A Bulk Arrival Retrial Queue with Starting Failures and Exponentially Distributed Multiple Working Vacation

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Abstract- This paper discusses a bulk arrival retrial queue with Starting Failures and exponentially distributed multiple working vacation. The retrial time and service time are assumed to follow an arbitrary distribution. The customers in the orbit access the server under FCFS discipline. After completion of the service there is no customer in the orbit the server takes vacation. The vacation time follows an exponentional distribution. During the vacation period, customers can be served at a lower rate. Whenever the starting failure occurs the server may breakdown and sent for repair immediately, the repair time is arbitrary. Using supplementary variable method, we obtain the probability generating function for the number of customers in the orbit. Some particular cases are discussed.

Keywords- Bulk arrival retrial queues, Linear retrial policy, Starting Failures, Working vacation and Supplementary variable method.

MSC: 60K25, 60K30

1. INTRODUCTION

Retrial queueing models are characterized by the feature that arriving calls which find the server busy, do not line up or leave the system immediately forever, but go to some virtual place called as orbit and try their luck agian after some random time. For references, see the book by Falin and Templeton[7], the survey papers of Yang and Templeton [22] and Artalejo[2]. The characteristic of a working vacation is that the server serves customers at a lower service rate during the vacation period. In the literature of queueing systems with working vacations has been discussed through a considerable amount of work in the recent past. Servi and Finn [19] studied an M/M/1 queue with multiple working vacation and obtained the probability generating function for the number of customers in the system and the waiting time distribution. Some other notable works were done by Wu and Takagi [20], Aftab Begum [1], Pazhani Bala Murugan and Santhi[16] and Pazhani Bala Murugan and Vijaykrishnaraj[18].

In practical situtations we often meet the case where service stations may fail or slow down, during the time, at which the repairing works are carried out. Kulkarni and Choi [12] have analysed the M/G/1 retrial queue with server subjected to repairs and breakdowns. Yang and Li [21] have studied the M/G/1 retrial queue with server subject to starting failures. Krishna Kumar et al. have studied M/G/1 retrial queue with feedback and starting failures. Vacation queueing models subject to breakdowns have been studied by Choudhury and Tadj [5], Pazhani Bala Murugan and Santhi [17] studied Working vacation queueing models subject to server breakdowns.

In this paper we study an $M^X/G/1$ retrial queue with Starting Failures and exponentially distributed multiple working vacation. The organization of the paper is as follows. In section 2, we describe the model. In section 3, we obtain the steady state probability gernerating function. Particular cases are discussed in section 4.

2. MODEL DESCRIPTION

We consider an $M^X/G/1$ queueing system where the primary customers arrive according to a compound poisson process with arrival rate $\lambda(\geq 0)$. The batch size X is a random variable $Pr(X=n)=g_n, n=1,2,3,\ldots$ with probability generating function $X(z)=\sum_{k=1}^{\infty} g_k z^k$ and the first and second factorial moments of X are defined by $X^{(1)}(1)=E(X)$ and $X^{(2)}(1)=E(X(X-1))$. We assume that there is no waiting space and therefore if an arriving customer (external or repeated) finds the server occupied, he leaves the service area and joins a pool of blocked customers called orbit.

The retrial time follows a general distribution, with distribution functions $B(x)$. Let $b(x)$ and $B^*(\theta)$ denote the probability density function and Laplace Stieltjes Transform of $B(x)$ respectively for regular service period and let $a(x), A^*(\theta)$ denote the probability density function and Laplace Stieltjes Transform of $A(x)$ respectively for working vacation period. If the server is idle, an arriving customer must start the server, which takes negligible time. If the server is started successfully the customer gets service immediately. Otherwise, the repair for the server commences immediately and the customer must leave for the orbit and make a retrial at a later time. We assume that the probability of successful commencement of service is δ for a new customer who finds the server idle and sees no other customer in the orbit and is α for all other new and returning customers.

The service time is assumed to follow a general distribution, with distribution function $S_b(x)$ and density function $s_b(x)$. Let $S_b^*(\theta)$ be the Laplace Stieltjes Transform (LST) of the service time S_b.

Whenever the orbit becomes empty at a service completion instant the server starts a working vacation and the duration of the vacation time follows an exponential distribution with rate η. At a vacation completion instant if there are customers in the orbit the server will start a new busy period. Otherwise he takes another working vacation. This type of vacation policy is called multiple working vacation.

During the working vacation period, the server provides service with service time S_v which follows a general distribution with distribution function $S_v(x)$. Let$s_v(x)$ be the probability density function and $S_v^*(\theta)$ be the Laplace Stieltjes Transform of $S_v(x)$. Further, it is noted that the service interrupted at the end of a vacation is lost and it is restarted with different distribution at the beginning of the following service period.

We assume that inter-arrival times, retrial times, service times, working vacation times and breakdown times are mutually independent.

Let us define the following random variables.

N(t)-the orbit size at time t.

$A^0(t)$-the remaining retrial time in working vacation period.

$B^0(t)$-the remaining retrial time in regular service period.

$S_v^0(t)$-the remaining service time in working vacation period.

$S_b^0(t)$-the remaining service time in regular service period.

$S_{r_1}^0(t)$-the remaining repair time in working vacation period.

$S_{r_2}^0(t)$-the remaining repair time in regular service period.

$Y(t) = 0$ if the server is on working vacation period at time t but not occupied, 1 if the server is in regular service period at time t but not occupied, 2 if the server is busy on working vacation period at time t, 3 if the server is busy in regular service period at time t, 4 if the server is under repair on working vacation period, 5 if the server under repair on regular service period.

so that the supplementary variables $A^0(t), B^0(t), S_v^0(t), S_b^0(t), S_{r_1}^0(t)$ and $S_{r_2}^0(t)$ are introduced in order to obtain the bivaritate Markov Process $\{N(t), \partial(t); t \geq 0\}$,

Where,

$$\partial(t) = A^0(t)\ if\ Y(t) = 0, B^0(t)\ if\ Y(t) = 1, S_v^0(t)\ if\ Y(t) = 2, S_b^0(t)\ if\ Y(t) = 3,$$
$$S_{r_1}^0(t) if Y(t) = 4, S_{r_2}^0(t) if Y(t) = 5$$

We define the following limiting probabilites:

$$Q_{0,0} = \lim_{t\to\infty} Pr\{N(t) = 0, Y(t) = 0\}$$
$$Q_{0,n}(x) = \lim_{t\to\infty} Pr\{N(t) = n, Y(t) = 0, \quad x < A^0(t) \leq x + dx\}; n \geq 1$$
$$P_{0,n}(x) = \lim_{t\to\infty} Pr\{N(t) = n, Y(t) = 1, \quad x < B^0(t) \leq x + dx\}; n \geq 1$$
$$Q_{1,n}(x) = \lim_{t\to\infty} Pr\{N(t) = n, Y(t) = 2, \quad x < S_v^0(t) \leq x + dx\}; n \geq 0$$
$$P_{1,n}(x) = \lim_{t\to\infty} Pr\{N(t) = n, Y(t) = 3, \quad x < S_b^0(t) \leq x + dx\}; n \geq 0$$
$$R_{1,n}(x) = \lim_{t\to\infty} Pr\{N(t) = n, Y(t) = 4, \quad x < S_{r_1}^0(t) \leq x + dx\}; n \geq 1$$
$$R_{2,n}(x) = \lim_{t\to\infty} Pr\{N(t) = n, Y(t) = 5, \quad x < S_{r_2}^0(t) \leq x + dx\}; n \geq 1$$

We define the Laplace Stieltjes Transform and the probability generating functions as follows, $S_b^*(\theta) = \int_0^\infty e^{-\theta x} s_b(x)dx;$ $\quad S_v^*(\theta) = \int_0^\infty e^{-\theta x} s_v(x)dx;$ $\quad A^*(\theta) = \int_0^\infty e^{-\theta x} a(x)dx;$

$$B^*(\theta) = \int_0^\infty e^{-\theta x} b(x)dx; \qquad S_{r_1}^*(\theta) = \int_0^\infty e^{-\theta x} S_{r_1}(x)dx; \quad S_{r_2}^*(\theta) = \int_0^\infty e^{-\theta x} S_{r_2}(x)dx;$$
$$Q_{0,n}^*(\theta) = \int_0^\infty e^{-\theta x} Q_{0,n}(x)dx; \; Q_{0,n}^*(0) = \int_0^\infty Q_{0,n}(x)dx; \qquad Q_{1,n}^*(\theta) = \int_0^\infty e^{-\theta x} Q_{1,n}(x)dx;$$
$$Q_{1,n}^*(0) = \int_0^\infty Q_{1,n}(x)dx; \qquad P_{0,n}^*(\theta) = \int_0^\infty e^{-\theta x} P_{0,n}(x)dx; P_{0,n}^*(0) = \int_0^\infty P_{0,n}(x)dx$$

$$R_{1,n}^*(\theta) = \int_0^\infty e^{-\theta x} R_{1,n}(x)dx;\;\; R_{1,n}^*(0) = \int_0^\infty R_{1,n}(x)dx; \qquad R_{2,n}^*(\theta) = \int_0^\infty e^{-\theta x} R_{2,n}(x)dx$$

$$R_{2,n}^*(0) = \int_0^\infty R_{2,n}(x)dx; Q_0^*(z,\theta) = \sum_{n=1}^\infty Q_{0,n}^*(\theta)z^n; Q_0^*(z,0) = \sum_{n=1}^\infty Q_{0,n}^*(0)z^n$$

$$Q_0(z,0) = \sum_{n=1}^\infty Q_{0,n}(0)z^n; Q_1^*(z,\theta) = \sum_{n=0}^\infty Q_{1,n}^*(\theta)z^n; Q_1^*(z,0) = \sum_{n=0}^\infty Q_{1,n}^*(0)z^n;$$

$$Q_1(z,0) = \sum_{n=0}^\infty Q_{1,n}(0)z^n; P_0^*(z,\theta) = \sum_{n=1}^\infty P_{0,n}^*(\theta)z^n; P_0^*(z,0) = \sum_{n=1}^\infty P_{0,n}^*(0)z^n;$$

$$P_0(z,0) = \sum_{n=1}^\infty P_{0,n}(0)z^n; P_1^*(z,\theta) = \sum_{n=0}^\infty P_{1,n}^*(\theta)z^n;\; P_1^*(z,0) = \sum_{n=0}^\infty P_{1,n}^*(0)z^n;$$

$$P_1(z,0) = \sum_{n=0}^\infty P_{1,n}(0)z^n; R_1^*(z,\theta) = \sum_{n=0}^\infty R_{1,n}^*(\theta)z^n; R_1^*(z,0) = \sum_{n=0}^\infty R_{1,n}^*(0)z^n;$$

$$R_1(z,0) = \sum_{n=0}^\infty R_{1,n}(0)z^n; R_2^*(z,\theta) = \sum_{n=0}^\infty R_{2,n}^*(\theta)z^n; R_2^*(z,0) = \sum_{n=0}^\infty R_{2,n}^*(0)z^n;$$

$$R_2(z,0) = \sum_{n=0}^\infty R_{2,n}(0)z^n;$$

3. THE ORBIT SIZE DISTRIBUTION

By assuming that the system is in steady state condition, the differential difference equations governing the system are as follows:

$$\lambda Q_{0,0} = P_{1,0}(0) + Q_{1,0}(0) \tag{1}$$

$$-\frac{d}{dx}Q_{0,n}(x) = -(\lambda+\eta)Q_{0,n}(x) + Q_{1,n}(0)a(x) + R_{1,n}(0)a(x); n \geq 1 \tag{2}$$

$$-\frac{d}{dx}Q_{1,0}(x) = -(\lambda+\eta)Q_{1,0}(x) + \alpha Q_{0,1}(0)s_v(x) + \lambda g_1 \delta Q_{0,0} s_v(x) \tag{3}$$

$$-\frac{d}{dx}Q_{1,n}(x) = -(\lambda+\eta)Q_{1,n}(x) + \alpha Q_{0,n+1}(0)s_v(x) + \lambda g_{n+1}\delta Q_{0,0}s_v(x) + \sum_{k=1}^n \lambda g_k Q_{1,n-k}(x) + \sum_{k=1}^n \int_0^\infty \lambda g_k \alpha Q_{0,n-k+1}(x)dx.\, s_v(x); n \geq 1 \tag{4}$$

$$-\frac{d}{dx}P_{0,n}(x) = -\lambda P_{0,n}(x) + P_{1,n}(0)b(x) + \eta b(x)\int_0^\infty Q_{0,n}(x)dx + R_{2,n}(0)b(x) \tag{5}$$

$$-\frac{d}{dx}P_{1,0}(x) = -\lambda P_{1,0}(x) + \alpha P_{0,1}(0)s_b(x) + \eta s_b(x)\int_0^\infty Q_{1,0}(x)dx \tag{6}$$

$$-\frac{d}{dx}P_{1,n}(x) = -\lambda P_{1,n}(x) + \alpha P_{0,n+1}(0)s_b(x) + \sum_{k=1}^n \lambda g_k P_{1,n-k}(x) + \eta s_b(x)\int_0^\infty Q_{1,n}(x)dx + \sum_{k=1}^n \int_0^\infty \lambda g_k \alpha P_{0,n-k+1}(x)dx.\, s_b(x); n \geq 1 \tag{7}$$

$$-\frac{d}{dx}R_{1,1}(x) = -(\lambda+\eta)R_{1,1}(x) + \lambda g_1 \overline{\delta} Q_{0,0}s_{r_1}(x) + \overline{\alpha}Q_{0,1}(0)s_{r_1}(x) \tag{8}$$

$$-\frac{d}{dx}R_{1,n}(x) = -(\lambda+\eta)R_{1,n}(x) + \lambda g_n \overline{\delta} Q_{0,0}s_{r_1}(x) + \overline{\alpha}Q_{0,n}(0)s_{r_1}(x) + \sum_{k=1}^{n-1}\lambda g_k R_{1,n-k}(x) + \sum_{k=1}^{n-1}\int_0^\infty \lambda g_k \overline{\alpha} Q_{0,n-k}(x)dx.\, s_{r_1}(x); n \geq 2 \tag{9}$$

$$-\frac{d}{dx}R_{2,1}(x) = -\lambda R_{2,1}(x) + \overline{\alpha}P_{0,1}(0)s_{r_2}(x) + \eta s_{r_2}(x)\int_0^\infty R_{1,1}(x)dx \tag{10}$$

$$-\frac{d}{dx}R_{2,n}(x) = -\lambda R_{2,n}(x) + \overline{\alpha}P_{0,n}(0)s_{r_2}(x) + \sum_{k=1}^{n-1}\lambda g_k R_{2,n-k}(x) + \eta s_{r_2}(x)\int_0^\infty R_{1,n}(x)dx + \sum_{k=1}^{n-1}\int_0^\infty \lambda g_k \overline{\alpha} P_{0,n-k}(x)dx.\, s_{r_2}(x); n \geq 2 \tag{11}$$

Taking LST on both sides of the equation from (2) to (7) we get,

$$\theta Q_{0,n}^*(\theta) - Q_{0,n}(0) = (\lambda+\eta)Q_{0,n}^*(\theta) - Q_{1,n}(0)A^*(\theta) - R_{1,n}(0)A^*(\theta); n \geq 1 \tag{12}$$

$$\theta Q_{1,0}^*(\theta) - Q_{1,0}(0) = (\lambda+\eta)Q_{1,0}^*(\theta) - \alpha Q_{0,1}(0)S_v^*(\theta) - \lambda g_1 \delta Q_{0,0} S_v^*(\theta) \tag{13}$$

$$\theta Q_{1,n}^*(\theta)-Q_{1,n}(0)=(\lambda+\eta)Q_{1,n}^*(\theta)-\alpha Q_{0,n+1}(0)S_v^*(\theta)-\lambda g_{n+1}\delta Q_{0,0}S_v^*(\theta)$$
$$-\sum_{k=1}^{n}\lambda g_k Q_{1,n-k}^*(\theta)-\sum_{k=1}^{n}Q_{0,n+1-k}^*(0)\lambda g_k\alpha S_v^*(\theta);n\geq 1 \quad (14)$$
$$\theta P_{0,n}^*(\theta)-P_{0,n}(0)=\lambda P_{0,n}^*(\theta)-P_{1,n}(0)B^*(\theta)-\eta B^*(\theta)Q_{0,n}^*(0)-R_{2,n}(0)B^*(\theta);n\geq 1 \quad (15)$$
$$\theta P_{1,0}^*(\theta)-P_{1,0}(0)=\lambda P_{1,0}^*(\theta)-\lambda P_{0,1}(0)S_b^*(\theta)-\eta S_b^*(\theta)Q_{1,0}^*(0) \quad (16)$$
$$\theta P_{1,n}^*(\theta)-P_{1,n}(0)=\lambda P_{1,n}^*(\theta)-\lambda P_{0,n+1}(0)S_b^*(\theta)-\sum_{k=1}^{n}\lambda g_k P_{1,n-k}^*(\theta)$$
$$-\eta S_b^*(\theta)Q_{1,n}^*(0)-\sum_{k=1}^{n}\lambda g_k\alpha P_{0,n-k+1}^*(0)S_b^*(\theta);n\geq 1 \quad (17)$$
$$\theta R_{1,1}^*(\theta)-R_{1,1}(0)=(\lambda+\eta)R_{1,1}^*(\theta)-\lambda g_1\overline{\delta}Q_{0,0}s_{r_1}^*(\theta)-\overline{\alpha}Q_{0,1}(0)s_{r_1}^*(\theta) \quad (18)$$
$$\theta R_{1,n}^*(\theta)-R_{1,n}(0)=(\lambda+\eta)R_{1,n}^*(\theta)-\lambda g_n\overline{\delta}Q_{0,0}s_{r_1}^*(\theta)-\overline{\alpha}Q_{0,n}(0)s_{r_1}^*(\theta)-\sum_{k=1}^{n-1}\lambda g_k R_{1,n-k}^*(\theta)$$
$$-\sum_{k=1}^{n-1}\lambda g_k\overline{\alpha}Q_{1,n-k}^*(0)s_{r_1}^*(\theta);n\geq 2 \quad (19)$$
$$\theta R_{2,1}^*(\theta)-R_{2,1}(0)=\lambda R_{2,1}^*(\theta)-\overline{\alpha}P_{0,1}(0)s_{r_2}^*(\theta)-\eta R_{1,1}^*(0)s_{r_2}^*(\theta) \quad (20)$$
$$\theta R_{2,n}^*(\theta)-R_{2,n}(0)=\lambda R_{2,n}^*(\theta)-\overline{\alpha}P_{0,n}(0)s_{r_2}^*(\theta)-\sum_{k=1}^{n-1}\lambda g_k R_{2,n-k}^*(\theta)-\eta s_{r_1}^*(\theta)R_{1,n}^*(0)$$
$$-\sum_{k=1}^{n-1}\lambda g_k\overline{\alpha}P_{0,n-k}^*(0)s_{r_2}^*(\theta);n\geq 2 \quad (21)$$

Multiplying (12) with z^n and summed over n from 1 to ∞, we get

$$[\theta-(\lambda+\eta)]Q_0^*(z,\theta)=Q_0(z,0)-A^*(\theta)Q_1(z,0)-A^*(\theta)R_1(z,0)+A^*(\theta)Q_{1,0}(0) \quad (22)$$

z^n times (14) summed over n from 1 to ∞ and added up with (13) gives

$$[\theta-(\lambda-\lambda X(z)+\eta)]Q_1^*(z,\theta)=Q_1(z,0)-\frac{S_v^*(\theta)}{z}[\alpha Q_0(z,0)+\lambda\delta X(z)Q_{0,0}+\lambda\alpha X(z)Q_0^*(z,0)] \quad (23)$$

Multiplying (15) with z^n and summed over n from 1 to ∞, we get

$$[\theta-\lambda]P_0^*(z,\theta)=P_0(z,0)-B^*(\theta)P_1(z,0)-\eta B^*(\theta)Q_0^*(z,0)-B^*(\theta)R_2(z,0)+B^*(\theta)P_{1,0}(0) \quad (24)$$

z^n times (17) summed over n from 1 to ∞ and added up with (16) gives

$$[\theta-(\lambda-\lambda X(z))]P_1^*(z,\theta)=P_1(z,0)-\frac{S_b^*(\theta)}{z}[\alpha P_0(z,0)+\eta zQ_1^*(z,0)+\lambda\alpha X(z)P_0^*(z,0)] \quad (25)$$

z^n times (19) summed over n from 2 to ∞ and added up with (18) gives

$$[\theta-(\lambda-\lambda X(z)+\eta)]R_1^*(z,\theta)=R_1(z,0)-\lambda\overline{\delta}X(z)S_{r_1}^*(\theta)Q_{0,0}-\overline{\alpha}S_{r_1}^*(\theta)Q_0(z,0)$$
$$-\lambda\overline{\alpha}X(z)S_{r_1}^*(\theta)Q_0^*(z,0) \quad (26)$$

z^n times (21) summed over n from 2 to ∞ and added up with (20) gives

$$\left[\theta-\left(\lambda-\lambda X(z)\right)\right]R_2^*(z,\theta)=R_2(z,0)-\overline{\alpha}S_{r_2}^*(\theta)P_0(z,0)-\eta S_{r_2}^*(\theta)R_1^*(z,0)$$
$$-\lambda\overline{\alpha}X(z)S_{r_2}^*(\theta)P_0^*(z,0) \quad (27)$$

Inserting $\theta=\lambda+\eta$ in (22), we get

$$Q_0(z,0)=A^*(\lambda+\eta)\left[Q_1(z,0)+R_1(z,0)-Q_{1,0}(0)\right] \quad (28)$$

Substituting $\theta=0$ in (22) and using (28), we get

$$Q_0^*(z,0)=\frac{[1-A^*(\lambda+\eta)]}{\lambda+\eta}[Q_1(z,0)+R_1(z,0)-Q_{1,0}(0)] \quad (29)$$

Substituting $\theta=\lambda-\lambda X(z)+\eta$ in (23), we get

$$Q_1(z,0)=\frac{S_v^*(\lambda-\lambda X(z)+\eta)]}{z}[\alpha Q_0(z,0)+\lambda\delta X(z)Q_{0,0}+\lambda\alpha X(z)Q_0^*(z,0)] \quad (30)$$

Substituting $\theta=\lambda-\lambda X(z)+\eta$ in (26), we get

$$R_1(z,0)=[\lambda\overline{\delta}X(z)Q_{0,0}+\overline{\alpha}Q_0(z,0)+\lambda\overline{\alpha}X(z)Q_0^*(z,0)]S_{r_1}^*(\lambda-\lambda X(z)+\eta) \quad (31)$$

Inserting $\theta=0$ in (23)and using (30), we get

$$Q_1^*(z,0)=\frac{[1-S_v^*(\lambda-\lambda X(z)+\eta)][\alpha Q_0(z,0)+\lambda\alpha X(z)Q_0^*(z,0)+\lambda\delta X(z)Q_{0,0}]}{z[(\lambda-\lambda X(z)+\eta)]} \quad (32)$$

Inserting $\theta=0$ in (26)and using (31), we get

$$R_1^*(z,0)=\frac{[1-S_{r_1}^*(\lambda-\lambda X(z)+\eta)]}{[\lambda-\lambda X(z)+\eta]}[\lambda\overline{\delta}X(z)Q_{0,0}+\overline{\alpha}Q_0(z,0)+\lambda\overline{\alpha}X(z)Q_0^*(z,0)] \quad (33)$$

Solving the equation (28),(29), (30)and (31), we get

$$Q_0(z,0)=\frac{(\lambda+\eta)A^*(\lambda+\eta)[\lambda X(z)[\delta S_v^*(\lambda-\lambda X(z)+\eta)+\overline{\delta}zS_{r_1}^*(\lambda-\lambda X(z)+\eta)]Q_{0,0}-zQ_{1,0}(0)]}{(\lambda+\eta)z-[\alpha S_v^*(\lambda-\lambda X(z)+\eta)+\overline{\alpha}zS_{r_1}^*(\lambda-\lambda X(z)+\eta)][\lambda X(z)+(\lambda-\lambda X(z)+\eta)A^*(\lambda+\eta)]} \quad (34)$$

Let as consider,

$f(z) = (\lambda+\eta)z - [\alpha S_v^*(\lambda-\lambda X(z)+\eta) + \overline{\alpha} z S_{r_1}^*(\lambda-\lambda X(z)+\eta)][\lambda X(z) + (\lambda-\lambda X(z)+\eta)A^*(\lambda+\eta)]$,

at $z=0, f(0) = \alpha S_v^*(\lambda+\eta)(\lambda+\eta)A^*(\lambda+\eta) < 0$

at $z=1, f(1) = (\lambda+\eta) - [\alpha S_v^*(\eta) + \overline{\alpha} S_{r_1}^*(\eta)][\lambda + \eta A^*(\lambda+\eta)] > 0$

Therefore their exist a real root z_1 in (0,1) satisfying the equation $f(z) = 0$

Hence at $z = z_1$, the equation (34) becomes,

$$\lambda X(z_1)[\delta S_v^*(\lambda-\lambda X(z_1)+\eta) + \overline{\delta} z_1 S_{r_1}^*(\lambda-\lambda X(z_1)+\eta)]Q_{0,0} - z_1 Q_{1,0}(0) = 0$$

$$Q_{1,0}(0) = \frac{\lambda X(z_1)}{z_1}[\delta S_v^*(\lambda-\lambda X(z_1)+\eta) + \overline{\delta} z_1 S_{r_1}^*(\lambda-\lambda X(z_1)+\eta)]Q_{0,0}$$

$$Q_{1,0}(0) = \lambda U(z_1)Q_{0,0} \quad (35)$$

Where,

$$U(z_1) = \frac{X(z_1)}{z_1}[\delta S_v^*(\lambda-\lambda X(z_1)+\eta) + \overline{\delta} z S_{r_1}^*(\lambda-\lambda X(z_1)+\eta)] \quad (36)$$

Therefore,

$$Q_0(z,0) = \frac{(\lambda+\eta)A^*(\lambda+\eta)}{D_2(z)}[\lambda X(z)[\delta S_v^*(\lambda-\lambda X(z)+\eta) + \overline{\delta} z S_{r_1}^*(\lambda-\lambda X(z)+\eta)] - \lambda U(z_1)z]Q_{0,0} \quad (37)$$

$$Q_1(z,0) = \frac{S_v^*(\lambda-\lambda X(z)+\eta)}{D_2(z)}[\lambda(\lambda+\eta)\delta\lambda X(z) + \left[(\alpha\overline{\delta} - \overline{\alpha}\delta)\lambda X(z) S_{r_1}^*(\lambda-\lambda X(z)+\eta) - \alpha\lambda U(z_1)\right]$$
$$[\lambda X(z) + (\lambda-\lambda X(z)+\eta)A^*(\lambda+\eta)]]Q_{0,0} \quad (38)$$

$$R_1(z,0) = \frac{S_{r_1}^*(\lambda-\lambda X(z)+\eta)}{D_2(z)}[\lambda(\lambda+\eta)\overline{\delta} z X(z) - \left[(\alpha\overline{\delta} - \overline{\alpha}\delta)\lambda X(z) S_v^*(\lambda-\lambda X(z)+\eta) - \overline{\alpha} z\lambda U(z_1)\right]$$
$$[\lambda X(z) + (\lambda-\lambda X(z)+\eta)A^*(\lambda+\eta)] \quad (39)$$

Subtituting (35), (36), (37),(38) and (39) in (29),(32) and (33), we get

$$Q_0^*(z,0) = \frac{[1-A^*(\lambda+\eta)][\lambda X(z)[\delta S_v^*(\lambda-\lambda X(z)+\eta) + \overline{\delta} z S_{r_1}^*(\lambda-\lambda X(z)+\eta)] - \lambda U(z_1)z]Q_{0,0}}{D_2(z)} \quad (40)$$

$$Q_1^*(z,0) = \frac{[1-S_v^*(\lambda-\lambda X(z)+\eta)]\lambda(\lambda+\eta)\delta X(z)Q_{0,0}}{D_1(z)D_2(z)}$$
$$+ \frac{[\lambda X(z)(\alpha\overline{\delta} - \overline{\alpha}\delta)S_{r_1}^*(\lambda-\lambda X(z)+\eta) - \alpha\lambda U(z_1)][\lambda X(z) + (\lambda-\lambda X(z)+\eta)A^*(\lambda+\eta)]Q_{0,0}}{D_1(z)D_2(z)} \quad (41)$$

$$R_1^*(z,0) = \frac{[1-S_{r_1}^*(\lambda-\lambda X(z)+\eta)]\lambda(\lambda+\eta)\overline{\delta} z X(z)}{D_1(z)D_2(z)}$$
$$- \frac{[\lambda X(z)(\alpha\overline{\delta} - \overline{\alpha}\delta)S_v^*(\lambda-\lambda X(z)+\eta) + \overline{\alpha}\lambda U(z_1)z][\lambda X(z) + (\lambda-\lambda X(z)+\eta)A^*(\lambda+\eta)]Q_{0,0}}{D_1(z)D_2(z)} \quad (42)$$

Where,

$D_1(z) = [\lambda - \lambda X(z) + \eta]$

$D_2(z) = [(\lambda+\eta)z - [\alpha S_v^*(\lambda-\lambda X(z)+\eta) + \overline{\alpha} z S_{r_1}^*(\lambda-\lambda X(z)+\eta)]$
$[\lambda X(z) + (\lambda-\lambda X(z)+\eta)A^*(\lambda+\eta)]]$

Inserting $\theta = \lambda$ in (24), we get

$$P_0(z,0) = B^*(\lambda)[P_1(z,0) + \eta Q_0^*(z,0) + R_2(z,0) - P_{1,0}(0)] \quad (43)$$

Inserting $\theta = 0$ in (24), we get

$$P_0^*(z,0) = [\frac{1-B^*(\lambda)}{\lambda}][P_1(z,0) + \eta Q_0^*(z,0) + R_2(z,0) - P_{1,0}(0)] \quad (44)$$

Put $\theta = \lambda - \lambda X(z)$ in (25), we get

$$P_1(z,0) = \frac{S_b^*(\lambda-\lambda X(z))}{z}[\alpha P_0(z,0) + \eta z Q_1^*(z,0) + \lambda\alpha X(z)P_0^*(z,0)] \quad (45)$$

Put $\theta = 0$ in (25), we get

$$P_1^*(z,0) = \frac{[1-S_b^*(\lambda-\lambda X(z))]}{z[\lambda-\lambda X(z))]}[\alpha P_0(z,0) + \eta z Q_1^*(z,0) + \lambda\alpha X(z)P_0^*(z,0)] \quad (46)$$

Put $\theta = (\lambda - \lambda X(z))$ in (27), we get

$$R_2(z,0) = S_{r_2}^*(\lambda-\lambda X(z))[\overline{\alpha} P_0(z,0) + \eta R_1^*(z,0) + \overline{\alpha}\lambda X(z)P_0^*(z,0)] \quad (47)$$

Put $\theta = 0$ in (25), we get

$$R_2^*(z,0) = \frac{[1-S_{r_2}^*(\lambda-\lambda X(z))]}{[\lambda-\lambda X(z))]}[\overline{\alpha} P_0(z,0) + \eta R_1^*(z,0) + \overline{\alpha}\lambda X(z)P_0^*(z,0)] \quad (48)$$

Using (35) in (1), we get

$$P_{1,0}(0) = [1 - U(z_1)]\lambda Q_{0,0} \tag{49}$$

Solving equation (43) to (48) and using (49), we get

$$P_0(z,0) = \frac{B^*(\lambda)[\eta z Q_0^*(z,0) + \eta z S_b^*(\lambda - \lambda X(z))Q_1^*(z,0) + \eta z S_{r_2}(\lambda - \lambda X(z))R_1(z,0) - zP_{1,0}(0)]}{z - [\alpha S_b^*(\lambda - \lambda X(z)) + \overline{\alpha} z S_{r_2}^*(\lambda - \lambda X(z))][B^*(\lambda) + (1 - B^*(\lambda))X(z)]} \tag{50}$$

$$P_1(z,0) = \frac{\alpha S_b^*(\lambda - \lambda X(z))[B^*(\lambda) + (1 - B^*(\lambda))X(z)][\eta Q_0^*(z,0) + \eta S_{r_2}^*(\lambda - \lambda X(z))R_1^*(z,0) - (1 - U(z_1))\lambda Q_{0,0}]}{z - [\alpha S_b^*(\lambda - \lambda X(z)) + \overline{\alpha} z S_{r_2}^*(\lambda - \lambda X(z))][B^*(\lambda) + (1 - B^*(\lambda))X(z)]}$$
$$+ \frac{\eta z S_b^*(\lambda - \lambda X(z))[1 - \overline{\alpha} S_{r_2}^*(\lambda - \lambda X(z))[B^*(\lambda) + (1 - B^*(\lambda))X(z)]]Q_1^*(z,0)}{z - [\alpha S_b^*(\lambda - \lambda X(z)) + \overline{\alpha} z S_{r_2}^*(\lambda - \lambda X(z))][B^*(\lambda) + (1 - B^*(\lambda))X(z)]} \tag{51}$$

and

$$R_2(z,0) = \frac{\overline{\alpha} z S_{r_2}^*(\lambda - \lambda X(z))[B^*(\lambda) + (1 - B^*(\lambda))X(z)][\eta Q_0^*(z,0) - (1 - U(z_1))\lambda Q_{0,0} + \eta S_b^*(\lambda - \lambda X(z))Q_1^*(z,0)]}{z - [\alpha S_b^*(\lambda - \lambda X(z)) + \overline{\alpha} z S_{r_2}^*(\lambda - \lambda X(z))][B^*(\lambda) + (1 - B^*(\lambda))X(z)]}$$
$$+ \frac{\eta S_{r_2}^*(\lambda - \lambda X(z))[z - \alpha S_b^*(\lambda - \lambda X(z))[B^*(\lambda) + (1 - B^*(\lambda))X(z)]]R_1^*(z,0)}{z - [\alpha S_b^*(\lambda - \lambda X(z)) + \overline{\alpha} z S_{r_2}^*(\lambda - \lambda X(z))][B^*(\lambda) + (1 - B^*(\lambda))X(z)]} \tag{52}$$

Substituting (50), (51), (52) in (44), (46) and (48), we get

$$P_0^*(z,0) = \frac{[1 - B^*(\lambda)][\eta z Q_0^*(z,0) + \eta z S_b^*(\lambda - \lambda X(z))Q_1^*(z,0) + \eta z S_{r_2}(\lambda - \lambda X(z))R_1^*(z,0) - zP_{1,0}(0)]}{\lambda D_4(z)} \tag{53}$$

$$P_1^*(z,0) = [\frac{1 - S_b^*\big(\lambda - \lambda X(z)\big)}{D_3(z)D_4(z)}][B^*(\lambda) + (1 - B^*(\lambda))X(z)]\alpha\eta Q_0^*(z,0)$$
$$+ \Big[1 - \overline{\alpha} S_{r_2}^*\big(\lambda - \lambda X(z)\big)\big[B^*(\lambda) + \big(1 - B^*(\lambda)\big)X(z)\big]\Big]\eta z Q_1^*(z,0)$$
$$+ \big[B^*(\lambda) + \big(-B^*(\lambda)\big)X(z)\big]\alpha\eta S_{r_2}^*\big(\lambda - \lambda X(z)\big)R_1^*(z,0)$$
$$-[B^*(\lambda) + (1 - B^*(\lambda))X(z)]\alpha P_{1,0}(0)] \tag{54}$$

$$R_2^*(z,0) = [\frac{1 - S_{r_2}^*\big(\lambda - \lambda X(z)\big)}{D_3(z)D_4(z)}][\overline{\alpha}[B^*(\lambda) + (1 - B^*(\lambda))X(z)]\eta z Q_0^*(z,0)$$
$$+\overline{\alpha}[B^*(\lambda) + (1 - B^*(\lambda))X(z)]\ \eta z S_b^*(\lambda - \lambda X(z))Q_1^*(z,0) + [z - \alpha S_b(\lambda - \lambda X(z))$$
$$[B^*(\lambda) + (1 - B^*(\lambda))X(z)]]\eta R_1^*(z,0) - \overline{\alpha}[B^*(\lambda) + (1 - B^*(\lambda))X(z)]zP_{1,0}(0)] \tag{55}$$

Where,

$D_3(z) = \lambda - \lambda X(z)$

$D_4(z) = z - [\alpha S_b^*(\lambda - \lambda X(z)) + \overline{\alpha} z S_{r_2}^*(\lambda - \lambda X(z))][B^*(\lambda) + (1 - B^*(\lambda))X(z)]$

Subtituting (40), (41) and (42) in (53), (54) and (55), we get

$$P_0^*(z,0) = [1 - B^*(\lambda)]\frac{N_1(z)}{D_1(z)D_2(z)D_4(z)}Q_{0,0} \tag{56}$$

$$\mathrm{P}_1^*(\mathrm{z},0) = \frac{[1 - \mathrm{S}_b^*(\lambda - \lambda \mathrm{X}(\mathrm{z}))]\mathrm{N}_2(\mathrm{z})}{\mathrm{D}_1(\mathrm{z})\mathrm{D}_2(\mathrm{z})\mathrm{D}_3(\mathrm{z})\mathrm{D}_4(\mathrm{z})}\lambda \mathrm{Q}_{0,0} \tag{57}$$

$$\mathrm{R}_2^*(\mathrm{z},0) = \frac{[1 - \mathrm{S}_{r_2}^*(\lambda - \lambda \mathrm{X}(\mathrm{z}))]\mathrm{N}_3(\mathrm{z})}{\mathrm{D}_1(\mathrm{z})\mathrm{D}_2(\mathrm{z})\mathrm{D}_3(\mathrm{z})\mathrm{D}_4(\mathrm{z})}\lambda \mathrm{Q}_{0,0} \tag{58}$$

Where,

$D_1(z) = [\lambda - \lambda X(z) + \eta]$

$D_2(z) = [(\lambda + \eta)z - [\alpha S_v^*(\lambda - \lambda X(z)) + \overline{\alpha} z S_{r_1}^*(\lambda - \lambda X(z))][\lambda x(z) + (\lambda - \lambda X(z) + \eta)A^*(\lambda + \eta)]]$

$D_3(z) = [\lambda - \lambda X(z)]$

$D_4(z) = z - [\alpha S_b^*(\lambda - \lambda X(z)) + \overline{\alpha} z S_{r_2}^*(\lambda - \lambda X(z))][B^*(\lambda) + (1 - B^*(\lambda))X(z)]$

$$N_1(z) = \eta z[1 - A^*(\lambda + \eta)][\lambda - \lambda X(z) + \eta][X(z)[\delta S_v^*(\lambda - \lambda X(z) + \eta) + \overline{\square} z S_{r_1}^*(\lambda - \lambda X(z) + \eta)] - U(z_1)z]$$
$$+\eta z S_b^*(\lambda - \lambda X(z))[1 - S_v^*(\lambda - \lambda X(z) + \eta)][(\lambda + \eta)\delta X(z) + [X(z)(\alpha\overline{\delta} - \overline{\alpha}\delta)S_{r_1}^*(\lambda - \lambda X(z) + \eta)$$
$$-\alpha U(z_1)] \qquad [\lambda X(z) + (\lambda - \lambda X(z) + \eta)A^*(\lambda + \eta)]] + \eta z S_{r_2}^*(\lambda - \lambda\square(z)+)[1 - S_{r_1}^*(\lambda - \lambda X(z) + \eta)]$$
$$[(\lambda + \eta)\overline{\delta} X(z)z - [X(z)(\alpha\overline{\delta} - \overline{\alpha}\delta)S_v^*(\lambda - \lambda X(z) + \eta) + \overline{\alpha} U(z_1)z]$$
$$[\lambda X(z) + (\lambda - \lambda X(z) + \eta)A^*(\lambda + \ \eta)]] - [1 - U(z_1)]z[\lambda - \lambda X(z) + \eta]$$
$$[(\lambda + \eta)z - [\alpha S_v^*(\lambda - \lambda X(z) + \eta) + \overline{\alpha} z S_{\square_1}^*(\lambda - \lambda X(z) + \eta)]$$
$$[\lambda X(z) + (\lambda - \lambda X(z) + \eta)A^*(\lambda + \eta)]]$$

$$N_2(z) = \alpha\eta[B^*(\lambda) + (1 - B^*(\lambda)X(z)][1 - A^*(\lambda + \eta)][\lambda - \lambda X(z) + \eta]$$
$$[X(z)[\delta S_v^*(\lambda - \lambda X(z) + \eta) + \overline{\delta} z S_{r_1}^*(\lambda - \lambda X(z) + \eta)] - U(z_1)z]$$
$$+\eta z[1 - \overline{\alpha} S_{r_2}^*(\lambda - \lambda X(z)[B^*(\lambda) + (1 - B^*(\lambda)X(z))]]$$
$$[1 - S_v^*(\lambda - \lambda X(z) + \eta)][(\lambda + \eta)\delta X(z) + [X(z)(\alpha\overline{\delta} - \overline{\alpha}\delta)S_{r_1}^*(\lambda - \lambda X(z) + \eta) - \alpha U(z_1)]$$
$$[\lambda X(z) + (\lambda - \lambda X(z) + \eta)A^*(\lambda + \eta)]] + \alpha\eta S_{r_2}^*(\lambda - \lambda X(z))[B^*(\lambda) + (1 - B^*(\lambda)X(z)]$$

$$[1-S_{r_1}^*(\lambda-\lambda X(z)+\eta)][(\lambda+\eta)\overline{\delta}X(z)z-[X(z)(\alpha\overline{\delta}-\overline{\alpha}\delta)S_v^*(\lambda-\lambda X(z)+\eta)+\overline{\alpha}U(z_1)z]$$
$$[\lambda X(z)+(\lambda-\lambda X(z)+\eta)A^*(\lambda+\eta)]]-\alpha[1-U(z_1)][B^*(\lambda)+(1-B^*(\lambda)X(z)][\lambda-\lambda X(z)+\eta]$$
$$[(\lambda+\eta)z-[\alpha S_v^*(\lambda-\lambda X(z)+\eta)+\overline{\alpha}zS_{r_1}^*(\lambda-\lambda X(z)+\eta)][\lambda X(z)+(\lambda-\lambda X(z)+\eta)A^*(\lambda+\eta)]]$$
$$N_3(z)=\eta z\overline{\alpha}[B^*(\lambda)+(1-B^*(\lambda)X(z)][1-A^*(\lambda+\eta)][\lambda-\lambda X(z)+\eta]$$
$$[X(z)[\delta S_v^*(\lambda-\lambda X(z)+\eta)+\overline{\delta}zS_{r_1}^*(\square-\lambda X(z)+\eta)]-U(z_1)z]$$
$$+\eta z\overline{\alpha}S_b^*(\lambda-\lambda X(z)[B^*(\lambda)+(1-B^*(\lambda)X(z))][1-S_v^*(\lambda-\lambda X(z)+\eta)]$$
$$[(\lambda+\eta)\delta X(z)+[X(z)(\alpha\overline{\delta}-\overline{\alpha}\delta)S_{r_1}^*(\lambda-\lambda X(z)+\eta)-\alpha U(z_1)][\lambda X(z)+(\lambda-\lambda X(z)+\eta)A^*(\lambda+\eta)]]$$
$$+\eta[z-\alpha S_b^*(\lambda-\lambda X(z))[B^*(\lambda)+(1-B^*(\lambda)X(z)][1-S_{r_1}^*(\lambda-\lambda X(z)+\eta)]$$
$$[(\lambda+\eta)\overline{\delta}X(z)z-[X(z)(\alpha\overline{\delta}-\overline{\alpha}\delta)S_v^*(\lambda-\lambda X(z)+\eta)+\overline{\alpha}U(z_1)z][\lambda X(z)+(\lambda-\lambda X(z)+\eta)A^*(\lambda+\eta)]]$$
$$-\overline{\alpha}z[1-U(z_1)][B^*(\lambda)+(1-B^*(\lambda)X(z)][\lambda-\lambda X(z)+\eta]$$
$$[(\lambda+\eta)z-[\alpha S_v^*(\lambda-\lambda X(z)+\eta)+\overline{\alpha}zS_{r_1}^*(\lambda-\lambda X(z)+\eta)][\lambda X(z)+(\lambda-\lambda X(z)+\eta)A^*(\lambda+\eta)]]$$

We define,

$$P_V(z)=Q_0^*(z,0)+Q_1^*(z,0)+R_1^*(z,0)+Q_{0,0} \qquad (59)$$

as the probability generating function for the number of customers in the orbit when the server is on working vacation Period. Where $Q_0^*(z,0)$, $Q_1^*(z,0)$ and $R_1^*(z,0)$ are given in equation (40), (41) and (42).

We define,

$$P_B(z)=P_0^*(z,0)+P_1^*(z,0)+R_2^*(z,0) \qquad (60)$$

as the probability generating function for the number of customers in the orbit when the server is on busy period (Not working vacation Period).Where $P_0^*(z,0)$, $P_1^*(z,0)$ and $R_2^*(z,0)$ are given in equation (56), (57) and (58).

We define,

$$P(z)=P_V(z)+P_B(z) \qquad (61)$$

as the probability generating function for the number of customers in the orbit irrespective of the server state. Where $P_V(z)$ and $P_B(z)$ are given in equation (59) and (60). We shall now use the normalizing condition $P(1)=1$ to determine the unknown $Q_{0,0}$ which appears in (61). Substituting $z=1$ in (61) and using L'Hospitals rule, we obtain

$$Q_{0,0}=\frac{1-\lambda E(X)E[S_b]-\frac{\overline{\alpha}}{\alpha}\lambda E(X)E(S_{r_2})-\frac{1-B^*(\lambda)}{\alpha}E(X)}{\frac{\lambda-\lambda U(z_1)+\eta}{\eta}-[\frac{\alpha-\delta}{\alpha}]\lambda S_{r_2}^{*(1)}(0)-[\frac{1-B^*(\lambda)}{\alpha}-\lambda S_b^{*(1)}(0)S_v^*(\eta)][\frac{(\lambda+\eta)\delta+[(\alpha-\delta)S_{r_1}^*(\eta)-\alpha U(z_1)][\lambda+\eta A^*(\lambda+\eta)]}{\lambda+\eta-[\alpha S_r^*(\eta)+\overline{\alpha}S_{r_1}^*(\eta)][\lambda+\eta A^*(\lambda+\eta)]}]}$$
$$+\frac{1-\lambda E(X)E[S_b]-\frac{\overline{\alpha}}{\alpha}\lambda E(X)E(S_{r_2})-\frac{1-B^*(\lambda)}{\alpha}E(X)}{\lambda S_{r_2}^{*(1)}(0)S_{r_1}^*(\eta)[\frac{(\lambda+\eta)\overline{\delta}-[(\alpha-\delta)S_r^*(\eta)+\overline{\alpha}U(z_1)][\lambda+\eta A^*(\lambda+\eta)]}{\lambda+\eta-[\alpha S_v^*(\eta)+\overline{\alpha}S_{r_1}^*(\eta)][\lambda+\eta A^*(\lambda+\eta)]}]} \qquad (62)$$

4. PARTICULAR CASES

Case (i):

Suppose that the arrival is single, there is no retrial and no starting failures by setting $B^*(\lambda)=1, A^*(\lambda+\eta)=1$, $X(z)=z$ and $\alpha=\delta=1$ in (61), we get

$$P(z)=P_V(z)+P_B(z) \qquad (63)$$

Where,

$$P_V(z)=\frac{[1-S_v^*(\lambda-\lambda z+\eta)][z-z_1]\lambda zQ_{0,0}}{[\lambda-\lambda z+\eta][z-S_v^*(\lambda-\lambda z+\eta)]}$$
$$P_B(z)=\frac{[1-S_v^*(\lambda-\lambda X(z)+\eta)]\eta z[1-S_v^*(\lambda-\lambda X(z)+\eta)]-[1-U(z_1)][\lambda-\lambda z+\eta][z-S_v^*(\lambda-\lambda z+\eta]}{[1-z][\lambda-\lambda z+\eta][z-S_b^*(\lambda-\lambda z][z-S_v^*(\lambda-\lambda z+\eta]}$$
$$Q_{0,0}=\frac{1-\lambda E(X)E[S_b]}{[\frac{\lambda-\lambda z_1+\eta}{\eta}-\frac{1-z_1}{1-S_v^*(\eta)}]}$$

Equation (63) is well known probability generating function for number of customers in the orbit of an $M/G/1$ queue with multiple working vacation (Takagi (2006)) irrespective of the notation.

Case (ii):

If there is no retrial, arrival is single, no starting failures and the server never takes vacation [by setting $B^*(\lambda)=1$, $A^*(\lambda+\eta)=1$, $X(z)=z$, $\alpha=\delta=1$ and $\eta\to\infty$ in (61)], we get

$$P(z)=\frac{S_b^*(\lambda-\lambda z)(1-z)[1-\lambda E[S_b]]}{S_b^*(\lambda-\lambda z)-z} \qquad (64)$$

Equation (64) is well known probability generating function of the steady state system length distribution of an $M/G/1$ queue (Medhi (2003)) irrespective of the notations.

Case (iii):

If the server never takes the vacation, the arrival is single and no Starting failures then taking limit $\eta \to \infty$ and put $X(z) = z,\ \alpha = \delta = 1$ in (61), we get

$$P(z) = \frac{[B^*(\lambda) - \lambda E[S_b]][1-z]S_b^*(\lambda - \lambda z)}{B^*(\lambda)[1-z]S_b^*(\lambda - \lambda z) - z[1 - S_b^*(\lambda - \lambda z)]} \tag{65}$$

Equation (65) is well known probability generating function of the orbit size distribution of an $M/G/1$ retrial queue (Gomez-Corral (1999)) irrespective of the notations.

Case (iv):

If no starting failures then by setting ($\alpha = \delta = 1$) in (61), we get

$$P(z) = P_V(z) + P_B(z) \tag{66}$$

Where, $P_V(z) = \frac{N_V(z)}{D_V(z)} Q_{0,0}$ and $P_B(z) = \frac{N_B(z)}{D_B(z)} Q_{0,0}$

$N_V(z) = \lambda[1 - A^*(\lambda + \eta)][\lambda - \lambda X(z) + \eta][X(z)S_v^*(\lambda - \lambda X(z) + \eta) - U(\square_1)z] + \lambda[1 - S_v^*(\lambda - \lambda X(z) + \eta)]$
$[(\lambda + \eta)X(z) - U(z_1)[\lambda X(z) + (\lambda - \lambda X(z) + \eta)A^*(\lambda + \eta)]] + [\lambda - \lambda X(z) + \eta][(\lambda + \eta)z$
$-S_v^*(\lambda - \lambda X(z) + \eta)[\lambda X(z) + (\lambda - \lambda X(z) + \eta)A^*(\lambda + \eta)]$

$D_V(z) = [\lambda - \lambda X(z) + \eta][(\lambda + \eta)z - S_v^*(\lambda - \lambda X(z) + \eta)[\lambda X(z) + (\lambda - \lambda X(z) + \eta)A^*(\lambda + \eta)]$

$N_B(z) = \lambda\eta z[1 - S_v^*(\lambda - \lambda X(z) + \eta)][(\lambda + \eta)X(z) - U(z_1)[\lambda X(z) + (\lambda - \lambda X(z) + \eta)A^*(\lambda + \eta)]]$
$[1 - S_b^*(\lambda - \lambda X(z))[B^*(\lambda) + (1 - B^*(\lambda)X(z)]] + \lambda\eta[1 - A^*(\lambda + \eta)][\lambda - \lambda X(z) + \eta]$
$[X(z)S_v^*(\lambda - \lambda X(z) + \eta) - U(z_1)z][z + [1 - z - S_v^*(\lambda - \lambda X(z))][B^*(\lambda) + (1 - B^*(\lambda)X(z)]$
$-\lambda[1 - U(z_1)][\lambda - \lambda X(z) + \eta][(\lambda + \eta)z - S_v^*(\lambda - \lambda X(z))[\lambda X(z) + (\lambda - \lambda X(z) + \eta)]]$
$[z + [1 - z - S_b^*(\lambda - \lambda X(z))][B^*(\lambda) + (1 - B^*(\lambda)X(z)]]$

$D_B(z) = [\lambda - \lambda X(z)][\lambda - \lambda X(z) + \eta][z - S_b^*(\lambda - \lambda X(z))[B^*(\lambda) + (1 - B^*(\lambda)X(z)]]$
$[(\lambda + \eta)z - S_v^*(\lambda - \lambda X(z) + \eta)[\lambda X(z) + (\lambda - \lambda X(z) + \eta)A^*(\lambda + \eta)]]$

$$Q_{0,0} = \frac{1 - \lambda E(X)E[S_b] - E(X)[1 - B^*(\lambda)]}{\frac{\lambda - \lambda U(z_1) + \eta}{\eta} - [\frac{\lambda + \eta - U(z_1)[\lambda + \eta A^*(\lambda + \eta)]}{\lambda + \eta - S_v^*(\eta)[\lambda + \eta A^*(\lambda + \eta)]}][\lambda E[S_b]S_v^*(\eta) + 1 - B^*(\lambda)]}$$

Equation (66) is well known generating function of the orbit size distribution of A bulk arrival retrial queue with exponentially distributed multiple working vacation studied by S.Pazhani Bala Murugan and R.Vijaykrishnaraj [17] irrespective of the notation.

Case(v):

If there is no retrial and no starting failures then on setting $A^*(\lambda + \eta) = 1,\ B^*(\lambda) = 1$ and $\alpha = \delta = 1$ in (61), we get

$$P(z) = P_V(z) + P_B(z) \tag{67}$$

Where, $P_V(z) = \frac{N_V(z)}{D_V(z)} Q_{0,0}$ and $P_B(z) = \frac{N_B(z)}{D_B(z)} Q_{0,0}$

$N_V(z) = \lambda[1 - S_v^*(\lambda - \lambda X(z) + \eta)][X(z) - U(z_1)z] + [\lambda - \lambda X(z) + \eta][z - S_v^*(\lambda - \lambda X(z) + \eta)]$

$D_V(z) = [\lambda - \lambda X(z) + \eta][z - S_v^*(\lambda - \lambda X(z) + \eta)]$

$N_B(z) = \lambda\eta z[1 - S_v^*(\lambda - \lambda X(z) + \eta)][X(z) - U(z_1)z][1 - S_b^*(\lambda - \lambda X(z))]$

$D_B(z) = [\lambda - \lambda X(z)][z - S_b^*(\lambda - \lambda X(z)][\lambda - \lambda X(z) + \eta][z - S_v^*(\lambda - \lambda X(z) + \eta)]$

$$Q_{0,0} = \frac{1 - \lambda E(X)E(S_b)}{[\frac{\lambda - \lambda U(z_1) + \eta}{\eta} - \frac{1 - U(z_1)}{1 - S_v^*(\eta)}[\lambda E[S_b]S_v^*(\eta)]]}$$

Equation (67) is well known generating function of the queue size distribution of an $M^X/G/1$ queue with multiple working vacation studied by M.I.Aftab Begum (2011) irrespective of the notation.

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