Comparing different Estimators of two Parameters for new Transmuted Rayleigh Distribution

Dr. Worood Basim Noor Bihea
University Dijlah College
worodbasim@gmail.com

Rehab. K. Hamza Al-Mafraji
Institute Management Al-Ressafa
Rehabhamza86@gmail.com

Abstract:

This paper deals with transforming a one parameter Rayleigh to a new family of two parameters called transmuted Rayleigh, all the derivations required for this work are explained like finding p.d.f, C.D.f, deriving formula for \( r^{th} \) moment, then comparing estimators by maximum likelihood, moment estimator and gran estimator by simulation procedure.

Keywords: Rayleigh distribution, Maximum likelihood, Grans method, Moment method.

1. Introduction

Rayleigh distribution is a special case of the type of failure model (Weibull distribution when shape parameter equal (2)). Rayleigh distribution has many applications in life testing experiment, attributed sampling and many other field of reliability and test of life. It is derived originally by physical world of England (Lord Rayleigh 1919), then many other scientists used this model to analyze the resulting data of different experiment.

Let the \( p.d.f \) of Rayleigh failure density function is;

\[
f_T(t; \theta) = \frac{2}{\theta^2} t e^{-\frac{(t)^2}{\theta}} \quad t, \theta > 0
\]  

(1)

Mean life and variance of \((T)\) is;

\[
E(T) = \frac{\sqrt{\pi \theta}}{2}
\]
\[
var(T) = \theta \left(1 - \frac{\pi}{4}\right)
\]
Rayleigh distribution has different applications which are signal analysis and error analysis of different type system.

The C.D.F of one parameter Rayleigh is;

\[ F(t) = 1 - e^{-\left(\frac{t}{\theta}\right)^2} \] (2)

The one parameter Rayleigh can be transformed to a new family called transmuted Rayleigh by using transmuted parameter \(|\lambda| \leq 1\), then applying the transformation in equation;

\[ G(t) = (1 + \lambda)F(t) - \lambda[F(t)]^2 \] (3)
\[ g(t) = (1 + \lambda)f(t) - 2\lambda[F(t)]f(t) \] (4)

Therefore;

\[ g(t) = f(t) \{ (1 + \lambda) - 2\lambda[F(t)] \} \]

\[ = \frac{2}{\theta^2} t e^{-\left(\frac{t}{\theta}\right)^2} \left\{ (1 + \lambda) - 2\lambda[1 - e^{-\left(\frac{t}{\theta}\right)^2}] \right\} \]

\[ g(t) = \frac{2}{\theta^2} t e^{-\left(\frac{t}{\theta}\right)^2} \left\{ 1 - \lambda + 2\lambda e^{-\left(\frac{t}{\theta}\right)^2} \right\} \] (5)

This is new generated transmuted one parameter Rayleigh into two parameters new family \((\lambda, \theta)\), while the derived C.D.F obtained from (3);

\[ G(t) = (1 + \lambda) \left( 1 - e^{-\left(\frac{t}{\theta}\right)^2} \right) - \lambda \left( 1 - e^{-\left(\frac{t}{\theta}\right)^2} \right)^2 \] (6)

This reduced to;

\[ G(t) = \left( 1 - e^{-\left(\frac{t}{\theta}\right)^2} \right) \left[ 1 + \lambda - \lambda \left( 1 - e^{-\left(\frac{t}{\theta}\right)^2} \right) \right] \]

\[ G(t) = \left( 1 - e^{-\left(\frac{t}{\theta}\right)^2} \right) \left[ 1 + \lambda e^{-\left(\frac{t}{\theta}\right)^2} \right] \] (7)

2. Methods of Estimation

2.1 Moment Estimation

Now we derive a formula for \(r^{th}\) moments about origin;
\[ \mu'_r = E(t^r) = \int_0^\infty t^r g(t) \] 

(8)

Then;

\[ \mu'_r = \int_0^\infty t^r \frac{2}{\theta^2} t e^{-\left(\frac{t}{\theta}\right)^2} \left\{ 1 - \lambda + 2\lambda e^{-\left(\frac{t}{\theta}\right)^2} \right\} dt \]

\[ = (1 - \lambda) \frac{2}{\theta^2} \int_0^\infty t^{r+1} e^{-\left(\frac{t}{\theta}\right)^2} dt + \frac{4\lambda}{\theta^2} \int_0^\infty t^{r+1} e^{-\left(\frac{t}{\theta}\right)^2} dt \]

Let \( u = \left(\frac{t}{\theta}\right)^2 \) \( \sqrt{u} = \frac{t}{\theta} \) \( t = \theta \sqrt{u} \) \( dt = \frac{\theta}{2\sqrt{u}} du \)

\[ \mu'_r = l_1 + l_2 \]

\[ l_1 = \int_0^\infty t^{r+1} e^{-\left(\frac{t}{\theta}\right)^2} dt \]

\[ = \int_0^\infty (\theta \sqrt{u})^{r+1} e^{-u} \frac{\theta}{2\sqrt{u}} du \]

\[ = \frac{\theta^{r+2}}{2} \int_0^\infty (u)^{r/2} e^{-u} du \]

\[ l_1 = \frac{\theta^{r+2}}{2} \Gamma\left(\frac{r}{2} + 1\right) = (1 - \lambda)\theta^r \Gamma\left(\frac{r}{2} + 1\right) \]

(9)

\[ l_2 = \frac{4\lambda}{\theta^2} \int_0^\infty t^{r+1} e^{-\left(\frac{t}{\theta}\right)^2} dt \]

\[ = \frac{4\lambda}{\theta^2} \int_0^\infty (\theta \sqrt{u})^{r+1} e^{-2u} \frac{\theta}{2\sqrt{u}} du \]

\[ = 2\lambda \theta^r \int_0^\infty (u)^{r/2} e^{-2u} du \]

\[ = \frac{\lambda \theta^r}{2^\frac{r}{2}} \int_0^\infty (2u)^{r/2} e^{-2u} du \]

\[ = \frac{\lambda \theta^r}{2^\frac{r}{2}} \Gamma\left(\frac{r}{2} + 1\right) \]

(10)

Then a formula for \( r^{th} \) moments about origin;

\[ \mu'_r = (1 - \lambda)\theta^r \Gamma\left(\frac{r}{2} + 1\right) + \frac{\lambda \theta^r}{2^\frac{r}{2}} \Gamma\left(\frac{r}{2} + 1\right) \]
\[
\mu'_r = \Gamma\left(\frac{r}{2} + 1\right) \theta^r \left[(1 - \lambda) + \frac{\lambda}{r^2}\right]
\]

(11)

Since \(|\lambda| \leq 1\) we can only estimate (\(\theta\)) by moment estimator (\(\hat{\theta}\)) by solving (\(\mu'_1 = \bar{x}\)), then;

\[
(1 - \lambda)\hat{\theta}_{mom} \Gamma\left(\frac{3}{2}\right) + \frac{\lambda \hat{\theta}_{mom}}{\sqrt{2}} \Gamma\left(\frac{3}{2}\right) = \bar{x}
\]

\[
\hat{\theta}_{mom} \Gamma\left(\frac{3}{2}\right) [(1 - \lambda) + \frac{\lambda}{\sqrt{2}}] = \bar{x}
\]

(12)

This equation gives \(\hat{\theta}_{mom}\)

### 2.2 Maximum Likelihood Estimators

Let \((t_1, t_2, ..., t_n)\) be a random sample from equation (5), then;

\[
L = \prod_{i=1}^{n} g(t_i)
\]

\[
L = \frac{2}{\theta^{2n}} \prod_{i=1}^{n} t_i e^{-\sum_{i=1}^{n} \left(\frac{t_i}{\theta}\right)^2} \prod_{i=1}^{n} \left\{1 - \lambda + 2\lambda e^{-\sum_{i=1}^{n} \left(\frac{t_i}{\theta}\right)^2}\right\}
\]

\[
\log L = n \log 2 - 2n \log \theta + \sum_{i=1}^{n} \log t_i - \sum_{i=1}^{n} \left(\frac{t_i}{\theta}\right)^2 + \sum_{i=1}^{n} (K)
\]

Where \(K = \left\{1 - \lambda + 2\lambda e^{-\left(\frac{t_i}{\theta}\right)^2}\right\}\)

\[
\frac{\partial L}{\partial \theta} = -\frac{2n}{\theta} + \frac{1}{\theta^2} \sum_{i=1}^{n} t_i^2 + \sum_{i=1}^{n} \frac{1}{k} \left\{2\lambda e^{-\left(\frac{t_i}{\theta}\right)^2} \left(\frac{t_i^2}{\theta^2}\right)\right\}
\]

\[
\sum_{i=1}^{n} \frac{2 \lambda t_i^2 e^{-\left(\frac{t_i}{\theta}\right)^2}}{\left(1 - \lambda + 2\lambda e^{-\left(\frac{t_i}{\theta}\right)^2}\right)} = 2n\theta + \sum_{i=1}^{n} t_i^2
\]

Therefore
\[ \hat{\theta}_{MLE} = \frac{\sum_{i=1}^{n} \left( \frac{2\lambda t_i^2 e^{-\left(\frac{t_i}{\theta}\right)^2}}{1 - \lambda + 2\lambda e^{-\left(\frac{t_i}{\theta}\right)^2}} \right) - \sum_{i=1}^{n} t_i^2}{2n} \]  

(13)

While

\[ \frac{\partial L}{\partial \lambda} = \sum_{i=1}^{n} \left( \frac{2e^{\left(\frac{t_i}{\theta}\right)^2}}{1 - \lambda + 2\lambda e^{\left(\frac{t_i}{\theta}\right)^2}} \right) = 0 \]  

(14)

Which can be solved numerically to find \( \hat{\lambda}_{MLE} \).

### 2.3 Crane's Method

The estimation by this method depends on using nonparametric estimators for sample moments. This is denoted by \( m'_k \), then equating \( m'_k \) with \( m'_k = E(t^r) \), where;

\[ m'_k = \int_0^\infty [1 - G_w(t)]^k dt = \sum_{r=0}^{n} \left(1 - \frac{r}{n}\right)^k (t_{r+1} - t_r) \]  

(15)

where \( t_0 = 0, \ t_1 \leq t_2 \leq \ldots \leq t_n, \ m'_1 = \bar{t} \)

The C.D.F estimated by;

\[ G_w(t) = \begin{cases} 
0 & t < t_{(1)} \\
\left(\frac{r}{n}\right) & t_{(r)} \leq t \leq t_{(r+1)} \\
1 & t_{n} \leq t 
\end{cases} \]  

(16)

Therefore \( \mu'_k = m'_k = E(t^r) \), where \( E(t^r) \) found in equation (11).

### 3. Simulation procedure

This section deals with introducing results of simulation experiments to find the best estimator of two parameters \( (\lambda, \theta) \) of transmuted Rayleigh distribution taking different sample size \( (n = 25, 50, 75) \) by replicate each experiment \( (R = 500) \), the comparison has been done using mean square error.
Table (1): Transmuted Rayleigh distribution two parameters estimators

<table>
<thead>
<tr>
<th>$n$</th>
<th>Method</th>
<th>$\lambda = -0.5$</th>
<th>$\theta = 1.5$</th>
<th>$\lambda = 0.8$</th>
<th>$\theta = 2$</th>
<th>$\lambda = -0.3$</th>
<th>$\theta = 3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>25</td>
<td>MLE</td>
<td>1.1511</td>
<td>0.8748</td>
<td>2.2106</td>
<td>1.4656</td>
<td>3.3133</td>
<td>1.9541</td>
</tr>
<tr>
<td></td>
<td>CRA</td>
<td>0.9830</td>
<td>0.8977</td>
<td>1.9825</td>
<td>1.4972</td>
<td>2.9788</td>
<td>2.0023</td>
</tr>
<tr>
<td></td>
<td>MOM</td>
<td>1.9999</td>
<td>2.9924</td>
<td>1.5016</td>
<td>1.9960</td>
<td>0.9011</td>
<td>0.9961</td>
</tr>
<tr>
<td>50</td>
<td>MLE</td>
<td>1.0962</td>
<td>0.8924</td>
<td>2.1319</td>
<td>1.4870</td>
<td>3.2063</td>
<td>1.9746</td>
</tr>
<tr>
<td></td>
<td>CRA</td>
<td>0.9951</td>
<td>0.9011</td>
<td>1.9950</td>
<td>1.5015</td>
<td>2.9914</td>
<td>1.9899</td>
</tr>
<tr>
<td></td>
<td>MOM</td>
<td>2.0024</td>
<td>2.9888</td>
<td>1.4982</td>
<td>1.9925</td>
<td>0.8987</td>
<td>0.9930</td>
</tr>
<tr>
<td>75</td>
<td>MLE</td>
<td>1.0848</td>
<td>0.8923</td>
<td>2.1213</td>
<td>1.4859</td>
<td>3.1148</td>
<td>1.9902</td>
</tr>
<tr>
<td></td>
<td>CRA</td>
<td>0.9953</td>
<td>0.9007</td>
<td>1.9945</td>
<td>1.5001</td>
<td>2.9974</td>
<td>1.9981</td>
</tr>
<tr>
<td></td>
<td>MOM</td>
<td>1.9991</td>
<td>2.9984</td>
<td>1.5001</td>
<td>1.9955</td>
<td>0.9008</td>
<td>0.9963</td>
</tr>
</tbody>
</table>

Table (2): Mean Square Error (MSE)

<table>
<thead>
<tr>
<th>$n$</th>
<th>Method</th>
<th>$\lambda = -0.5$</th>
<th>$\theta = 0.9$</th>
<th>$\lambda = 0.8$</th>
<th>$\theta = 1.5$</th>
<th>$\lambda = -0.3$</th>
<th>$\theta = 2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>25</td>
<td>MLE</td>
<td>0.0007</td>
<td>0.0015e-003</td>
<td>0.0001</td>
<td>0.0033e-003</td>
<td>0.0001</td>
<td>0.0006</td>
</tr>
<tr>
<td></td>
<td>CRA</td>
<td>0.0208</td>
<td>0.0077e-003</td>
<td>0.0374</td>
<td>0.0240e-003</td>
<td>0.0964</td>
<td>0.0002</td>
</tr>
<tr>
<td></td>
<td>MOM</td>
<td>0.0008</td>
<td>0.0005</td>
<td>0.0034e-003</td>
<td>0.0001</td>
<td>0.0016e-003</td>
<td>0.0005</td>
</tr>
<tr>
<td>50</td>
<td>MLE</td>
<td>0.0006</td>
<td>0.0126e-004</td>
<td>0.0003</td>
<td>0.0025e-003</td>
<td>0.0001</td>
<td>0.0033e-003</td>
</tr>
<tr>
<td></td>
<td>CRA</td>
<td>0.0105</td>
<td>0.0128e-004</td>
<td>0.0195</td>
<td>0.0050e-003</td>
<td>0.0339</td>
<td>0.0012e-003</td>
</tr>
<tr>
<td></td>
<td>MOM</td>
<td>0.0000e-003</td>
<td>0.0001</td>
<td>0.0025e-003</td>
<td>0.0002</td>
<td>0.0126e-004</td>
<td>0.0002</td>
</tr>
<tr>
<td>75</td>
<td>MLE</td>
<td>0.0001</td>
<td>0.0048e-004</td>
<td>0.0001</td>
<td>0.0001e-003</td>
<td>0.0001</td>
<td>0.0075e-004</td>
</tr>
<tr>
<td></td>
<td>CRA</td>
<td>0.0067</td>
<td>0.0058e-004</td>
<td>0.0125</td>
<td>0.0236e-003</td>
<td>0.0079</td>
<td>0.1464e-004</td>
</tr>
<tr>
<td></td>
<td>MOM</td>
<td>0.0075e-004</td>
<td>0.0006</td>
<td>0.0000e-003</td>
<td>0.0001</td>
<td>0.0048e-004</td>
<td>0.0006</td>
</tr>
</tbody>
</table>

Conclusion

From results of simulation illustrated in table (2), for $(n = 25)$ we find the best estimator is $(\hat{\lambda}_{MLE})$ and for $(\hat{\theta}_{MLE})$ and $(\hat{\theta}_{MOM})$. For $(n = 50)$ we...
find the best estimator is \((\hat{\lambda}_{MLE})\) and for \((\hat{\theta}_{MLE})\) and \((\hat{\theta}_{CRA})\). For \((n = 75)\) we find the best estimator is \((\hat{\lambda}_{MOM})\) and for \((\hat{\theta}_{MLE})\). The transmuted distribution gives a new family which allows representing the data in a more flexible way.

**References**


