Application of Variational Iteration Method to solve one dimensional wave equation and wave-like equation with variable coefficients

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Abstract- Present work is devoted to application of variational iteration method to obtain the solutions of wave equation and wave-like equation. Numerical examples are taken to test the efficiency of this method. We have shown that the successive approximations of each problem are converging to their exact solution. Further we have shown graphically the fourth approximation values and exact values.

Keywords – Variational iteration method, successive approximation, one dimensional wave equation

I. INTRODUCTION

Many problems in various fields of sciences and engineering yield partial differential equation. For their physical interpretation we need their solutions. It may not possible to find the exact solution of certain partial differential equations. There are many methods to find the approximate solutions of such equation. For example, numerical methods, Adomain decomposition method, Homotopy analysis method etc. J.H.He[2-3] developed the variational iteration method. Several researchers [1,4,6] working on application of this method to find the solutions of linear and nonlinear partial differential equations. E.Rama, K.Somaiah and K.Sambaiah [5] studied variational iteration method for solving various types of problems. In this paper we found the successive approximations of one dimensional wave equation and wavelike equation using Varaitional iteration method. Further it was shown that they are converging to their exact solution.

II. DESCRIPTION OF THE METHOD

Here we describe briefly the basic idea of the VIM to solve partial differential equation. Consider the following partial differential equation

$$L_t u(x,t) + L_x u(x,t) + Nu(x,t) = g(x,t) \tag{1}$$

where Lt is a linear operator involving partial derivatives with respect to t only, Lx is linear operator involving partial derivatives with respect to x only, N is non-linear operator and g(x,t) is a continuous function. According to VIM the correctional functional of equation (1) in t and x directions respectively are given by

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda_1(x,s)\left[L_s u_n + \{L_x + N\}\tilde{u}_n(x,s) - g(x,s)\right]ds \tag{2}$$

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda_2(x,s)\left[L_s u_n + \{L_t + N\}\tilde{u}_n(x,s) - g(x,s)\right]ds \qquad (3)$$

where $u_n(x,t)$ denotes nth approximation of $u(x,t)$, λ_1 and λ_2 are Lagrangian multipliers which can be identified optimally using the variational theory and $\tilde{u}_n$ is the restricted variation. i.e., $\tilde{\delta} u_n = 0$. After determination of Lagrange multipliers λ_1 and λ_2 the successive approximations $u_{n+1}(x,t)$ of $u(x,t)$ can be obtained by suitable choice of initial approximation $u_0(x,t)$. The solution of equation (1) is obtained as

$$u(x,t) = \underset{n\to\infty}{Lt}\, u_n(x,t) \qquad (4)$$

III. ILLUSTRATIVE EXAMPLES

3.1 Example –

Consider one dimensional wave equation

$$\frac{\partial^2 u(x,t)}{\partial t^2} - 9\,\frac{\partial^2 u(x,t)}{\partial x^2} = 0 \qquad (5)$$

with the following initial and boundary conditions

$$(i).u(0,t) = 0 \quad (ii).u(1,t) = 0 \quad (iii).u(x,t) = 6\sin(\pi x) \quad (iv).\frac{\partial u(x,t)}{\partial t} = 0 \;\; at \;\; t = 0$$

According to the VIM the correctional formula for the wave equation (5) becomes

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(s,t)\left(\frac{\partial^2}{\partial s^2}u_n(x,s) - 9\frac{\partial^2}{\partial x^2}\tilde{u}_n(x,s)\right)ds \qquad (6)$$

The equation (6) gives the following stationary conditions noting that $\tilde{\delta} u_n = 0$.

$$1 - \lambda'(s,t) = 0 \;\; at \;\; s = t$$

$$\lambda(s,t) = 0 \;\; at \;\; s = t$$

$$\lambda''(s,t) = 0 \;\; at \;\; s = t$$

where dash denotes partial derivative with respect to s. From the above three equations we have

$$\lambda(s,t) = s - t \qquad (7)$$

Hence the variational iteration formula for the problem considered is

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t (s-t)\left(\frac{\partial^2}{\partial s^2}u_n(x,s) - 9\frac{\partial^2}{\partial x^2}u_n(x,s)\right)ds \qquad (8)$$

Choose $u_0(x,t) = u(x,0) = 6\sin(\pi x)$. Substituting n=0 and the initial approximation in equation (8) we get

$$u_1(x,t) = u_0(x,t) + \int_0^t (s-t)\left(\frac{\partial^2}{\partial s^2}u_0(x,s) - 9\frac{\partial^2}{\partial x^2}u_0(x,s)\right)ds$$

$$u_1(x,t) = 6\sin(\pi x) + \int_0^t (s-t)\left(0 - 9\frac{\partial^2}{\partial x^2}(6\sin(\pi x))\right)ds$$

$$u_1(x,t) = 6\sin(\pi x)\left[1 - \frac{3^2\pi^2 t^2}{2!}\right] \tag{9}$$

Continuing this process the successive approximation are

$$u_2(x,t) = 6\sin(\pi x)\left[1 - \frac{3^2\pi^2 t^2}{2!} + \frac{3^4\pi^4 t^4}{4!}\right] \tag{10}$$

$$u_3(x,t) = 6\sin(\pi x)\left[1 - \frac{3^2\pi^2 t^2}{2!} + \frac{3^4\pi^4 t^4}{4!} - \frac{3^6\pi^6 t^6}{6!}\right] \tag{11}$$

$$u_4(x,t) = 6\sin(\pi x)\left[1 - \frac{3^2\pi^2 t^2}{2!} + \frac{3^4\pi^4 t^4}{4!} - \frac{3^6\pi^6 t^6}{6!} + \frac{3^8\pi^8}{8!}\right] \tag{12}$$

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$$u(x,t) = \underset{n\to\infty}{Lt}\, u_n(x,t) = 6\sin(\pi x)\cos(3\pi)$$

The successive approximations obtained are converging to the exact solution of the equation (5).

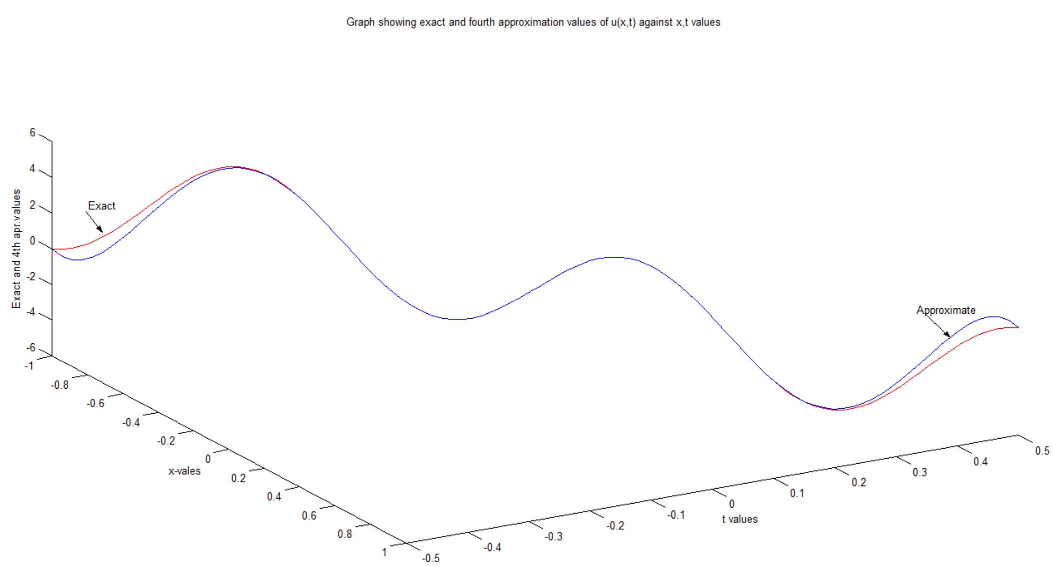

Fig.1.

For small values of t and x, the fourth approximation values are close to the exact values of u(x,t). However the limit of these approximations is approaching to the exact solution.

3.2 Example

Now consider wave-like wave equation with variable coefficients

$$\frac{\partial^2 u(x,t)}{\partial t^2}+\frac{x^2}{2}\frac{\partial^2 u(x,t)}{\partial x^2}=0 \tag{13}$$

With the following initial and boundary conditions.

$$(i).u(0,t)=x^2 \quad (ii).u(1,t)=\cos(t) \quad (iii).u(x,t)=0 \quad (iv).\frac{\partial u(x,t)}{\partial t}=0 \quad at \quad t=0$$

Proceeding as in case of earlier problem we get

$$u_{n+1}(x,t)=u_n(x,t)+\int_0^t (s-t)\left(\frac{\partial^2}{\partial s^2}u_n(x,s)+\frac{x^2}{2}\frac{\partial^2}{\partial x^2}u_n(x,s)\right)ds \tag{14}$$

Choose the initial approximation $u_0(x,t)=u(x,0)=x^2$.

Substituting n=0 and using the initial approximation we get

$$u_1(x,t)=x^2(1-\frac{t^2}{2!}) \tag{15}$$

$$u_2(x,t)=x^2(1-\frac{t^2}{2!}+\frac{t^4}{4!}) \tag{16}$$

$$u_3(x,t)=x^2(1-\frac{t^2}{2!}+\frac{t^4}{4!}-\frac{t^6}{6!}) \tag{17}$$

$$u_3(x,t)=x^2(1-\frac{t^2}{2!}+\frac{t^4}{4!}-\frac{t^6}{6!}+\frac{t^8}{8!}) \tag{18}$$

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$$u(x,t)=\underset{n\to\infty}{Lt}\,u_n(x,t)=x^2\cos(t).$$

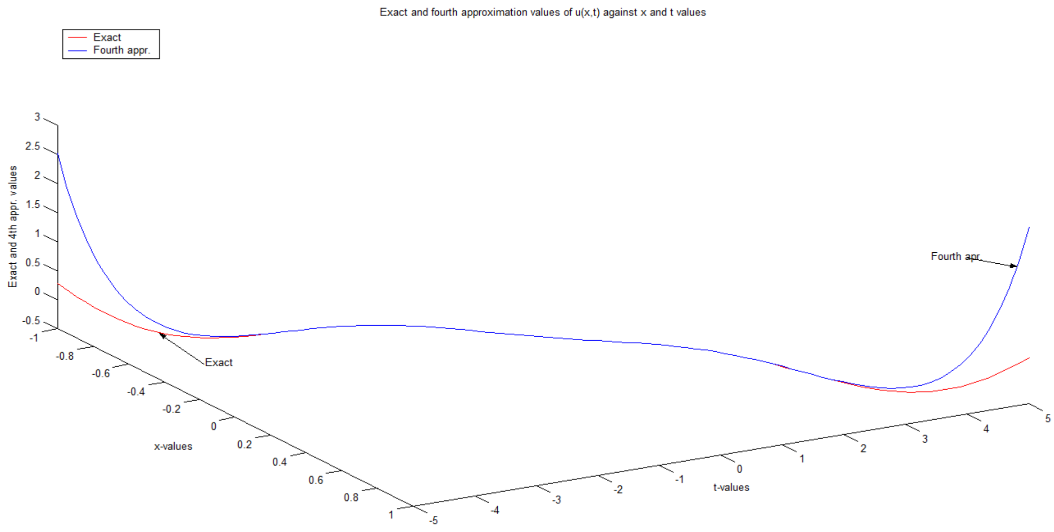

Fig.2

3.3 Example

Now consider the homogeneous equation

$$\frac{\partial^2 u(x,t)}{\partial t^2} - \frac{\partial^2 u(x,t)}{\partial x^2} + 8u(x,t) = 0 \tag{19}$$

With the initial and boundary conditions

$$(i).u(x,0) = 0 \qquad (ii).u(\pi,t) = 0 \qquad (iii).u(x,0) = \sin(x) \qquad (iv).\frac{\partial u(x,t)}{\partial t} = 0 \quad at \quad t = 0$$

The variational iteration formula corresponding to equation (19) is given by

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t (s-t)\left(\frac{\partial^2}{\partial s^2}u_n(x,s) - \frac{\partial^2}{\partial x^2}u_n(x,s) + 8u_n(x,s)\right)ds \tag{20}$$

Choose the initial approximation $u_0(x,t) = u(x,0) = \sin(x)$. Substituting n=0 and the initial approximation in equation (20) we get

$$u_1(x,t) = u_0(x,t) + \int_0^t (s-t)\left(\frac{\partial^2}{\partial s^2}u_0(x,s) - \frac{\partial^2}{\partial x^2}u_0(x,s) + 8u_0(x,s)\right)ds$$

$$= \sin(x) + \int_0^t (s-t)\left(\frac{\partial^2}{\partial s^2}\sin(x) - \frac{\partial^2}{\partial x^2}\sin(x) + 8\sin(x)\right)ds$$

$$= \sin(x) + \int_0^t (s-t)\left(0 + \sin(x) + 8\sin(x)\right)ds$$

Thus,

$$u_1(x,t) = \sin(x)\left(1 - \frac{(3t)^2}{2!}\right) \tag{21}$$

Continuing this process we obtain

$$u_2(x,t) = \sin(x)(1 - \frac{(3t)^2}{2!} + \frac{(3t)^4}{4!}) \tag{22}$$

$$u_3(x,t) = \sin(x)(1 - \frac{(3t)^2}{2!} + \frac{(3t)^4}{4!} - \frac{(3t)^6}{6!}) \tag{23}$$

$$u_4(x,t) = \sin(x)(1 - \frac{(3t)^2}{2!} + \frac{(3t)^4}{4!} - \frac{(3t)^6}{6!} + \frac{(3t)^8}{8!}) \tag{24}$$

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$$u(x,t) = \underset{n\to\infty}{Lt}\, u_n(x,t) = \sin(x)\cos(3t)$$

The successive approximations are converging to the exact solution of (19).

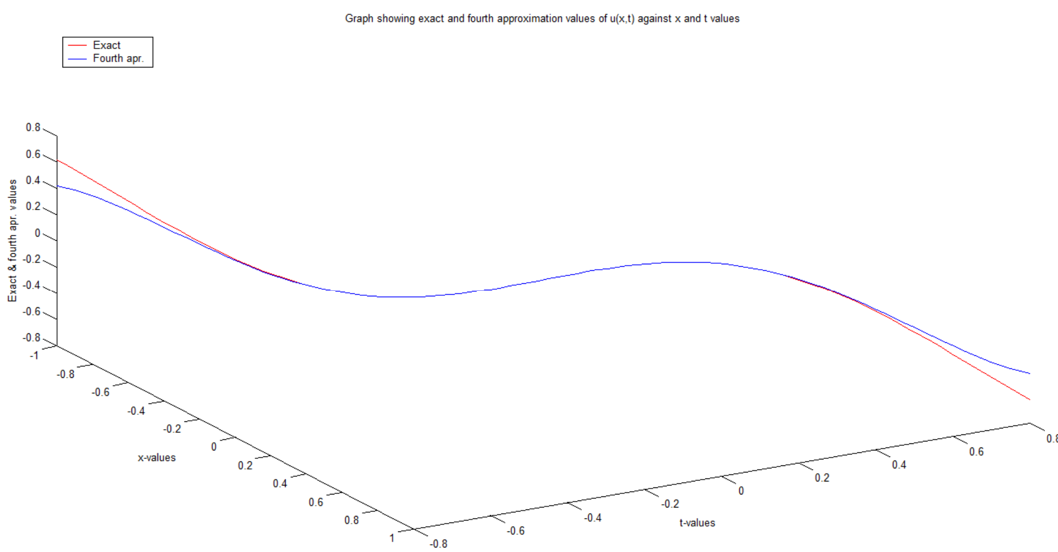

Fig.3

IV.CONCLUSION

Thus VIM can be easily applicable to obtain the solutions of one-dimensional wave equation and wave-like equation. It can be observed that the approximations are moving close to the exact solution. The same can be observed from the graphs. This shows the efficacy of this method.

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