The Study of Linear Surface Instability of type Rayleigh-Taylor in a Non-Newtonian Fluid

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Abstract : The paper deals with linear surface instability of type Rayleigh-Taylor to predict the nature of surface instability namely marginally stable, stable and unstable in a non-Newtonian Fluid using complementary Ramanujan's partition procedure with the help of the following theorem:

Theorem: "The system is stable, unstable or marginally stable depending on the positive number *n*, the power law index obtained by modeling a non-Newtonian fluid, obeying the power law model".

Keywords: Rayleigh-Taylor instability, power-law fluid, Ramanujan's partition theory, Saffman-slip condition.

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I. INTRODUCTION

Rayleigh-Taylor instability (RTI) deals with the instability between a less dense fluid below a high dense fluid, which has gained tremendous interest in both Newtonian and non-Newtonian fluids[1]. In several natural processes extending from coastal upwelling, which supports to renew the nutrients near sea surface [2] to the ignition of a supernova at the end of life of some stars [3], RTI plays a significant role. The RTI is present in the formation of some astrophysical structures such as the supernova remnants in the Eagle and Crab-nebula [4], in the air bubble formation, in the blood of deep-sea divers [5] and in various industrial processes [6]. It is one of the main subjects of concern in Inertial confinement fusion (ICF) [7]. Babchin et al.,[8] have studied the non-linear RT instability in a thin Newtonian fluid film when the wavelength is much greater than the film thickness. Later, Brown[9] relaxed this assumption on the wavelength and studied the RT instability in a finite thin layer of a viscous fluid using the combined Stokes and lubrication approximations as Babchin et al.[8]. Also, RTI has gained importance as the results obtained are of immense use in the design of:

(i) Inertial Fusion Energy (IFE) target to reduce the asymmetry caused due to laser radiation by reducing the growth rate at the ablative surface. Inertial fusion energy is considered by International Atomic Energy Agency (IAEA) of United Nations, Vienna, Austria as affordable, effective, efficient, everlasting, practical and provide uninterrupted supply of energy which do not have greater adverse impact on the environment as do fossil fuel.

For efficient extraction of fusion energy, it is essential to control RTI at the ablatively accelerated IFE targets. This RTI is one of the physical mechanisms limiting the performance of laser fusion target and hence additional mitigation of RTI growth rate to achieve high gain in IFE target is needed. Mechanisms like gradual variation of density instead of abrupt change, compression of fluid duct to laser impinge and target shell with porous foam have been generally used by different nuclear scientists to reduce the growth rate. Of these, the work of Takabe et al [10] of Japan is considered as effective mechanism to reduce the growth rate. Using compression effect resulting from laser radiation, Takabe et al [10] have shown that the maximum growth rate was reduced to 45% of the classical value and is called Japanese Takabe model. The work (See Rudraiah [11]) has proposed a mechanism of providing a nano-structured smart material lining at the ablative surface of IFE target and shown that the maximum growth rate has been reduced to 78.5% of the classical value. This reduction due to nano-structured smart lining is much greater and efficient than 45% reduction shown by Takabe et al [10]. The work on this is to improve the efficiency by using the external constraints of magnetic field and /or Coriolis force, electric field in a poorly conducting fluid, non-Newtonian fluid and so on(see Rudraiah[12,13]).

(ii) Coronary artery disease (CAD):

Recently in CAD, bypass surgery is replaced by laser surgery. The high intensity laser no 14the walls of the endothelium. To overcome this erosion there is a need to reduce the growth rate of this surface instability mainly of the RTI type. This understanding of nature of RTI in a non-Newtonian fluid is essential as the blood in the arteries is regarded as non-Newtonian fluid bounding the porous nature of endothelium.

(iii) Synovial Joints:

Another example where surface instabilities play a significant role is the synovial joints, which are freely movable joints. They consist of an articulate cartilage (CA) which is a two-phase deformable porous material having fixed electrical charges embedded in the tissue and a positively charged liquid (see Rudraiah et al [14,15]) and the synovial fluid (SF) which in general is a non-Newtonian fluid having viscosity 1000 times that of water. One of the causes for degenerative changes, evolving through Osteoarthritis due to old age, Traumatic arthritis due to injuries, Rheumatoid arthritis due to diseases, is due to surface instabilities of the type RTI occurring at the interface between CA and SF. Since the CA is a fluid saturated porous media lining the SF, the effective design of artificial joints there is a need to understand the nature of surface instability at the cartilage lined with the non-Newtonian nature of SF.

The RTI of inviscid, electrically conducting compressible fluid layer of finite thickness in the presence of magnetic field has being investigated by Vijayalakshmi[16]. The linear growth rate for the instability that occurs when the density in the region above the interface is greater than that of fluid below is calculated, by solving the linear eigenvalue problem obtained using the normal mode analysis. This problem has been solved separately in both the regions filled with constant temperature, ideal polytrope exponentially stratified electrically conducting fluids and the eigen frequencies are obtained by using the kinematic and dynamic pressure matching conditions at the interface. Thus, we have obtained the solution of the eigenvalue problem for the frequencies and investigated its dependence on the wave number *k* and other parameters. The main objective of this paper is study the nature of RTI in a non-Newtonian fluid using complementary Ramanujan's partition theory.

II. MATHEMATICAL FORMULATION

To study the nature of surface instability of the RTI type we consider a two-dimensional configuration shown in the Figure 1.

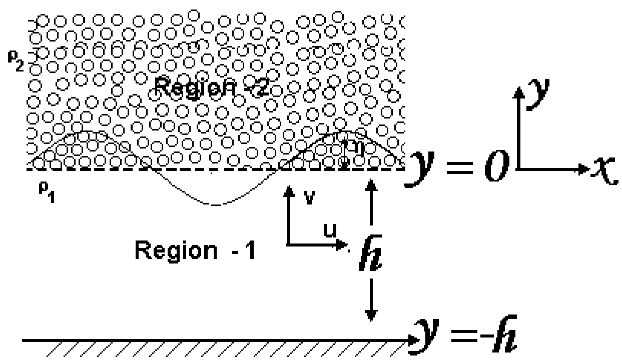

Figure 1: Physical Configuration

It consists of a lighter non-Newtonian incompressible fluid in region-1 and a non-Newtonian heavy fluid saturated in porous layer in region-2. The non-Newtonian fluid is modeled as power law fluid. The Cartesian co-ordinate system aligned with the plane describes power-law fluid, *x* points in horizontal direction and *y* is vertical to it. The fluid properties of two layers are different; we use subscripts to distinguish them as ρ_1 and ρ_2 which denote densities in upper and lower fluids respectively. The system involves two semi-infinite inviscid, incompressible power-law fluids differentiated by an interface in the existence of fluid porous layer. Since gravity is present; the effective acceleration is in the positive *y*- direction (upwards). Therefore, light fluid (region-1) thrusts on heavy fluid (region-2).

2.1. LINEAR RTI USING COMPLEMENTARY RAMANUJAN'S PARTITION

Since the flow in region - 2 is heavy and creeping, we assume that it is almost static. Further, following Rudraiah et al[15], we assume the flow in region-1 obeys combined lubrication and Stokes approximations and hence the basic equations for region-1 are:

$$\frac{\partial p}{\partial x} = \frac{\partial}{\partial y}\left(K \left|\frac{\partial u}{\partial y}\right|^{n-1} \frac{\partial u}{\partial y} \right) \tag{1}$$

$$\frac{\partial p}{\partial y} = 0 \tag{2}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{3}$$

where *K* is the consistency index, (u, v) are the horizontal and vertical components of velocity and *p* is the pressure. Using the following Boundary and Surface Conditions these equations can be solved.

The Saffman[17]- slip condition

$$\frac{\partial u}{\partial y} = -\frac{\alpha u}{\sqrt{k}} \text{ at } y=0 \tag{4}$$

The no-slip Condition

$$u = v = 0 \text{ at } y = -h \tag{5}$$

The normal-stress Condition

$$p = \delta\eta - \gamma \frac{\partial^2 \eta}{\partial x^2} \text{ at } y = 0 \tag{6}$$

where $\delta = \left(\rho_2 - \rho_1\right) g$, η is the interface elevation and γ is the surface tension.
Also, the kinematic Condition

$$\frac{\partial \eta}{\partial t} + u \frac{\partial \eta}{\partial x} = v(0) \tag{7}$$

In this part, we consider linear stability and hence Eq.(7) takes the form

$$\frac{\partial \eta}{\partial t} = v(0) \tag{8}$$

The solution of (1) satisfying the condition (4) and (5) is

$$u = \left(\frac{P}{K}\right)^{1/n} \frac{n}{(n+1)} \left[(-h)^{\frac{n+1}{n}} - \frac{(-h)^{\frac{n+1}{n}}}{(1+\alpha\sigma)} \left(1 - \alpha\sigma\left(y/h\right)\right) \right] \tag{9}$$

where $\sigma = \frac{h}{\sqrt{k}}$ which appears in Eq.(4) when we made it dimension by using the length scale *h* in *y* and $P = \frac{\partial p}{\partial x}$.

Integrating (3) w.r.t *y* from $-h$ to 0, and using (5), we get

$$v(0) = -\int_{-h}^{0} \frac{\partial u}{\partial x} dy = \frac{n}{(n+1) K^{\frac{1}{n}}} \frac{\partial}{\partial x} (p)^{1/n} (-h)^{\frac{2n+1}{n}} \left[\frac{2+\alpha\sigma}{1+\alpha\sigma} - \frac{n}{2n+\alpha} \right] \tag{10}$$

we can write

$$(P)^{1/n} = \left(\frac{\partial p}{\partial x}\right)^{1/n} = \left(\frac{\partial p}{\partial x}\right)^{\frac{1-n}{n}} \frac{\partial p}{\partial x}$$

Following Rudraiah et al (2000) we approximate

$$(P)^{1/n} = b(n) \frac{\partial p}{\partial x} + c(n) \tag{11}$$

where the fitting constants $b(n)$ and $c(n)$ are assumptions of power law index as well as normal stress rate. Using (11) then equation (10) becomes

$$v(0)=-\frac{n}{(n+1)K^{\frac{1}{n}}}b(n)\frac{\partial^2 p}{\partial x^2}(-h)^{\frac{2n+1}{n}}\frac{(2+2n+\alpha\sigma)}{2(2n+1)(1+\alpha\sigma)} \tag{12}$$

Using (12) and (16), equation (8), becomes

$$\frac{\partial\eta}{\partial t}=-\frac{nb(n)}{(2n+1)K^{\frac{1}{n}}}(-h)^{\frac{2n+1}{n}}\left[-\delta\frac{\partial^2\eta}{\partial x^2}-\gamma\frac{\partial^4\eta}{\partial x^4}\right]\frac{(2n+2+\alpha\sigma)}{2(2n+1)(1+\alpha\sigma)} \tag{13}$$

Assuming perturbation of the form

$$\eta=\eta_0\exp(\omega t+i\ell x) \tag{14}$$

where $\omega=\omega_r+i\omega_1,\ \frac{\omega_r}{\ell}$ is the phase velocity and ω_i is the growth rate. Then (13), using (14), takes the term

$$\omega=\frac{n}{(2n+1)K^{\frac{1}{n}}}(-1)^{\frac{3n+1}{n}}(h)^{2n+1/n}\frac{(2n+2+\alpha\sigma)}{2(n+1)(1+\alpha\sigma)}\ell^2\left(\delta-\gamma\ell^2\right) \tag{15}$$

Making this dimensionless, using

$$\omega^*=\frac{\omega}{b(n)\delta}\left(\frac{K}{\lambda}\right)^{\frac{1}{n}},\quad h^*=\frac{h}{\lambda},\ell^*=\ell\lambda,\quad \lambda=\sqrt{\frac{\gamma}{\delta}},\ x^*=\frac{x}{h},y^*=\frac{y}{h} \tag{16}$$

where the asterisks (*) denote dimensionless quantities. Equation (15), using (16) and neglecting asterisks for simplicity, we get

$$\omega=\frac{n}{(2n+1)}(-1)^{\frac{3n+1}{n}}h^{\frac{2n+1}{n}}\frac{(2n+2+\alpha\sigma)}{2(n+1)(1+\alpha\sigma)}\ell^2\left(1-\ell^2\right) \tag{17}$$

when $\sigma = 0$, implying absence of porous lining the growth rate ω_1, for non-Newtonian fluid, from (17) is

$$\omega_1=\frac{n}{(2n+1)}(-1)^{\frac{3n+1}{n}}h^{\frac{2n+1}{n}}\ell^2\left(1-\ell^2\right) \tag{18}$$

which is the same as the one given by Rudraiah et al[15]. Further when n = 1 implying Newtonian fluid in the presence of porous lining, the growth rate, ω_2, from, (17), is

$$\omega_2=\frac{h^3}{12}\frac{(4+\alpha\sigma)}{(1+\alpha\sigma)}\ell^2\left(1-\ell^2\right) \tag{19}$$

This is the same as the one given by Rudraiah[11]. Similarly, in the absence of porous lining with Newtonian Fluid (i.e., n=1), the growth rate, ω_3, either from (17) or from (19) is

$$\omega_3 = \frac{h^3}{3}\ell^2\left(1-\ell^2\right) \tag{20}$$

which is same as the one given by Brown[9] for Newtonian fluid in the absence of Porous lining.

III. NATURE OF STABILITY USING COMPLEMENTARY RAMANUJAN'S PARTITION THEOREM

In non-Newtonian Fluid satisfying the power law model there exists two types of fluids namely pseudo-plastic for which *n<1* and dilatant for which *n>1* (see Rudraiah and Kaloni[18]). We note that the dispersion relation (17) is valid for both pseudo-plastic and dilatant fluids. Also the nature of stability, namely magically (or neutrally) stable implying real part of ω, that is ω_r is zero, stable for $\omega_r < 0$ and unstable for $\omega_r > 0$, will depend on the values of n in $(-1)^{\frac{3n+1}{n}}$ appearing in (17). This is established in the following complimentary Ramanujan's Partition theorem.

➢ Complimentary Ramanujan's Partition Theorem:

The system is marginally stable, unstable or stable depending on the values of *n*.

Proof: Let

$$\omega = (-1)^{\frac{3n-1}{n}} = (-1)^{3+\frac{1}{n}} = -(-1)^{\frac{1}{n}} = e^{i\frac{\pi}{n}} \tag{21}$$

then, Re $\omega = \omega_r = -\cos\frac{\pi}{n}$. Hence, the following situations arise.

***Case 1*:** From (21), $\omega_r = 0$ iff $\frac{\pi}{n} = (2m+1)\,\frac{\pi}{n}$, where m = 0, 1, 2, 3, 4 ….

That is, $n \in \left\{\frac{2}{2m+1},\ for\ m = 0,1,2,3.....\right\}$

That is, iff $n \in \left\{2, 2/3, \frac{2}{5}, \frac{2}{7}, \frac{2}{9}, ...\right\}$

Pictorially, $\omega_r = 0$ at the points in the following Figure 2.

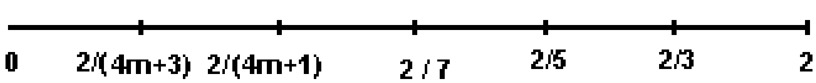

Figure 2: Values of *n* for which the system is marginally stable

***Case 2*:** From (21), $\omega_r > 0$ iff $\cos\frac{\pi}{n} < 0$.

That is, iff $\frac{\pi}{2} + 2\pi m < \frac{\pi}{n} < \frac{3\pi}{2} + 2m\pi$.

That is iff $\dfrac{2}{4m+3} < n < \dfrac{2}{2m+1}$

That is, iff $n \in \left(\dfrac{2}{4m+1}, \dfrac{1}{2m}\right) \cup \left(\dfrac{1}{2m+2}, \dfrac{2}{4m+3}\right), \quad m = 0,1,2,3$

That is, iff $n \in \left\{\left(\dfrac{2}{3}, 2\right) \cup \left(\dfrac{2}{7}, \dfrac{2}{5}\right) \cup \left(\dfrac{2}{11}, \dfrac{2}{9}\right) \cup \ldots \cup \left(\dfrac{2}{4m+3}, \dfrac{2}{4m+1}\right) \ldots\right\}$

Pictorially, $\omega_r > 0$ in the open intervals indicated by (++++) in the following Figure 3.

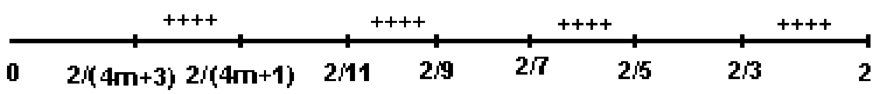

Figure 3: Values of *n* for which the system is unstable

***Case 3*:** From (21), $\omega_r < 0$ iff $\cos\dfrac{\pi}{n} > 0$.

That is iff $2m\pi \le \dfrac{\pi}{n} < 2m + \dfrac{\pi}{2} \; or \; 2m\pi + \dfrac{3\pi}{2} < \dfrac{\pi}{n} \le 2(m+1)\pi$

That is, iff $n \in \left(\dfrac{2}{4m+1}, \dfrac{1}{2m}\right) \cup \left(\dfrac{2}{2m+2}, \dfrac{2}{4m+3}\right), m = 0,1,2,3, \ldots\ldots$

That is, iff

$$n \in (2,\infty) \cup \left(\frac{2}{5}, \frac{2}{4}\right) \cup \left(\frac{2}{9}, \frac{2}{8}\right) \cup \ldots. \cup \left(\frac{2}{4m+5}, \frac{2}{4m+1}\right) \cup \ldots \cup \ldots \left(\frac{2}{4}, \frac{2}{3}\right) \cup \ldots \left(\frac{2}{4m+5}, \frac{2}{4m+3}\right) \cup \ldots.$$

That is $n \in (2,\infty) \cup \left(\dfrac{2}{5}, \dfrac{2}{3}\right) \cup \left(\dfrac{2}{9}, \dfrac{2}{7}\right) \cup \ldots. \cup \left(\dfrac{2}{4m+5}, \dfrac{2}{4m+3}\right) \cup \ldots$

Note : $\left(\dfrac{2}{5}, \dfrac{2}{4}\right) \cup \left(\dfrac{2}{4}, \dfrac{2}{3}\right) = \left(\dfrac{2}{5}, \dfrac{2}{3}\right) \ldots$

$$\left(\frac{2}{4m+5}, \frac{2}{4m+4}\right) \cup \left(\frac{2}{4m+4}, \frac{2}{4m+3}\right) = \left(\frac{2}{4m+5}, \frac{2}{4m+3}\right)$$

Pictorially, $\omega_r < 0$ in the open interval indicated by (- - - -) in Figure 4.

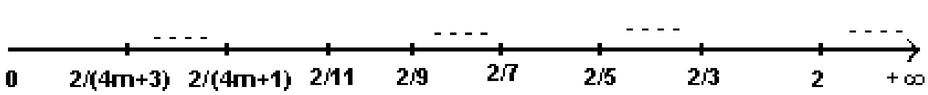

Figure 4: Values of *n* for which the system is stable

IV. CONCLUSION

The dispersion relation (17), is computed for different vales of film thickness *h*, power law index *n* and the consistence parameter *K.* We found that the nature of dispersion relation is influenced by both the reciprocal of the characteristic length $\frac{\delta}{\gamma}$ *and n*, with *h* just affecting the nature of growth rate of instability, in the sense that an increase in *h* increases the growth rate.

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