λ is fixed, the expected values E (T) and E (S) increase. As k if fixed and the parameter λ increases, the expected values E (T) and E (S) decrease.

REFERENCES

[1] Anderson Sweeney and Williams, (2009), “Inventory models-quantitative methods for business” (1st ed.) Thompson Southwest.

[2] Ramanarayanan R., and Jacob M.J., (1998), “General analysis of (S, s) inventory systems with random lead time and bulk demands”, *Cashieru.C.E.R.O*, Vol.3 No. 4, pp. 119-123.

[3] Tijms H. C., (1986), “Stochastic Modeling and Analysis: A Computational Approach”, Wiley Series in Probability and Mathematical Statistics. John Wiley & Sons, Chichester New York Brisbane Toronto Singapore.

[4] Parvathi S. Srinivasa Raghavan S., and Ramanarayanan R., (2013), “General analysis of two products inventory system with seasonal production and sales”, *International Journal of pure and Applied Mathematics*, Vol 87, No. 2, pp.195-204.

[5] Bagchi U., and Hayya J.C., (1984), “Demand during lead time for normal unit demand and Erlang lead time”, *Journal of the Operational Research Society*, Vol. 35, pp. 131-135.

[6] David J.B., and Robert F.L., (1985), “A (Q, R) inventory model with lost sales and Erlang-distributed lead times”, Navel *Research Logistics Quarterly*, vol. 32 No. 4, pp. 605-611.

[7] Kadri T., and Smaili K., (2015), “Convolutions of Hyper-Erlang and of Erlang distributions”, *International Journal of Pure and Applied Mathematics*, Vol. 98 No. 1, pp. 81-98.

[9] Kalpakam S., and Arivarignan G., (1985), “A continuous review inventory system with arbitrary inter arrival times between demands and an exhibiting item subject to random failure”, *Operation Research*, Vol. 22, pp. 153–168.