Super weak sum labeling of some families of graphs

Sheetal Thakare[1] and Manisha Acharya[2]

[1]*Assistant professor, Dept. of App. Mathematics, Lokmanya Tilak College of Engg.,Navi Mumbai/ thakare.sheetal@gmail.com*

[2]*Associate professor and Head, Dept. of Maths, M.D.College of Arts,Commerce and Science, Parel, Mumbai/ macharya13@gmail.com*

Abstract

A graph $G = (V, E)$ is called super weak sum (briefly *sw-sum*) graph, if there is a bijection $\phi : V \mapsto \{1, 2, ..., |V|\}$ such that for every edge uv in G, there is a vertex w in G with $\phi(u) + \phi(v) = \phi(w)$. A mapping ϕ here is called a *sw-sum labeling* of graph G. A sw-sum graph must have at least one isolated vertex, namely the vertex with the largest label, so can not be connected. The sw-sum number, $\omega(G)$ of a connected graph G is the least number of isolated vertices needs to be added such that $G \cup \overline{K_{\omega(G)}}$ is a sw-sum graph. A lower bound for sw-sum number is the minimum degree δ of a vertex in the graph. A graph is termed as δ-optimal sw-summable if it needs δ isolates to sw-sum label a graph. We in this paper provide labeling schemes for some graphs named as Book graph, Prism Graph, Shell butterfly graph, Bistar graph proving that they are δ-optimal sw-summable.

Key words : super weak sum labeling, super weak sum number, δ-optimal sw-summable .

1 Introduction

In 1990 Harary [1] introduced the notation of sum graph. A graph $G = (V, E)$ is called sum graph if there exists an injective function θ called sum labeling, from V to a set of positive integers S such that, $uv \in E$ if and only if there exists $w \in V$ such that $\theta(u)+\theta(v) = \theta(w)$. In every sum graph there must be at least one isolated vertex, namely the vertex with the largest label so every sum graph is disconnected. The sum number, $\sigma(G)$ of a graph G is the minimum number of isolated vertices required to be added to the graph so that $G \cup \overline{K_{\sigma(G)}}$ is a sum graph.

In [2] Javaid, Khalid, Ahmad and Imran introduced a weaker version of sum labeling of graphs as follows. Let $G = (V, E)$ be a simple finite undirected graph with $|V| = p$. G is *weak sum* graph if there exists a labeling L (called a w-sum labeling) of the vertices of V by distinct positive integers such that, for $uv \in E$ there exists a vertex $w \in V$ such that, $L(w) = L(u) + L(v)$. This is weaker in the sense that sum graph requires the "only if " condition. If G w-sum graph with the additional constraint that all the labels fall in set $[p]$ (that is set of integers from 1 to p), then G is called *super weak sumgraph* (sw-sum graph) and the corresponding labeling is called *super weak sum labeling*. A sw-sum graph has to be disconnected as it has at least one isolated vertex, namely the vertex with highest label. So in order to sw-sum label a connected graph, one needs to add a set of isolated vertices known as *isolates* as a disjoint union and the labeling scheme that requires the fewest isolates is termed as *optimal*. The minimum number of isolates required for a graph G to support a sw-sum labeling is known as *sw-sum number* of a graph, denoted by $\omega(G)$. In [2] Javaid, Khalid, Ahmad and Imran have proved following lemma about the lower bound of sw-sum number of a graph.

Lemma 1. *A lower bound for the sw-sum number $\omega(G)$ of a graph G is the minimum degree δ of a vertex in the graph.*

A sw-sum graph is termed as δ-optimal sw-summable if it needs δ isolates to be added to the graph to yield sw-sum graph. Javaid et al.

[2] have also shown that families of graphs like paths, cycles, wheels, complete graph, complete multipartite graphs so in perticular star graphs are δ-optimal superweak summable.
We in this paper will explore some more families of graphs which are δ-optimal sw-summable and also provide corresponding optimal labeling for them.

2 Super weak sum labeling of Book graph $B_n, n \geq 2$

The *Book graph* (B_n) is a cartesian product of star and single edge that is $S_n \times P_2$ [3],[9]. It may be also pictured as n quadilaterals sharing a common edge known as spine or base or edge of the book. The graph has $2n + 2$ vertices and its δ is 2.
For our convenience we refer to the vertices in the following way: The two vertices consisting spine of book graph are named v_1 ,v_2 and remaining two vertices of each page are adressed as p_1^i and p_2^i where i denote the page number, $1 \leq i \leq n$ such that p_1^i is adjacent to v_1 and p_2^i is adjacent to v_2.

Theorem 1. *B_n is 2-optimal sw-summable,for $n \geq 2$ that is $\omega(B_n) = 2$.*

Proof. As $\omega(B_n) \geq 2$ (by lemma 1), it just needs to develope a sw-sum labeling of Book graph with 2 isolates. Begin the labeling for B_n by labeling the spine vertex $v_1 = 1$. Label the vertices as $p_2^i = i + 1, 1 \leq i \leq n$, thus p_2^1 will get the label 2 and p_2^n will get the label $n + 1$. Now assign $v_2 = n + 2$ and $p_1^i = 2n + 3 - i, 1 \leq i \leq n$ that is p_1^n will get the label $n + 3$ and so on p_1^1 will be labeled as $2n + 2$, which is the maximum label. This vertex p_1^1 is adjacent to two vertices v_1 and p_2^1 with labels 1 and 2 respectively. So we require two isolates namely $2n+3, 2n+4$ to witeness these two edges. For all edges independent of spine verices, sum of their end vertices is $2n + 4$. Further it is easy to see that all edges incident on spine vertices are witnessed by the labeling and maximum label is $2n + 4$ including two isolates. Thus we are through. □

Fig. 1 and Fig. 2 show a labeling for B_4 and B_7 respectively.

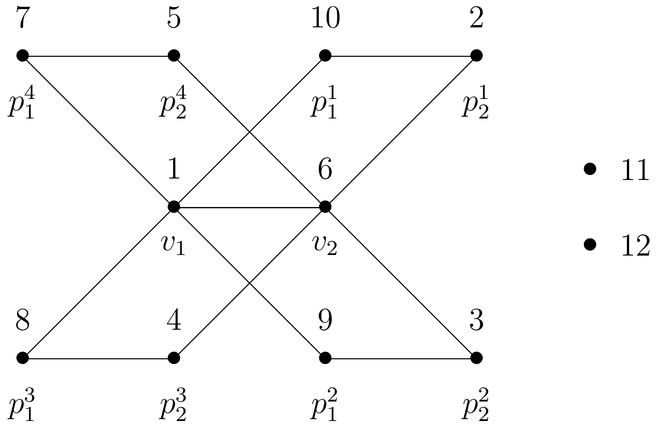

Figure 1: A 2-optimal sw-sum labeling for Book graph B_4

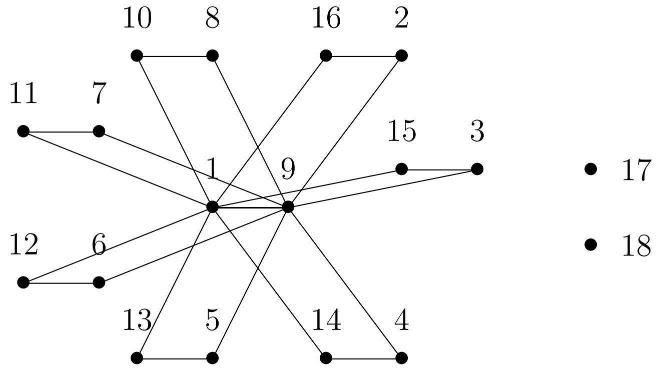

Figure 2: A 2-optimal sw-sum labeling for Book graph B_7

3 Super weak sum labeling of Prism graph $D_n, n \geq 3$

The *prism graph* $D_n, n \geq 3$ is the Cartesian product $C_n \times K_2$ of the cycle C_n and the complete graph of order 2 [4],[5],[6].

Theorem 2. *The Prism graph $D_n, n \geq 3$ is δ-optimal sw-summable that is $\omega(D_n) = 3$, for all $n \geq 3$.*

Proof. From lemma 1, $\omega(D_n) \geq 3$. So next we provide a sw-sum labeling of D_n with three isolates and get through the proof.
Let v_i for $1 \leq i \leq n$ be the vertices on outer cycle of the Prism and v_i' for $1 \leq i \leq n$ be the corresponding adjacent vertices on the inner cycle of the Prism D_n. There are $2n$ verices. We label them using numbers from the set $[2n]$ as follows,

$$v_i = \begin{cases} 2n - i + 1 & \text{for } i \text{ odd}, \ 1 \leq i < n \\ i + 1 & \text{for } i \text{ even}, \ 1 < i < n \end{cases}$$

where, $v_n = 2$

and

$$v'_i = \begin{cases} i+1 & \text{for } i \text{ odd}, \ 1 < i \leq n \\ 2n-i+1 & \text{for } i \text{ even}, \ 1 < i \leq n \end{cases}$$

with, $v'_1 = 1$

Note here that, v_1 has label $2n$, and v'_1, v_2, v_n, are neighbouring vertices so to witness these edges three sums are required as below, $v_1 + v'_1 = 2n+1, v_1+v_n = 2n+2, v_1+v_2 = 2n+3$. We need to introduce three isolates with these labels. Next we will show that with this labeling the graph $D_n \cup \overline{K_3}$ is weak sum graph and we are through. Here we check the sums for remaining pairs of adjacent vertices by considering following five cases,

Case 1. Sum of the labels of every vertex v_i, i even and its previous vertex in the cycle that is v_{i-1},

$$\begin{aligned} v_i + v_{i-1} &= i+1+2n-(i-1)+1 \\ &= 2n+3 \ \text{ if } \ 2 \leq i < n \\ &= n+4 \ \text{ if } \ i = n \ \text{ is even.} \end{aligned}$$

Case 2. Sum of the labels of every vertex v_i, i even and its next vertex in the cycle that is v_{i+1},

$$\begin{aligned} v_i + v_{i+1} &= i+1+2n-(i+1)+1 \\ &= 2n+1 \ \text{ if } \ 2 \leq i < n-1 \\ &= n+2 \ \text{ if } \ i = n-1 \ \text{ is even.} \end{aligned}$$

Case 3. Sum of the labels of every vertex v'_i, i even and its previous vertex in the cycle that is v'_{i-1},

$$\begin{aligned} v'_i + v'_{i-1} &= 2n-i+1+(i-1)+1 \\ &= 2n+1 \ \text{ for } \ 2 < i \leq n \\ &= 2n \quad \text{ for } \ i = 2. \end{aligned}$$

Case 4. Sum of the labels of every vertex v'_i, i even and its next vertex in the cycle that is v'_{i+1},

$$\begin{aligned} v'_i + v'_{i+1} &= 2n-i+1+(i+1)+1 \\ &= 2n+3 \quad \text{ for } \ 2 \leq i < n. \end{aligned}$$

If $i = n$ is even, as next vertex of v'_n in the cycle is v'_1, the sum $v'_n + v'_1 = n+2$.

Case 5. Sum of the labels of vertex v_i and v'_i ,

$$\begin{aligned} v_i + v'_i &= i+1+2n-i+1 \\ &= 2n+2 \quad \text{ for } \ 2 \leq i < n \\ &= n+3 \quad \text{ for } \ i = n. \end{aligned}$$

Thus we can say that D_n with set of three isolates $\{2n+1, 2n+2, 2n+3\}$ is a super weak sum graph and hence with reference to lemma 1 we can conclude the result. □

Figure 3, figure 4 below show a labeling for prism graphs D_7 and D_6.

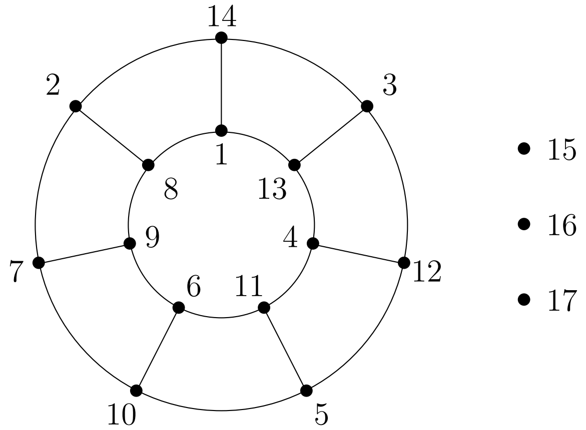

Figure 3: sw-sum labeling for D_7

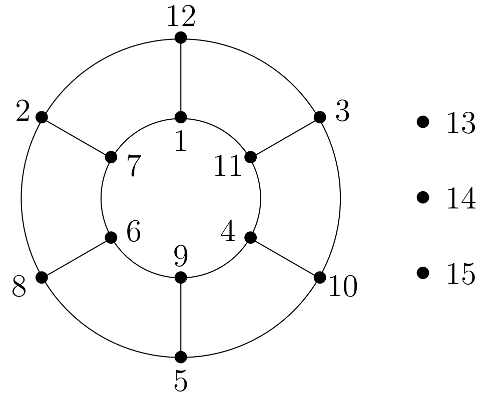

Figure 4: sw-sum labeling for D_6

4 Super weak sum labeling of Shell butterfly Graph

Deb and Limaye have defined a shell graph as a cycle C_n with $(n-3)$ chords sharing a common end point called the apex [7]. Jeba Jesintha and Hilda [8] define Shell -butterfly graph as a double shell in which each shell has any order with exactly two pendant edges at the apex.

Theorem 3. *Shell butterfly graph is δ-optimal sw-summable that is ω(Shell butterfly graph) = 1.*

Proof. Let G be a shell butterfly graph of order $m+n+3$. We denote the apex of G as v_0. Denote the vertices in the path of the right shell of G from bottom to top as $v_1, v_2, ..., v_m$ and that of the left shell of G from top to bottom as $v_{m+1}, v_{m+2}, ..., v_{m+n}$. The pendant vertices are denoted as v_{m+n+1} and v_{m+n+2}.

We label the graph with numbers from set $[m + n + 3]$. We start by labeling apex v_0 as 1 and distribute the labels $2, 3, 4, ...$ alternately along right shell from bottom to top. The maximum label at the end of right shell is $\lfloor \frac{m}{2} \rfloor$ on v_m or v_{m-1} depending upon whether m is even or odd. Then continue labeling sequence on left shell altenatively from top to bottom starting with vertex v_{m+2} and distributing next labels $\lfloor \frac{m}{2} \rfloor + 1, \lfloor \frac{m}{2} \rfloor + 2, ...$. Thus the maximum label at end of this is $\lfloor \frac{m}{2} \rfloor + \lfloor \frac{n}{2} \rfloor + 1$ and it is on v_n or v_{n-1} depending upon whether n is even or odd. If n is odd assign next label to v_n. Now continue labeling for remaining vertices in reverse direction on left shell followed by right. The maximum label at the end of this is $m+n+1$ on the vertex v_1. The two pendants are labeled as $m+n+2, m+n+3$. It is easy to see here that all edges are witnessed by the labeling except the edge between apex v_0 and pendant vertex with label $m + n + 3$. So we can conclude that G with one isolated vertex labeled as $m + n + 4$ yield a sw-sum graph. □

Fig. 5 below shows a labeling for shell butterfly graph of order 16.

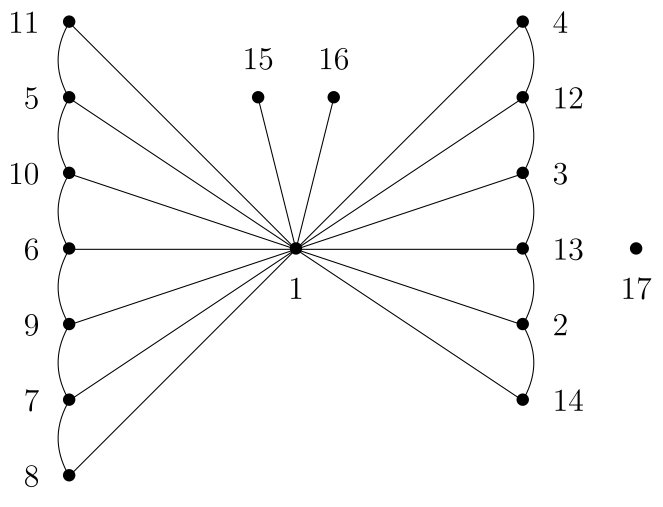

Figure 5: A 1-optimal sw-sum labeling for Shell butterfly Graph

5 Super weak sum labeling of Bistar graph

The bistar $B_{m,n}$ is graph obtained by joining the center (apex) vertices of two star graphs S_m and S_n by an edge, where $S_n = K_{1,n}$.

Theorem 4. *Bistar graph is δ-optimal sw-summable that is $\omega(B_{m,n}) = 1$, for all $n \geq 3$.*

Proof. Similar to earliar proofs here also it is enough to produce a sw-sum labeling of graph with $\delta = 1$ isolate. As the graph $B_{m,n}$ has $m + n + 2$ vertices we must give labeling using numbers from set $[m + n + 3]$ including isolates. W.l.o.g. let $m \geq n$, assign label 1 and 2 to apex of S_m and S_n respectively. Allot labels 3 to $n+2$ to pendant vertices of star S_n and then from $n+3$ to $n+m+2$ to pendant vertices of star S_m. It is easy to observe that with this labeling the only isolate required is $m + n + 3$. So we are through. □

References

[1] F. Harary, Sum graphs and difference graphs, Congr. Numer. 72 (1990) 101108.

[2] I. Javaid, F. Khalid, A. Ahmad, and M. Imran, On a weaker version of sum labeling of graphs, Math. Rep. (Bucur.) 16(66) (2014), no. 3, 413-420.

[3] B.N. Kavitha, Indrani Kelkar Split and equitable domination in Book graph and stacked book graph,International Journal of Advanced Research in Computer Science Volume 8, No. 6, July 2017 (Special Issue III)

[4] M. Acharya, Even Vertex Graceful Labeling for Two Graphs,International Journal of Mathematics Trends and Technology, Vol 9- May 2014.

[5] C. Barrientos and H. Hevia,On 2- Equitable Labelings Of Graphs, Research supported in part by FONDECYT project 1941219(94).

[6] Sheetal Thakare and M. Acharya, Exclusive sum labeling and exclusive sum number of Prism graph (D_n), accepted for publication in Ars Combinatoria.

[7] J.A. Gallian, A dynamic survey of graph labeling, Electron. J. Combin., 19 (2016).

[8] J. Jeba Jesintha and K. Ezhilarasi Hilda, Shell-butterfly graphs are harmonious, Adv. Appl. Disc. Math., 14 no. 1 (2014) 39-51.

[9] 'Book Graph.' From MathWORLD- A Wolfram Web Resource. http://mathworld.wolfarm.com/Book graph.html.