Zero-divisor conjector of R[G] in fields of characteristic zero.

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ABSTRACT

As we know that, when there will be a complex group ring C[G], then there exists no non-trivial zero-divisor. Similarly, we want to show that there will be no non-trivial zero-divisor, for group ring K[G] in which char K = 0. Therefore, this research paper will present zero-divisor concept of R[G] in fields of characteristic 0.

KEYWORDS : Zero-divisor, complex group-ring C[G], non-trivial zero-divisor, **characteristic of field[2],** field of characteristic zero.

1. INTRODUCTION :- Let us suppose that R be a ring and G be a group then there will be some conditions, which will declare the non-existence of non-trivial **zero-divisors[1]** in the algebraic structure R[G]. There are some conditions upon which the algebraic structure R[G] has no non-trivial zero – divisors.

(i) When group G will be torsion-free group, then for an arbitrary field on integral domain K, the algebraic structure K[G] has no non-trivial zero-divisors.

(ii) Let us suppose that G be a **torsion-free group[3]** and K be any field, then the algebraic structure K[G] will contain no non-trivial central zero-divisor.

(iii) Let us suppose that G be a unique product group and R be a **G-graded ring[13]** in which non-zero homogeneous elements are not zero-divisors, the R has no non-trivial zero-divisors.

(iv) Let us take G as group, then $\overline{Q}$[G] and $\overline{Z}$[G] have no non-trivial zero-divisors if only if C[G] has no non-trivial zero divisors. Here $\overline{Q}$ is conjugate of Q (Rational field) and $\bar{Z}$ is conjugate of z (integral field) while C is a complex field.

Now, we will try to show that any field K of char 0, the algebraic structure K[G] contains no non-trivial zero-divisors, as it is known that any complex group ring C[G] contains no non-trivial zero-divisors.

2. PROPOSITION:- There exists an embedding of countable field F into the complex field C.

Proof:- Since F be a **countable field**[6]. So we can embed the prime subfield of F into C. It is clear from **Zorn's Lemma**[4, 5, 7, 8] F will be a partially ordered set. Let us take subfield K of F with **embedding mapping**[11, 12] σ. Thus we can embed K into C as σ(K) $\rightarrow$C. We consider it as **maximal**[10] if K $\subseteq$ K' then there exists an another embedding σ' such that σ'(K') $\rightarrow$C. But σ' on K' coincides with σ on K, so we have K = K' as well as $\sigma = \sigma'$.

Let us suppose that K be a field which is maximal. If K $\subset$ F, then we take some $a \in$ F/K.

Now we can focus upon two cases.

(i) If a is **algebraic over K**[9] then there exists a minimal polynomial $p(x) \in$ K[x] such that the degree of $p(x)$ must be 2 since $a \in$ F/K. As $\sigma(K) \rightarrow C$ induces an isomorphism of rings $\mathrm{K}[x] \cong \sigma(\mathrm{K})[x]$ an isomorphism image of $p(x)$ must be irreducible of degree at least 2. As it is clear that $\sigma(\mathrm{K}) \subset \mathrm{C}$ and C is algebraically closed so we have $b \in \mathrm{C}/\sigma(\mathrm{K})$. Thus we can get an extension isomorphism $\mathrm{K}(a) \cong \sigma(\mathrm{K})(b)$ by mapping a to b. But we obtain contradiction, because we have taken K as maximal and maximal of pair (K, σ).

(ii) If a is **transcendental over K**[9] . As K is countable, so σ(K) has a countable algebraically closed in C. Hence there are a number of elements in C which are transcendental over σ(K). Now we choose any one such element and extend the isomorphism σ by mapping a to chosen above element of C, which implies that it is also transcendental over σ(K). But such concept contradicts the maximality of K. Thus we can say that field K = field F.

3. Corollary:- If a complex group ring C [G] contains no non-trivial zero-divisor. Then any field K of characteristic 0, the group ring K[G] contains no non-trivial zero-divisor.

Proof:- Let us suppose that a, $b \in$ K[G] and we construct a finitely generated subfield F such that F = Q (Supp (a), Supp(b)) of K. For which a, $b \in$ F[G] $\subseteq$ K[G]. As from above proposition, F is countable and we have an embedding mapping $\phi(F) \to C$. Further we get a ring embedding of ϕ(F)[G] into C[G]. As we know that, F[G]$\cong \phi$(F)[G] $\subseteq$C[G]. So, we have $a.b \neq 0$. Hence K[G] contains no non-trivial zero divisor if **Char K = 0[2].**

4. CONCLUSIONS:- This research paper is based on the non-existence of zero-divisor conjector, but not in trivial case. We have found that if there exists a countable field F of char 0, then F[G] will possess no non-trivial zero-divisor if only if C[G] the complex group ring contains no non-trivial zro-divisor. For this purpose we have got an embedding mapping ϕ such that ϕ: F$\to$C. Then we induce a ring imbedding of ϕ(F)[G] to C[G]. But F[G] $\cong \phi$(F)[G]. Again ϕ(F)[G] $\subseteq$ C[G] so we get that K[G] has no non-trivial zero-divisor as it is known that F[G] $\subseteq$ K[G].

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