On the Solution of Wave Equation using Direct Substitution Method

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Abstract: In this paper, solutions of the One, Two, Three dimensional wave equation with boundary Conditions on Cartesian Co-ordinates and eight Parameter Lie Group of Projective Transformations in $\boldsymbol{R}^2$ are studied. Also, we discussed the 3-D wave equation using Partial differential equation(PDE).

Keywords: Lie Group, Lie Algebra, Symmetries of Wave Equation.

1. Introduction

Symmetry analysis plays an important role in the theory of Differential equations. Symmetry of a Differential Equation is a transformation that maps any solution to another solution of the system. Solutions of the wave equation with boundary conditions have many practical applications in Engineering and Physics.

The procedure for finding lie point symmetries of a given differential equation is well known [2]. For one-dimensional solutions of nonlinear ordinary differential equations in [7] proposed a method of quasi-linearization. With this method a decision along the nonlinear problem is reduced to solving a sequence of linear problems. In this paper, we have to find the point symmetries admitted by a scalar PDE or system of PDEs. Use admitted point symmetries of PDEs to construct resulting invariant solutions. Invariant solutions for scalar PDEs or systems of PDEs can be determined from admitted point symmetry in two ways.

2. INVARIANCE OF A PDE

We consider a k^{th} order scalar PDE by

$$F(x, u, \partial u, \partial^2 u, \ldots\ldots\ldots\ldots, \partial^k u) = 0 \qquad \ldots.(1)$$

where $x = (x_1,x_2,\ldots\ldots\ldots\ldots,x_n)$ denotes the coordinates corresponding to its n independent variables, u denotes the coordinates corresponding to its dependent variable and $\partial^j u$ denotes the corresponding coordinates with components

$\partial^j u / \partial x_{i_1} \; \partial x_{i_2} \ldots\ldots \partial x_{i_j} = u_{i_1, i_2 \ldots\ldots i_j}$

$i_j = 1,2,\ldots\ldots n$, for $j = 1,2,\ldots\ldots\ldots k$, corresponding to all j^{th} order partial derivatives of u with respect to x.

2.1.ONE PARAMETER LIE GROUP

The one-parameter Lie group of point transformation

$$x^* = X(x,u;\xi) \qquad \ldots(2)$$

$$u^* = U(x,u;\xi) \qquad \ldots(3)$$

leaves invariant the PDE (1) i.e is a point symmetry admitted by PDE (1), if and only if its k^{th} extension, defined by

$$x^* = X(x,u;\xi)$$

$$u^* = U(x,u;\xi)$$

$$\partial u^* = \partial U(x,u,\partial u;\xi)$$

$$\vdots$$

$$\partial^k u^* = \partial^k U(x,u,\partial u, \ldots\ldots \partial^k u;\xi)$$

and

$$\begin{bmatrix} u_1{}^* \\ u_2{}^* \\ \cdot \\ \cdot \\ \cdot \\ u_n{}^* \end{bmatrix} = \begin{bmatrix} U_1 \\ U_2 \\ \cdot \\ \cdot \\ \cdot \\ U_n \end{bmatrix} = A^{-1} \begin{bmatrix} D_1 U \\ D_2 U \\ \cdot \\ \cdot \\ \cdot \\ D_n U \end{bmatrix}$$

$$\begin{bmatrix} u^*_{i_1 i_2 \ldots\ldots i_{k-1}} 1 \\ u^*_{i_1 i_2 \ldots\ldots i_{k-1}} 2 \\ \cdot \\ \cdot \\ \cdot \\ u^*_{i_1 i_2 \ldots\ldots i_{k-1}} n \end{bmatrix} = \begin{bmatrix} U_{i_1 i_2 \ldots\ldots i_{k-1}} 1 \\ U_{i_1 i_2 \ldots\ldots i_{k-1}} 2 \\ \cdot \\ \cdot \\ \cdot \\ U_{i_1 i_2 \ldots\ldots i_{k-1}} n \end{bmatrix} = A^{-1} \begin{bmatrix} D_1 U_{i_1 i_2 \ldots i_{k-1}} \\ D_2 U_{i_1 i_2 \ldots i_{k-1}} \\ \cdot \\ \cdot \\ \cdot \\ D_n U_{i_1 i_2 \ldots i_{k-1}} \end{bmatrix}$$

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leaves invariant the surface (1).

2.2. INFINITESIMAL CRITERION FOR THE INVARIANCE OF A PDE

Let $X = \xi_i (x,u) \dfrac{\partial}{\partial x_i} + \eta(x,u) \dfrac{\partial}{\partial u}$ ….(4)

be the infinitesimal generator of the Lie group of point transformations.

$x^* = X(x,u;\xi)$;

$u^* = U(x,u;\xi)$; ….(5)

Let

$$X^{(k)} = \xi_i (x,u)\frac{\partial}{\partial x_i} + \eta(x,u)\frac{\partial}{\partial u} + \eta_i^{(1)}(x,u,\partial u)\frac{\partial}{\partial u_i}$$

$$+ \ldots\ldots\ldots + \eta_{i_1 i_2 \ldots\ldots i_k}^{(k)} (x,u,\partial u,\partial^2 u,\ldots\ldots\ldots\partial^k u)\frac{\partial}{\partial u_{i_1 i_2 \ldots\ldots i_k}} \quad \text{....(6).}$$

be the k^{th} extended infinitesimal generator of (4),

where $\eta_i^{(1)}$ is given by

$\eta_i^{(1)} = D_i\eta - (D_i\xi_j)u_j$, i = 1,2,......... n ….(7)

and $\eta^{(k)}_{i_1 i_2 \ldots\ldots i_j}$ by

$$\eta^{(k)}_{i_1 i_2 \ldots\ldots i_k} = D_{i_k} \eta_{i_1 i_2 \ldots\ldots i_{k-1}}^{(k)} - (D_{i_k} \xi_j)\, u_{i_1 i_2 \ldots\ldots i_{k-1}}^{\;j}$$

$i_{\mathbf{l}} = 1,2$ ....n for $\mathbf{l}$ = 1,2,...........k with $k \geq 2$

i_j = 1,2........n for j = 1,2,.........k

in terms of

$\xi(x,u) = (\xi_1 (x,u), \xi_2 (x,u), \ldots\ldots, \xi_n (x,u)), \eta (x,u)$

Then the one-parameter lie group of point transformation (2) is admitted by PDE (1) i-e, is a point symmetry of PDE (1) if and only if

$X^{(k)} (F(x,u,\partial u,\partial^2 u,\ldots \partial^k u)) = 0$ when

$F(x,u,\partial u, \partial^2 u,\ldots \partial^k u) = 0$ ….(8)

2.3.DETERMINING EQUATIONS FOR SYMMETRIES OF A Kth – ORDER PDE

Consider a kth – order PDE ($k \geq 2$, $\ell \leq k$)

$$u_{i_1 i_2 \ldots i_l} = f(x, u, \partial u, \partial^2 u, \ldots \partial^k u) \qquad (9)$$

where $f(x,u,\partial u,\partial^2 u, \ldots,\partial^k u)$ does not depend explicitly on $u_{i_1 i_2 \ldots\ldots i_l}$. The PDE (9) admits the one-parameter Lie group of point transformations with the infinitesimal generator.

$$X = \xi_i(x,u)\frac{\partial}{\partial x_i} + \eta(x,u)\frac{\partial}{\partial u} \qquad (10)$$

and with its kth extension given by (6), if and only if $\xi(x,u)$ and η (x,u) satisfy the symmetry determining equation

$$\eta_{i_1 i_2 \ldots\ldots i_l}^{(l)} = \xi_j \frac{\partial f}{\partial x_j} + \eta\frac{\partial f}{\partial u} + \eta_j^{(\ell)}\frac{\partial f}{\partial u_j} + \ldots\ldots + \eta_{j_1 j_2 \ldots\ldots j_k}^{(k)}\frac{\partial f}{\partial u_{j_1 j_2 \ldots\ldots j_k}}$$

when u satisfies (9). …(11)

It is easy to show that

(i) $\eta^{(p)}_{j_1 j_2 \ldots\ldots j_p}$ is linear in the components of the coordinates $\partial^p u$ if $p \geq 2$; and

(ii) $\eta^{(p)}_{j_1 j_2 \ldots\ldots j_p}$ is a polynomial in the components of the coordinates

∂u, $\partial^2 u$, ... $\partial^p u$ with coefficients that are linear homogeneous in $\xi(x,u)$, $\eta(x, u)$ and their derivatives with respect to x and u to order p.

From (i) and (ii), it follows that if $f(x, u, \partial u, \partial^2 u, \ldots. \partial^k u)$ is a polynomial in the components of ∂u, $\partial^2 u$, ... $\partial^k u$, then the symmetry determining equation (11) is a polynomial equation in the components of ∂u, $\partial^2 u$, ... $\partial^k u$ with coefficients that are linear homogeneous in $\xi(x, u)$, $\eta(x, u)$ and their derivatives to order k.

When PDE (9) is not a polynomial equation in the components of ∂u, $\partial^2 u$,..... $\partial^k u$, one can still split the symmetry determining equation (1.43) into a system of linear homogeneous PDEs for $\xi(x,u)$ and $\eta(x,u)$ by using the independence of the components of ∂u, $\partial^2 u$,..... $\partial^k u$ after substitution for the component $u_{i_1 i_2 \ldots\ldots i_l}$. Typically, the resulting set of determining equations will be an over determined system for $\xi(x,u)$ and $\eta(x,u)$.

When the set of determining equations is over determined, it often happens that its only solution is the trivial solution $\xi(x,u) = \eta(x,u) = 0$.

In this case, the given PDE (9) does not admit point symmetries could still admit contact symmetries, higher - order symmetries or non-local symmetries, which result from considering a more general infinitesimal generator than (10).

When the general solution of the set of determining equations is non-trivial, two cases arise:

If the general solution contains atmost a finite number r of essential arbitrary constants, then it yields an r-parameter Lie group of point transformations admitted by PDE (9).

If the general solution cannot be expressed in terms of a finite number of essential constants, then it yields an infinite parameter Lie group of point transformations admitted by PDE (9).

3. EXAMPLES OF LIE ALGEBRAS

3.1. EIGHT PARAMETER LIE GROUP OF PROJECTIVE TRANSFORMATIONS IN R^2

Projective transformations in R^2 map straight lines into straight lines. In particular, they are defined by the eight-parameter Lie group of transformations.

$$x^* = \frac{(1+\xi_3)x + \xi_4 y + \xi_5}{\xi_1 x + \xi_2 y + 1} \quad \text{.....(12)}$$

$$y^* = \frac{\xi_6 x + (1+\xi_7)y + \xi_8}{\xi_1 x + \xi_2 y + 1} \quad \text{....(13)}$$

with parameters $\xi_\ell \; \varepsilon \; R, \ell = 1,2\ldots8$.

The infinitesimal generators of the corresponding Lie algebra L^8 are given by,

$$X_1 = x^2 \frac{\partial}{\partial x} + xy \frac{\partial}{\partial y}$$

$$X_2 = xy \frac{\partial}{\partial x} + y^2 \frac{\partial}{\partial y}$$

$$X_3 = x \frac{\partial}{\partial x}$$

$$X_4 = y \frac{\partial}{\partial x}$$

$$X_5 = \frac{\partial}{\partial x}$$

$$X_6 = x\frac{\partial}{\partial y}$$

$$X_7 = y\frac{\partial}{\partial y}$$

$$X_8 = \frac{\partial}{\partial y} \qquad \text{....(14)}$$

3.2. GROUP OF RIGID MOTIONS IN R^2

The group of rigid motions in R^2 preserves distances between any two points in R^2. This group is the three-parameter Lie group of transformations of rotations and translations in R^2 given by

$$x^* = x\cos\xi_1 - y\sin\xi_1 + \xi_2 \qquad \text{............ (15)}$$

$$y^* = x\sin\xi_1 + y\cos\xi_1 + \xi_3 \qquad \text{....... ...... (16)}$$

The corresponding infinitesimal generators are given by

$$X_1 = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$$

$$X_2 = \frac{\partial}{\partial x}$$

$$X_3 = \frac{\partial}{\partial y}$$

3.3. SIMILITUDE GROUP IN R^2

The similitude group in R^2 consists of uniform scaling and rigid motion in R^2. It is the four parameter Lie group of transformations given by,

$$x^* = e^{\xi_4}\left(x\cos\xi_1 - y\sin\xi_1\right) + \xi_2 \qquad \text{....(17)}$$

$$y^* = e^{\xi_4}\left(x\sin\xi_1 + y\cos\xi_1\right) + \xi_3 \qquad \text{....(18)}$$

The infinitesimal generators are,

$$X_1 = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$$

$$X_2 = \frac{\partial}{\partial x}$$

$$X_3 = \frac{\partial}{\partial y}$$

$$X_4 = x\frac{\partial}{\partial y} + y\frac{\partial}{\partial y} \quad \text{....(19)}$$

3.4.1.RESULT

Suppose a K[th], order PDE

$$U_{i_1 i_2 \ldots i_k} = F(x, u, \partial u, \partial^2 u, ..\partial^k u) \quad \text{....(20)}$$

is of the form

$$B_{i_1 i_2 \ldots i_k}{}^{(x)} U_{i_1 i_2 \ldots\ldots i_k} = g(x, u, \partial u......, \partial^{k-1} u) \quad \text{......(21)}$$

$k \geq 2$ and admits a Lie group of point transformations with the infinitesimal generator

$$X = \xi_i(x, u)\frac{\partial}{\partial x_i} + \eta(x, u)\frac{\partial}{\partial u}$$

If there does not exist a change of independent variables x under which PDE (20) is equivalent to a PDE.

$$\frac{\partial^k u}{\partial x_i{}^k} = G(x, u, \partial u, ...\partial^{k-1} u) \quad \text{....(22)}$$

for some function G $(x,u,\partial u, ... \partial^{k-1}u)$.

$$\frac{\partial \xi_i}{\partial u} = 0, \qquad i = 1,2, ... n.$$

3.4.2. RESULT

Suppose a PDE of the form

$\frac{\partial^k u}{\partial x_i{}^k} = G(x, u,\partial u,...\partial^{k-1}u)$ is of order $k \geq 2$ and admits a Lie group of point transformations with the infinitesimal generator

$$X = \xi_i (x,u) \frac{\partial}{\partial x_j} + \eta(x,u) \frac{\partial}{\partial u}$$

Then $\frac{\partial \xi_i}{\partial u} = 0,$ i = 1,2,... n. ….(23)

3.4.3. RESULT

Suppose a PDE $U_{i_1 i_2 \dots i_k} = F(x, u, \partial u, \partial^2 u, .. \partial^k u)$ of order K ≥ 3 is of the form $A_{i_1 i_2 \dots\dots i_k}(x_i, u)\, U_{i_1 i_2 \dots\dots i_k} = C_{j_1 j_2 \dots\dots j_{k-1}}(x_i, u)\, U_{j_1 j_2 \dots\dots j_{k-1}} + h(x, u, \partial u, \dots\dots, \partial^{k-2} u)$ ……….(24)

and admits a Lie group of point transformations with the infinitesimal generator (23) Then

$$\frac{\partial \xi_i}{\partial u} = 0, i = 1,2,\dots n \text{ and } \frac{\partial^2 \eta}{\partial u^2} = 0$$

3.4.4. RESULT

Suppose a second order PDE (23) is of the form,

$$A_{i_j}^{(x,u)} U_{i\,j} = C_k (x,u)\, U_k + h(x,u),$$

and admits a Lie group of point transformation with the infinitesimal generator (23) that,

$$\frac{\partial \xi_i}{\partial u} = 0, i = 1,2,\dots n.$$

Then $\frac{\partial^2 \eta}{\partial u^2} = 0$

3.4.5. RESULT

Suppose a PDE (20) of order k ≥ 2 is a linear PDE that admits a Lie group of point transformations with infinitesimal generator (23), then

$$\frac{\partial \xi_i}{\partial u} = 0, i = 1,2,\dots n;\ \frac{\partial^2 \eta}{\partial u^2} = 0$$

4. SYMMETRIES OF ONE –DIMENSIONAL WAVE EQUATION

Consider the Wave equation

$$U_{xx} = U_{tt}$$ ….(25)

From result 1 of (20), it immediately follows that the infinitesimal generator of a point symmetry

$$X = \xi(x, t, u)\frac{\partial}{\partial x} + \tau(x, t, u)\frac{\partial}{\partial t} + \eta(x, t, u)\frac{\partial}{\partial u} \qquad \text{….(26)}$$

admitted by PDE $u_{tt} = u_{xx}$ must be of the form.

$$X = \xi(x, t)\frac{\partial}{\partial x} + \tau(x, t)\frac{\partial}{\partial t} + [f(x,t)\,u + g(x,t)]\frac{\partial}{\partial u} \qquad \text{….(27)}$$

We now find all infinitesimal generators of point symmetries (27) admitted by (25). For PDE $u_{tt} = u_{xx}$, the symmetry determining equation (11) becomes,

$$\eta_{xx}^{(2)} = \eta_{tt}^{(2)} \text{ when } u_{xx} = u_{tt} \qquad \text{….(28)}$$

An infinitesimal generator of the form of (27),

We have $\eta = fu + g$

$$\eta_x^{(1)} = \frac{\partial\eta}{\partial x} - u_x\frac{\partial\xi}{\partial x} - u_t\frac{\partial\tau}{\partial x}$$

$$= u\frac{\partial f}{\partial x} + f\frac{\partial u}{\partial x} + \frac{\partial g}{\partial x} - u_x\frac{\partial\xi}{\partial x} - u_t\frac{\partial\tau}{\partial x}$$

$$\eta_x^{(1)} = \frac{\partial g}{\partial x} + \frac{\partial f}{\partial x}u + \left(f - \frac{\partial\xi}{\partial x}\right)u_x - \frac{\partial\tau}{\partial x}u_t \qquad \text{…(29)}$$

$$\eta_{xx}^{(2)} = \frac{\partial^2 g}{\partial x^2} + u_x\frac{\partial f}{\partial x} + u\frac{\partial^2 f}{\partial x^2} + \frac{\partial f}{\partial x}u_x + f\,u_{xx}$$

$$- \frac{\partial^2\xi}{\partial x^2}u_x - 2\frac{\partial\xi}{\partial x}u_{xx} - 2u_{tx}\frac{\partial\tau}{\partial x} - u_t\frac{\partial^2\tau}{\partial x^2}$$

$$\eta_{xx}^{(2)} = \frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 f}{\partial x^2}u + \left(2\frac{\partial f}{\partial x} - \frac{\partial^2\xi}{\partial x^2}\right)u_x - \frac{\partial^2\tau}{\partial x^2}u_t$$

$$+\left(f - 2\frac{\partial\xi}{\partial x}\right)u_{xx} - 2\frac{\partial\tau}{\partial x}u_{tx} \qquad \text{…..(30)}$$

$$\eta_t^{(1)} = \frac{\partial\eta}{\partial t} - u_x\frac{\partial\xi}{\partial t} - u_t\frac{\partial\tau}{\partial t}$$

$$= u\frac{\partial f}{\partial t} + f\ \frac{\partial u}{\partial t} + \frac{\partial g}{\partial t} - u_x\frac{\partial \xi}{\partial t} - u_t\frac{\partial \tau}{\partial t}$$

$$\eta_t^{(1)} = \frac{\partial g}{\partial t} + \frac{\partial f}{\partial t}u + \left(f - \frac{\partial \tau}{\partial t}\right) u_t - \frac{\partial \xi}{\partial t}u_x \qquad \text{....(31)}$$

$$\eta_{tt}^{(2)} = \frac{\partial^2 g}{\partial t^2} + \frac{\partial^2 f}{\partial t^2}u + \frac{\partial f}{\partial t}u_t + \frac{\partial f}{\partial t}\ u_t + f\,u_{tt} - \frac{\partial^2 \tau}{\partial t^2}\ u_t$$

$$-2\ \frac{\partial \tau}{\partial t}u_{tt} - \frac{\partial^2 \xi}{\partial t^2}\ u_x - 2\frac{\partial \xi}{\partial t}u_{xt}$$

$$\eta_{tt}^{(2)} = \frac{\partial^2 g}{\partial t^2} + \frac{\partial^2 f}{\partial t^2}\ u - \frac{\partial^2 \xi}{\partial t^2}u_x + \left(2\frac{\partial f}{\partial t} - \frac{\partial^2 \tau}{\partial t^2}\right) u_t$$

$$-2\ \frac{\partial \xi}{\partial t}\ u_{xt} + \left(f - 2\frac{\partial \tau}{\partial t}\right) u_{tt} \qquad \text{...(32)}$$

Substituting the values of(30) and (32) in (25), we get,

$$\eta_{xx}^{(2)} - \eta_{tt}^{(2)} = 0$$

$$\left(\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 f}{\partial x^2}\ u + \left(2\frac{\partial f}{\partial x} - \frac{\partial^2 \xi}{\partial x^2}\right) u_x - \frac{\partial^2 \tau}{\partial x^2}\ u_t + \left(f - 2\frac{\partial \xi}{\partial x}\right) u_{xx} - 2\frac{\partial \tau}{\partial x}\ u_{tx}\right)$$

$$-\left(\frac{\partial^2 g}{\partial t^2} + \frac{\partial^2 f}{\partial t^2}\ u - \frac{\partial^2 \xi}{\partial t^2}\ u_x + \left(2\frac{\partial f}{\partial t} - \frac{\partial^2 \tau}{\partial t^2}\right) u_t - 2\ \frac{\partial \xi}{\partial t}\ u_{xt} + \left(f - 2\frac{\partial \tau}{\partial t}\right) u_{tt}\right) = 0$$

(i.e.) $$\left(\frac{\partial^2 g}{\partial x^2} - \frac{\partial^2 g}{\partial t^2}\right) + \left(\frac{\partial^2 f}{\partial x^2} - \frac{\partial^2 f}{\partial t^2}\right) u + \left(2\frac{\partial f}{\partial x} - \frac{\partial^2 \xi}{\partial x^2} + \frac{\partial^2 \xi}{\partial t^2}\right) u_x$$

$$+\left(\frac{\partial^2 \tau}{\partial t^2} - 2\frac{\partial f}{\partial t} - \frac{\partial^2 \tau}{\partial x^2}\right) u_t + \left(f - 2\frac{\partial \xi}{\partial x}\right) u_{xx}$$

$$-\left(f - 2\frac{\partial \tau}{\partial t}\right) u_{tt} - 2\frac{\partial \tau}{\partial x}\ u_{tx} + 2\frac{\partial \xi}{\partial t}u_{xt} = 0 \qquad \text{....(33)}$$

The symmetry determining equation (33) must hold for all values of x, t, u, u_x, u_{xt}. Hence, from setting to zero the coefficients of u_{tt}, u_{xt}, u_t, u_x, u and the first bracketed term of (33), we obtain the following set of determining equations, for $\xi(x,t)$, $\tau(x,t)$, $f(x,t)$, $g(x,t)$:

$$\frac{\partial^2 g}{\partial x^2} - \frac{\partial^2 g}{\partial t^2} = 0 \quad ....(34)$$

$$\frac{\partial^2 f}{\partial x^2} - \frac{\partial^2 f}{\partial t^2} = 0 \quad ....(35)$$

$$2\frac{\partial f}{\partial x} - \frac{\partial^2 \xi}{\partial x^2} + \frac{\partial^2 \xi}{\partial t^2} = 0 \quad ....(36)$$

$$\frac{\partial^2 \tau}{\partial t^2} - 2\frac{\partial f}{\partial t} - \frac{\partial^2 \tau}{\partial x^2} = 0 \quad ....(37)$$

$$\frac{\partial \tau}{\partial x} = \frac{\partial \xi}{\partial t} = 0 \quad ....(38)$$

$$\left(f - 2\frac{\partial \xi}{\partial x} \right) = 0 \quad ....(39)$$

$$\left(f - 2\frac{\partial \tau}{\partial t} \right) = 0 \quad ....(40)$$

The solution of PDE (34) corresponds to the trivial infinite parameter Lie group of point symmetries

$x^* = x$,

$u^* = u + \xi w(x)$ with $w(x) = g(x, t)$.

Non-trivial point symmetries arise from solving the system of linear PDEs from (34) to (40).

Assume $\frac{\partial \tau}{\partial x} = a(t)$

From this assumption, we get,

$$\tau = a(t)\,x + c(t)$$

$$f = 2a'(t)\,x + 2c'(t)$$

$$\xi = a'(t)\frac{x^2}{2} + c'(t)\,x + b(t)$$

By solving the above equations, we can find the values of the constants such that,

$a(t) = \alpha_1 t^2 + \alpha_2 t + \alpha_3$

$b(t) = -\alpha_1 t^3 - \frac{3}{2}\alpha_2 t^2 + \alpha_6 t + \alpha_7$

$c(t) = \alpha_1 xt^2 + \alpha_4 t + \alpha_5$

Thus one can show that the solution of the determining equations (34) to (40) is given by,

$$\xi(x,t) = \left(3x^2 t - t^3\right)\alpha_1 + \left(\frac{x^2}{2} - \frac{3}{2}t^2\right)\alpha_2 + \alpha_4 x + \alpha_6 t + \alpha_7 \quad ....(41)$$

$$\tau(x,t) = 2xt^2\,\alpha_1 + xt\,\alpha_2 + x\alpha_3 + t\alpha_4 + \alpha_5 \quad ....(42)$$

$$f(x,t) = 8xt\,\alpha_1 + 2\alpha_2 x + 2\alpha_4 \quad ....(43)$$

Where α_1, α_2, α_3, α_4, α_5, α_6, α_7, are seven arbitrary parameters.

Hence the point symmetry generators admitted by the wave equation (25) are given by,

$$X_1 = \left(3x^2 t - t^3\right)\frac{\partial}{\partial x} + 2xt^2\frac{\partial}{\partial t} + 8xt\frac{\partial}{\partial u}$$

$$X_2 = \left(\frac{x^2}{2} - \frac{3}{2}t^2\right)\frac{\partial}{\partial x} + xt\frac{\partial}{\partial t} + 2x\frac{\partial}{\partial u}$$

$$X_3 = x\frac{\partial}{\partial t}$$

$$X_4 = x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} + 2\frac{\partial}{\partial u}$$

$$X_5 = \frac{\partial}{\partial t}$$

$$X_6 = t\frac{\partial}{\partial x}$$

$$X_7 = \frac{\partial}{\partial x} \quad ....(44)$$

The infinitesimal generator (44) corresponds to a seven parameter Lie group of non-trivial point transformation acting on (x, t, u) – space.

5. SYMMETRIES OF TWO-DIMENSIONAL WAVE EQUATION

Consider the wave equation,

$u_{tt} = u_{xx} + u_{yy}$ ....(45)

it immediately follows that the infinitesimal generator of a point symmetry

$$X = \xi_1(x,y,t)\frac{\partial}{\partial x} + \xi_2(x,y,t)\frac{\partial}{\partial y} + \tau(x,y,t)\frac{\partial}{\partial t} + \eta(x,y,t,u)\frac{\partial}{\partial u} \quad ....(46)$$

admitted by PDE $u_{tt} = u_{xx}$ must be of the form

$$X = \xi_1(x,t)\frac{\partial}{\partial x} + \xi_2(y,t)\frac{\partial}{\partial y} + \tau(x,t)\frac{\partial}{\partial t} + \left[f(x,t)u + g(x,t)\right]\frac{\partial}{\partial u} \quad ....(47)$$

We now find the infinitesimal generators of point symmetries (47) admitted by (45). For PDE $u_{tt} = u_{xx}$, the symmetry determining equation (11) becomes,

$$\eta_{tt}^{(2)} = \eta_{xx}^{(2)} + \eta_{yy}^{(2)} \text{ when } u_{tt} = u_{xx} + u_{yy} \quad ....(48)$$

An infinitesimal generator of the form of (2.23), we have

$$\eta = fu + g$$

$$\eta_x^{(1)} = \frac{\partial \eta}{\partial x} - \frac{\partial \xi_1}{\partial x}u_x - \frac{\partial \xi_2}{\partial x}u_y - \frac{\partial \tau}{\partial x}u_t$$

$$= u\frac{\partial f}{\partial x} + fu_x + \frac{\partial g}{\partial x} - \frac{\partial \xi_1}{\partial x}u_x - \frac{\partial \xi_2}{\partial x}u_y - \frac{\partial \tau}{\partial x}u_t$$

$$\eta_x^{(1)} = \frac{\partial g}{\partial x} + u\frac{\partial f}{\partial x} + \left(f - \frac{\partial \xi_1}{\partial x}\right)u_x - \frac{\partial \xi_2}{\partial x}u_y - \frac{\partial \tau}{\partial x}u_t \quad ....(49)$$

$$\eta_{xx}^{(2)} = \frac{\partial^2 g}{\partial x^2} + u_x\frac{\partial f}{\partial x} + u\frac{\partial^2 f}{\partial x^2} + u_x\frac{\partial f}{\partial x} + fu_{xx}$$

$$-\frac{\partial^2 \xi_1}{\partial x^2}u_x - 2\frac{\partial \xi_1}{\partial x}u_{xx} - \frac{\partial^2 \xi_2}{\partial x^2}u_y - 2\frac{\partial \xi_2}{\partial x}u_{yx} - 2\frac{\partial^2 \tau}{\partial x^2}u_t - 2\frac{\partial \tau}{\partial x}u_{tx}$$

$$\eta_{xx}^{(2)} = \frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 f}{\partial x^2}u + \left(f - 2\frac{\partial \xi_1}{\partial x}\right)u_{xx} + \left(2\frac{\partial f}{\partial x} - \frac{\partial^2 \xi_1}{\partial x^2}\right)u_x$$

$$-2\frac{\partial \tau}{\partial x}u_{xt} - 2\frac{\partial^2 \tau}{\partial x^2}u_t - \frac{\partial^2 \xi_2}{\partial x^2}u_y - 2\frac{\partial \xi_2}{\partial x}u_{xy} \quad ....(50)$$

$$\eta_y^{(1)} = \frac{\partial \eta}{\partial y} - \frac{\partial \xi_1}{\partial y} u_x - \frac{\partial \xi_2}{\partial y} u_y - \frac{\partial \tau}{\partial y} u_t$$

$$= f \frac{\partial u}{\partial y} + u \frac{\partial f}{\partial y} + \frac{\partial g}{\partial y} - \frac{\partial \xi_1}{\partial y} u_x - \frac{\partial \xi_2}{\partial y} u_y - \frac{\partial \tau}{\partial y} u_t$$

$$\eta_y^{(1)} = \frac{\partial g}{\partial y} + u \frac{\partial f}{\partial y} + \left(f - \frac{\partial \xi_2}{\partial y} \right) u_y - \frac{\partial \xi_1}{\partial y} u_x - \frac{\partial \tau}{\partial y} u_t \qquad \text{….(51)}$$

$$\eta_{yy}^{(2)} = \frac{\partial^2 g}{\partial y^2} + u_y \frac{\partial f}{\partial y} + u \frac{\partial^2 f}{\partial y^2} + \frac{\partial f}{\partial y} u_y + f u_{yy}$$

$$-2 \frac{\partial \xi_2}{\partial y} u_{yy} - \frac{\partial^2 \xi_1}{\partial y^2} u_x - \frac{\partial \xi_1}{\partial y} u_{xy} - 2 \frac{\partial^2 \tau}{\partial y^2} u_t - 2 \frac{\partial \tau}{\partial y} u_{ty}$$

$$\eta_{yy}^{(2)} = \frac{\partial^2 g}{\partial y^2} + \frac{\partial^2 f}{\partial y^2} u + \left(f - 2 \frac{\partial \xi_2}{\partial y} \right) u_{yy} + \left(2 \frac{\partial f}{\partial y} - \frac{\partial^2 \xi_2}{\partial y^2} \right) u_y$$

$$-2 \frac{\partial \tau}{\partial y} u_{ty} - 2 \frac{\partial^2 \tau}{\partial y^2} u_t - \frac{\partial^2 \xi_1}{\partial y^2} u_x - \frac{\partial \xi_1}{\partial y} u_{xy} \qquad \text{....(52)}$$

$$\eta_t^{(1)} = \frac{\partial \eta}{\partial t} - \frac{\partial \xi_1}{\partial t} u_x - \frac{\partial \xi_2}{\partial t} u_y - \frac{\partial \tau}{\partial t} u_t$$

$$= f \frac{\partial u}{\partial t} + u \frac{\partial f}{\partial t} + \frac{\partial g}{\partial t} - \frac{\partial \xi_1}{\partial t} u_x - \frac{\partial \xi_2}{\partial t} u_y - \frac{\partial \tau}{\partial t} u_t$$

$$\eta_t^{(1)} = \frac{\partial g}{\partial t} + \frac{\partial f}{\partial t} u + \left(f - \frac{\partial \tau}{\partial t} \right) u_t - \frac{\partial \xi_1}{\partial t} u_x - \frac{\partial \xi_2}{\partial t} u_y \qquad \text{....(53)}$$

$$\eta_{tt}^{(2)} = \frac{\partial^2 g}{\partial t^2} + \frac{\partial^2 f}{\partial t^2} u + \frac{\partial f}{\partial t} u_t + \frac{\partial f}{\partial t} u_t + f u_{tt}$$

$$-\frac{\partial^2 \tau}{\partial t^2} u_t - 2 \frac{\partial \tau}{\partial t} u_{tt} - 2 \frac{\partial \xi_1}{\partial t} u_{xt} - \frac{\partial^2 \xi_1}{\partial t^2} u_x - 2 \frac{\partial \xi_2}{\partial t} u_{yt} - \frac{\partial^2 \xi_2}{\partial t^2} u_y$$

$$\eta_{tt}^{(2)} = \frac{\partial^2 g}{\partial t^2} + \frac{\partial^2 f}{\partial t^2} u + \left(f - 2 \frac{\partial \tau}{\partial t} \right) u_{tt} + \left(2 \frac{\partial f}{\partial t} - \frac{\partial^2 \tau}{\partial t^2} \right) u_t$$

$$-2 \frac{\partial \xi_1}{\partial t} u_{xt} - 2 \frac{\partial \xi_2}{\partial t} u_{yt} - \frac{\partial^2 \xi_1}{\partial t^2} u_x - \frac{\partial^2 \xi_2}{\partial t^2} u_y \qquad \text{….(54)}$$

Substituting the values of (50), (52) and (54) in (45), we get,

$$\eta_{xx}^{(2)} + \eta_{yy}^{(2)} - \eta_{tt}^{(2)} = 0$$

$$\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 f}{\partial x^2}u + \left(f - 2\frac{\partial \xi_1}{\partial x}\right)u_{xx} + \left(2\frac{\partial f}{\partial x} - \frac{\partial^2 \xi_1}{\partial x^2}\right)u_x$$
$$-2\frac{\partial \tau}{\partial x}u_{xt} - 2\frac{\partial^2 \tau}{\partial x^2}u_t - \frac{\partial^2 \xi_2}{\partial x^2}u_y - 2\frac{\partial \xi_2}{\partial x}u_{xy} +$$
$$\frac{\partial^2 g}{\partial y^2} + \frac{\partial^2 f}{\partial y^2}u + \left(f - 2\frac{\partial \xi_2}{\partial y}\right)u_{yy} + \left(2\frac{\partial f}{\partial y} - \frac{\partial^2 \xi_2}{\partial y^2}\right)u_y$$

$$-2\frac{\partial \tau}{\partial y}u_{ty} - 2\frac{\partial^2 \tau}{\partial y^2}u_t - \frac{\partial^2 \xi_1}{\partial y^2}u_x - \frac{\partial \xi_1}{\partial y}u_{xy} -$$

$$\frac{\partial^2 g}{\partial t^2} - \frac{\partial^2 f}{\partial t^2}u - \left(f - 2\frac{\partial \tau}{\partial t}\right)u_{tt} - \left(2\frac{\partial f}{\partial t} - \frac{\partial^2 \tau}{\partial t^2}\right)u_t$$

$$-2\frac{\partial \xi_1}{\partial t}u_{xt} - 2\frac{\partial \xi_2}{\partial t}u_{yt} - \frac{\partial^2 \xi_1}{\partial t^2}u_x - \frac{\partial^2 \xi_2}{\partial t^2}u_y = 0$$

(i.e.) $\left[\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} - \frac{\partial^2 g}{\partial t^2}\right] + \left[\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} - \frac{\partial^2 f}{\partial t^2}\right]u +$

$$+\left(2\frac{\partial f}{\partial x} - \frac{\partial^2 \xi_1}{\partial x^2} - \frac{\partial^2 \xi_1}{\partial y^2} + \frac{\partial^2 \xi_1}{\partial t^2}\right)u_x + \left[2\frac{\partial f}{\partial y} - \frac{\partial^2 \xi_2}{\partial y^2} - \frac{\partial^2 \xi_2}{\partial x^2} + \frac{\partial^2 \xi_2}{\partial t^2}\right]u_y$$

$$+\left[\frac{\partial^2 \tau}{\partial t^2} - 2\frac{\partial f}{\partial t} - 2\frac{\partial^2 \tau}{\partial x^2} - 2\frac{\partial^2 \tau}{\partial y^2}\right]u_t$$

$$+\left(f - 2\frac{\partial \xi_1}{\partial x}\right)u_{xx} + \left(f - 2\frac{\partial \xi_2}{\partial y}\right)u_{yy} + \left(f - 2\frac{\partial \tau}{\partial t}\right)u_{tt}$$
$$-\left(2\frac{\partial \xi_2}{\partial x} + \frac{\partial \xi_1}{\partial y}\right)u_{xy} + \left(2\frac{\partial \xi_1}{\partial t} - 2\frac{\partial \tau}{\partial x}\right)u_{xt} - \left(2\frac{\partial \tau}{\partial y} + 2\frac{\partial \xi_2}{\partial t}\right)u_{yt} = 0 \quad \text{.....(55)}$$

The symmetry determining equation (56) must hold for all values of x, t, y, u, u_x, u_t, u_y, u_{xx}, u_{tt}, u_{xt}, u_{yt}, u_{xy}. Hence, from setting to zero the coefficients of u, u_x, u_t, u_{xt}, u_{yt}, u_{xy}, u_{xx}, u_{yy}, u_{tt} and the first bracketed term of (56), we obtain the following set of determining equations for ξ_1 (x,t), ξ_2(y,t), τ (x,t), f(x, t) g(x, t) :

$$\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} - \frac{\partial^2 g}{\partial t^2} = 0 \quad \text{....(56)}$$

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} - \frac{\partial^2 f}{\partial t^2} = 0 \qquad ....(57)$$

$$2\frac{\partial f}{\partial x} - \frac{\partial^2 \xi_1}{\partial x^2} - \frac{\partial^2 \xi_1}{\partial y^2} + \frac{\partial^2 \xi_1}{\partial t^2} = 0 \qquad ....(58)$$

$$2\frac{\partial f}{\partial x} - \frac{\partial^2 \xi_2}{\partial x^2} - \frac{\partial^2 \xi_2}{\partial y^2} + \frac{\partial^2 \xi_2}{\partial t^2} = 0 \qquad ....(59)$$

$$\frac{\partial^2 \tau}{\partial t^2} - \frac{\partial^2 \tau}{\partial x^2} - \frac{\partial^2 \tau}{\partial y^2} - 2\frac{\partial f}{\partial t} = 0 \qquad ....(60)$$

$$f - 2\frac{\partial \xi_1}{\partial x} = 0 \qquad ....(61)$$

$$f - 2\frac{\partial \xi_2}{\partial y} = 0 \qquad .... (62)$$

$$f - 2\frac{\partial \tau}{\partial t} = 0 \qquad ....(63)$$

$$2\frac{\partial \xi_2}{\partial x} + 2\frac{\partial \xi_1}{\partial y} = 0 \qquad ....(64)$$

$$2\frac{\partial \xi_1}{\partial t} - 2\frac{\partial \tau}{\partial x} = 0 \qquad ....(65)$$

$$2\frac{\partial \xi_2}{\partial t} - 2\frac{\partial \tau}{\partial y} = 0 \qquad ....(66)$$

The solution of PDE (57) corresponds to the trivial infinite parameter Lie group of point symmetries,

$x^* = x$

$u^* = u + \xi w(x)$ with $w(x) = g(x, t)$.

Non-trivial point symmetries arise from solving the system of linear PDEs from (57) to (67).

By solving the above equations, one can show that the solution of the determining equations (57) to (67) is given by,

$$\xi_1(x,y,t) = \alpha_1 \frac{x^2}{2} + 2tx\alpha_3 + x\alpha_4 + t\,\alpha_6 + \alpha_7$$

$$\xi_2(x,y,t) = \left(xy - t^2\right)\alpha_1 + 2ty\alpha_2 + y\alpha_4 + t\,\alpha_8 + \alpha_9$$

$$\tau(x,t) = xt\,\alpha_1 + x\alpha_2 + t^2\,\alpha_3 + t\,\alpha_4 + \alpha_5$$

$$f(x,t) = 2x\,\alpha_1 + 4t\alpha_3 + 2\,\alpha_4$$

where α_1, α_2, α_3, α_4, α_5, α_6, α_7, α_8, α_9 are an arbitrary parameters.

Hence, the point symmetry generators admitted by the wave equation (41) are given by,

$$X_1 = \frac{x^2}{2}\frac{\partial}{\partial x} + \left(xy - t^2\right)\frac{\partial}{\partial y} + xt\frac{\partial}{\partial t} + 2x\frac{\partial}{\partial u}$$

$$X_2 = 2ty\frac{\partial}{\partial y} + x\frac{\partial}{\partial t}$$

$$X_3 = 2xt\frac{\partial}{\partial x} + t^2\frac{\partial}{\partial t} + 4t\frac{\partial}{\partial u}$$

$$X_4 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + t\frac{\partial}{\partial t} + 2\frac{\partial}{\partial u}$$

$$X_5 = \frac{\partial}{\partial t}$$

$$X_6 = t\frac{\partial}{\partial x}$$

$$X_7 = \frac{\partial}{\partial x}$$

$$X_8 = t\frac{\partial}{\partial y}$$

$$X_9 = \frac{\partial}{\partial y} \qquad \text{....(67)}$$

The infinitesimal generator (67) corresponds to a nine-parameters Lie group of non-trivial point transformations acting on (x, y, t) – space.

6. SYMMETRIES OF THREE DIMENSIONAL WAVE EQUATION

Consider the wave equation

$$u_{tt} = u_{xx} + u_{yy} + u_{zz} \quad \text{….(68)}$$

it immediately follows that the infinitesimal generator of a point symmetry

$$X = \xi_1(x,t,u)\frac{\partial}{\partial x} + \xi_2(y,t,u)\frac{\partial}{\partial y} + \xi_3(z,t,u)\frac{\partial}{\partial z} + \tau(x,t,u)\frac{\partial}{\partial t}$$

$$+\eta(x,t,u)\frac{\partial}{\partial u} \quad \text{….(69)}$$

admitted by PDE $u_{tt} = u_{xx}$ must be of the form,

$$X = \xi_1(x,t)\frac{\partial}{\partial x} + \xi_2(y,t)\frac{\partial}{\partial y} + \xi_3(z,t)\frac{\partial}{\partial z} + \tau(x,t)\frac{\partial}{\partial t}$$

$$+[f(x,t)u + g(x,t)]\frac{\partial}{\partial u} \quad \text{….(70)}$$

We now find all infinitesimal generators of point symmetries (70) admitted by (68). For PDE $u_{tt} = u_{xx}$, the symmetry determining equation (11) becomes,

$$\eta_{tt}^{(2)} = \eta_{xx}^{(2)} + \eta_{yy}^{(2)} + \eta_{zz}^{(2)} \quad \text{when } u_{tt} = u_{xx} + u_{yy} + u_{zz} \quad \text{….(71)}$$

An infinitesimal generator of the form of (2.33), we have

$$\eta = fu + g$$

$$\eta_x^{(1)} = \frac{\partial \eta}{\partial x} - \frac{\partial \xi_1}{\partial x}u_x - \frac{\partial \xi_2}{\partial x}u_y - \frac{\partial \xi_3}{\partial x}u_z - \frac{\partial \tau}{\partial x}u_t$$

$$= u\frac{\partial f}{\partial x} + f\frac{\partial u}{\partial x} + \frac{\partial g}{\partial x} - \frac{\partial \xi_1}{\partial x}u_x - \frac{\partial \xi_2}{\partial x}u_y - \frac{\partial \xi_3}{\partial x}u_z - \frac{\partial \tau}{\partial x}u_t$$

$$\eta_x^{(1)} = \frac{\partial g}{\partial x} + \frac{\partial f}{\partial x}u + \left(f - \frac{\partial \xi_1}{\partial x}\right)u_x - \frac{\partial \xi_2}{\partial x}u_y - \frac{\partial \xi_3}{\partial x}u_z - \frac{\partial \tau}{\partial x}u_t \quad \text{…(72)}$$

$$\eta_{xx}^{(2)} = \frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 f}{\partial x^2}u + \frac{\partial f}{\partial x}u_x + fu_{xx} + \frac{\partial f}{\partial x}u_x - \frac{\partial^2 \xi_1}{\partial x^2}u_x$$
$$- \frac{\partial \xi_1}{\partial x}u_{xx} - \frac{\partial^2 \xi_2}{\partial x^2}u_y - \frac{\partial \xi_2}{\partial x}u_{yx} - \frac{\partial^2 \xi_3}{\partial x^2}u_z - 2\frac{\partial \xi_3}{\partial x}u_{zx}$$

$$-2\frac{\partial^2\tau}{\partial x^2}u_t-2\frac{\partial\tau}{\partial x}u_{tx}$$

$$\eta_{xx}^{(2)}=\frac{\partial^2 g}{\partial x^2}+\frac{\partial^2 f}{\partial x^2}u+\left(f-2\frac{\partial\xi_1}{\partial x}\right)u_{xx}+\left(2\frac{\partial f}{\partial x}-\frac{\partial^2\xi_1}{\partial x^2}\right)u_x$$

$$-2\frac{\partial\tau}{\partial x}u_{tx}-2\frac{\partial\xi_2}{\partial x}u_{yx}-\frac{\partial^2\xi_3}{\partial x^2}u_z-2\frac{\partial\xi_3}{\partial x}u_{zx}-2\frac{\partial^2\tau}{\partial x^2}u_t \quad ....(73)$$

$$\eta_y^{(1)}=\frac{\partial\eta}{\partial y}-\frac{\partial\xi_1}{\partial y}u_x-\frac{\partial\xi_2}{\partial y}u_y-\frac{\partial\xi_3}{\partial y}u_z-\frac{\partial\tau}{\partial y}u_t$$

$$\eta_y^{(1)}=\frac{\partial g}{\partial y}+\frac{\partial f}{\partial y}u+\left(f-\frac{\partial\xi_2}{\partial y}\right)u_y-\frac{\partial\xi_1}{\partial y}u_x-\frac{\partial\xi_3}{\partial y}u_z-\frac{\partial\tau}{\partial y}u_t \quad ….(74)$$

$$\eta_{yy}^{(2)}=\frac{\partial^2 g}{\partial y^2}+\frac{\partial^2 f}{\partial y^2}u+\left(f-2\frac{\partial\xi_2}{\partial y}\right)u_{yy}+\left(2\frac{\partial f}{\partial y}-\frac{\partial^2\xi_2}{\partial y^2}\right)u_y$$

$$-2\frac{\partial\tau}{\partial y}u_{yt}-2\frac{\partial\xi_1}{\partial y}u_{yx}-\frac{\partial^2\xi_3}{\partial y^2}u_z-2\frac{\partial\xi_3}{\partial y}u_{zy}-2\frac{\partial^2\tau}{\partial y^2}u_t \quad ….(75)$$

$$\eta_z^{(1)}=\frac{\partial\eta}{\partial z}-\frac{\partial\xi_1}{\partial z}u_x-\frac{\partial\xi_2}{\partial z}u_y-\frac{\partial\xi_3}{\partial z}u_z-\frac{\partial\tau}{\partial z}u_t$$
$$=\frac{\partial g}{\partial z}+u\frac{\partial f}{\partial z}+\left(f-\frac{\partial\xi_3}{\partial z}\right)u_z-\frac{\partial\xi_1}{\partial z}u_x-\frac{\partial\xi_2}{\partial z}u_y-\frac{\partial\tau}{\partial z}u_t \quad ….(76)$$

$$\eta_{zz}^{(2)}=\frac{\partial^2 g}{\partial z^2}+\frac{\partial^2 f}{\partial z^2}u+\left(f-2\frac{\partial\xi_3}{\partial z}\right)u_{zz}+\left(2\frac{\partial f}{\partial z}-\frac{\partial^2\xi_3}{\partial z^2}\right)u_z$$

$$-2\frac{\partial\xi_1}{\partial z}u_{xz}-\frac{\partial^2\xi_2}{\partial z^2}u_y-2\frac{\partial\xi_2}{\partial z}u_{yz}-2\frac{\partial\tau}{\partial z}u_{tz}-\frac{\partial^2\tau}{\partial z^2}u_t \quad ….(77)$$

$$\eta_t^{(1)}=\frac{\partial\eta}{\partial t}-\frac{\partial\xi_1}{\partial t}u_x-\frac{\partial\xi_2}{\partial t}u_y-\frac{\partial\xi_3}{\partial t}u_z-\frac{\partial\tau}{\partial t}u_t$$

$$\eta_t^{(1)}=\frac{\partial g}{\partial t}+u\frac{\partial f}{\partial t}+\left(f-\frac{\partial\tau}{\partial t}\right)u_t-\frac{\partial\xi_1}{\partial t}u_x-\frac{\partial\xi_2}{\partial t}u_y-\frac{\partial\xi_3}{\partial t}u_z \quad ….(78)$$

$$\eta_{tt}^{(2)}=\frac{\partial^2 g}{\partial t^2}+\frac{\partial^2 f}{\partial t^2}u+\left(f-2\frac{\partial\tau}{\partial t}\right)u_{tt}+\left(2\frac{\partial f}{\partial t}-\frac{\partial^2\tau}{\partial t^2}\right)u_t$$

$$-2\frac{\partial\xi_1}{\partial t}u_{xt}-\frac{\partial^2\xi_3}{\partial t^2}u_z-2\frac{\partial\xi_2}{\partial t}u_{yt}-\frac{\partial\xi_3}{\partial t}u_{zt}-\frac{\partial^2\xi_1}{\partial t^2}u_z \quad ....(79)$$

Substituting the values of(73), (75), (77) and (79) in (68), we get,

$$\eta_{xx}^{(2)}+\eta_{yy}^{(2)}+\eta_{zz}^{(2)}-\eta_{tt}^{(2)}=0$$

$$\frac{\partial^2 g}{\partial x^2}+\frac{\partial^2 f}{\partial x^2}u+\frac{\partial f}{\partial x}u_x+fu_{xx}+\frac{\partial f}{\partial x}u_x-\frac{\partial^2\xi_1}{\partial x^2}u_x$$

$$-\frac{\partial\xi_1}{\partial x}u_{xx}-\frac{\partial^2\xi_2}{\partial x^2}u_y-\frac{\partial\xi_2}{\partial x}u_{yx}-\frac{\partial^2\xi_3}{\partial x^2}u_z-2\frac{\partial\xi_3}{\partial x}u_{zx}-2\frac{\partial^2\tau}{\partial x^2}u_t$$

$$-2\frac{\partial\tau}{\partial x}u_{tx}+\frac{\partial^2 g}{\partial y^2}+\frac{\partial^2 f}{\partial y^2}u+\left(f-2\frac{\partial\xi_2}{\partial y}\right)u_{yy}+\left(2\frac{\partial f}{\partial y}-\frac{\partial^2\xi_1}{\partial y^2}\right)u_y$$

$$-2\frac{\partial\tau}{\partial y}u_{yt}-2\frac{\partial\xi_1}{\partial y}u_{yx}-\frac{\partial^2\xi_3}{\partial y^2}u_z-2\frac{\partial\xi_3}{\partial y}u_{zy}-2\frac{\partial^2\tau}{\partial y^2}u_t+$$

$$\frac{\partial^2 g}{\partial z^2}+\frac{\partial^2 f}{\partial z^2}u+\left(f-2\frac{\partial\xi_3}{\partial z}\right)u_{zz}+\left(2\frac{\partial f}{\partial z}-\frac{\partial^2\xi_3}{\partial z^2}\right)u_z$$

$$-2\frac{\partial\xi_1}{\partial z}u_{xz}-\frac{\partial^2\xi_2}{\partial z^2}u_y-2\frac{\partial\xi_2}{\partial z}u_{yz}-2\frac{\partial\tau}{\partial z}u_{tz}-\frac{\partial^2\tau}{\partial z^2}u_t$$

$$-\frac{\partial^2 g}{\partial t^2}-\frac{\partial^2 f}{\partial t^2}u-\left(f-2\frac{\partial\tau}{\partial t}\right)u_{tt}-\left(2\frac{\partial f}{\partial t}-\frac{\partial^2\tau}{\partial t^2}\right)u_t$$

$$-2\frac{\partial\xi_1}{\partial t}u_{xt}-\frac{\partial^2\xi_3}{\partial t^2}u_z-2\frac{\partial\xi_2}{\partial t}u_{yt}-\frac{\partial\xi_3}{\partial t}u_{zt}-\frac{\partial^2\xi_1}{\partial t^2}u_z=0$$

$$\left(\frac{\partial^2 g}{\partial x^2}+\frac{\partial^2 g}{\partial y^2}+\frac{\partial^2 g}{\partial z^2}-\frac{\partial^2 g}{\partial t^2}\right)+\left(\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial z^2}-\frac{\partial^2 f}{\partial t^2}\right)u$$

$$+\left(2\frac{\partial f}{\partial x}-\frac{\partial^2\xi_1}{\partial x^2}-\frac{\partial^2\xi_1}{\partial y^2}-\frac{\partial^2\xi_1}{\partial z^2}+\frac{\partial^2\xi_1}{\partial t^2}\right)u_x+\left(f-2\frac{\partial\xi_1}{\partial x}\right)u_{xx}$$

$$+\left(f-2\frac{\partial\xi_2}{\partial y}\right)u_{yy}+\left(f-2\frac{\partial\xi_3}{\partial z}\right)u_{zz}+\left(\frac{\partial\xi_1}{\partial t}-\frac{\partial\tau}{\partial x}\right)u_{xt}$$

$$+\left(2\frac{\partial f}{\partial y}-\frac{\partial^2\xi_2}{\partial x^2}-\frac{\partial^2\xi_2}{\partial y^2}-\frac{\partial^2\xi_2}{\partial z^2}+\frac{\partial^2\xi_2}{\partial t^2}\right)u_y$$

$$+\left(\frac{\partial^2\tau}{\partial t^2}-2\frac{\partial f}{\partial y}-\frac{\partial^2\tau}{\partial x^2}-\frac{\partial^2\tau}{\partial y^2}-\frac{\partial^2\tau}{\partial z^2}\right)u_t$$

$$+\left(f-2\frac{\partial\tau}{\partial t}\right)u_{tt}-\left(2\frac{\partial\xi_2}{\partial x}+2\frac{\partial\xi_1}{\partial y}\right)u_{xy}+\left(-\frac{\partial\xi_3}{\partial t}-2\frac{\partial\tau}{\partial z}\right)u_{zt}$$

$$-\left(2\frac{\partial\xi_2}{\partial z}+2\frac{\partial\xi_3}{\partial y}\right)u_{zy}+\left(-2\frac{\partial\tau}{\partial y}-2\frac{\partial\xi_2}{\partial t}\right)u_{yt}$$

$$+\left(2\frac{\partial f}{\partial z}-\frac{\partial^2\xi_3}{\partial x^2}-\frac{\partial^2\xi_3}{\partial y^2}-\frac{\partial^2\xi_3}{\partial z^2}+\frac{\partial^2\xi_3}{\partial t^2}\right)u_z$$

$$-\left(2\frac{\partial\xi_1}{\partial z}+2\frac{\partial\xi_3}{\partial x}\right)u_{zx}=0 \qquad \text{….(80)}$$

The symmetry determining equation (80) must hold for all values of x, t, u, u_x, u_y, u_z, u_{xx}, u_{yy}, u_{zz}, u_{tt}, u_{xt}, u_{yz}, u_{zt}, u_{xy}, u_{xz}. Hence, from setting to zero the coefficients of u_{xt},u_{yz},u_{xz},u_{zy},u_{xt} ,u_{tt},u_{xx} ,u_{yy} ,u_x,u_y,u_z,u_t,u and the first bracketed term of (80) we obtain the following set of determining eqations for ξ_1(x,t), ξ_2(y,t), ξ_3(z,t), τ(x,t), f(x,t), g(x,t),

$$\frac{\partial^2 g}{\partial x^2}+\frac{\partial^2 g}{\partial y^2}+\frac{\partial^2 g}{\partial z^2}-\frac{\partial^2 g}{\partial t^2}=0 \qquad \text{….(81)}$$

$$\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial z^2}-\frac{\partial^2 f}{\partial t^2}=0 \qquad \text{….(82)}$$

$$2\frac{\partial f}{\partial x}-\frac{\partial^2\xi_1}{\partial x^2}-\frac{\partial^2\xi_1}{\partial y^2}-\frac{\partial^2\xi_1}{\partial z^2}+\frac{\partial^2\xi_1}{\partial t^2}=0 \qquad \text{….(83)}$$

$$f-2\frac{\partial\xi_1}{\partial x}=0 \qquad \text{….(84)}$$

$$f-2\frac{\partial\xi_2}{\partial y}=0 \qquad \text{….(85)}$$

$$f-2\frac{\partial\xi_3}{\partial z}=0 \qquad ….(86)$$

$$2\frac{\partial f}{\partial x}-\frac{\partial^2\xi_3}{\partial x^2}-\frac{\partial^2\xi_3}{\partial y^2}-\frac{\partial^2\xi_3}{\partial z^2}+\frac{\partial^2\xi_3}{\partial t^2}=0 \qquad ….(87)$$

$$2\frac{\partial f}{\partial y}-\frac{\partial^2\xi_2}{\partial x^2}-\frac{\partial^2\xi_2}{\partial y^2}-\frac{\partial^2\xi_2}{\partial z^2}+\frac{\partial^2\xi_2}{\partial t^2}=0 \qquad ….(88)$$

$$\frac{\partial^2\tau}{\partial t^2}-2\frac{\partial f}{\partial t}-\frac{\partial^2\tau}{\partial x^2}-\frac{\partial^2\tau}{\partial y^2}-\frac{\partial^2\tau}{\partial z^2}=0 \qquad ....(89)$$

$$f-2\frac{\partial\tau}{\partial t}=0 \qquad ….(90)$$

$$2\frac{\partial\xi_2}{\partial x}+2\frac{\partial\xi_1}{\partial y}=0 \qquad ….(91)$$

$$\frac{\partial\xi_3}{\partial t}+2\frac{\partial\tau}{\partial z}=0 \qquad ….(92)$$

$$2\frac{\partial\xi_3}{\partial z}+2\frac{\partial\xi_3}{\partial y}=0 \qquad ….(93)$$

$$2\frac{\partial\tau}{\partial y}+2\frac{\partial\xi_2}{\partial t}=0 \qquad ….(94)$$

$$2\frac{\partial\xi_1}{\partial z}+2\frac{\partial\xi_3}{\partial x}=0 \qquad …. (95)$$

The solution of PDE (81) corresponds to the trivial infinite parameter Lie group of point symmetries

$x^* = x$,

$u^* = u + \xi\, w(x)$ with $w(x) = g(x,t)$.

Non-trivial point symmetries arise from solving the system of linear PDEs from (81) to (95).

By solving the above equations, one can show that the solution of the determining equations (81) to (95) is given by,

$$\xi_1(x,t)=(x^2+t^2)\alpha_1+2tx\,\alpha_3+x\,\alpha_4+t\,\alpha_8+\alpha_9$$

$$\xi_2(y,t) = 2xy\,\alpha_1 + 4ty\,\alpha_3 + 2y\,\alpha_4 + t\,\alpha_6 + \alpha_7$$

$$\xi_3(z,t) = xz\,\alpha_1 + 2tx\,\alpha_3 + z\,\alpha_4 + t\,\alpha_{10} + \alpha_{11}$$

$$\tau(x,t) = xt\,\alpha_1 + x\,\alpha_2 + t^2\,\alpha_3 + t\,\alpha_4 + \alpha_5$$

$$f(x,t) = 2x\,\alpha_1 + 4t\,\alpha_3 + 2\,\alpha_4$$

where α_1, α_2, α_3, α_4, α_5, α_6, α_7, α_8, α_9, α_{10}, α_{11} are eleven arbitrary parameters.

Hence, the point symmetry generators admitted by the wave equation(45) are given by,

$$X_1 = \left(x^2 + t^2\right)\frac{\partial}{\partial x} + 2xy\frac{\partial}{\partial y} + xt\frac{\partial}{\partial t} + 2x\frac{\partial}{\partial u} + xz\frac{\partial}{\partial z}$$

$$X_2 = x\frac{\partial}{\partial t}$$

$$X_3 = 2xt\frac{\partial}{\partial x} + t^2\frac{\partial}{\partial t} + 4t\frac{\partial}{\partial u} + 4ty\frac{\partial}{\partial y} + 2tz\frac{\partial}{\partial z}$$

$$X_4 = x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y} + t\frac{\partial}{\partial t} + 2\frac{\partial}{\partial u} + z\frac{\partial}{\partial z}$$

$$X_5 = \frac{\partial}{\partial t}$$

$$X_6 = t\frac{\partial}{\partial y}$$

$$X_7 = \frac{\partial}{\partial y}$$

$$X_8 = t\frac{\partial}{\partial x}$$

$$X_9 = \frac{\partial}{\partial x}$$

$$X_{10} = t\frac{\partial}{\partial z}$$

$$X_{11} = \frac{\partial}{\partial z} \quad \text{....(96)}$$

The infinitesimal generator (96) corresponds to a eleven parameters Lie group of non-trivial point transformations acting on (x, y,z,t) – space.

CONCLUSION

In this paper, the point symmetries admitted by a scalar PDE or system of PDEs are discussed. Using admitted point symmetries of PDEs to construct resulting invariant solutions. Invariant solutions for scalar PDEs or systems of PDEs can be determined from admitted point symmetry in two ways.

(1) Direct substitution method

(2) Invariant form method.

The resulting invariant solutions are determined by solving the reduced PDEs and then substituting solutions of the reduced PDEs into either the invariant surface conditions or the given PDEs.

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