Hermite-Hadamard Inequalities for Riemann-Liouville Fractional Integrals for $\eta - m - convex$ functions

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Abstract

We introduce the notion of $\eta - m - convex$ functions. The basic purpose of this paper is to establish a variant of Hermite-Hadamard inequalities for the Riemann-Liouville fractional integral using $\eta - m$ convex functions.

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1 Introduction and Preliminaries

Inequalities is one of the important notions of mathematics having a large number of applications in mathematics and other sciences. Several mathematicians have been working on the notion of inequalities with different types of convex functions satisfying certain integral conditions, for ready reference one can see [10, 8, 6, 7, 11, 15, 1]. In this connection Hermite–Hadamard type inequalities are very well know which along with different parameters have been discussed, refined and generalized for different types of convex functions. A considerable number of integral inequalities of the Hermite−Hadamard type for convex functions via fractional integrals have been established (see [3, 2, 13, 5, 10, 8, 6, 11, 1]. S.S Dragomir and G.H Toader [4, 14] introduced the concept of $m-$convex functions and the concept of $\eta-$convex functions was introduced by Gordji [9]. In this paper, we introduce the concept of $\eta - m-$convex functions and prove some results established in [12]. Now we give some basics and preliminaries which are very useful for the completion. Let $I = [a, b]$ be a closed interval, a function $g : I \longrightarrow R$ is said to be convex if $g(\alpha x + (1 - \alpha)y) \leq \alpha g(x) + (1 - \alpha)g(y), \forall x, y \in [a, b], \alpha \in [0, 1]$. For a convex function g, the inequality $g\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b g(x)dx \leq \frac{g(a)+g(b)}{2}$ is known as the Hermite-Hadamard inequality. Let g be a function defined on $[a, b]$. Then the Riemann-Liouville fractional integrals having order $\alpha > 0$ with $a \geqslant 0$ are defined as

$$J_{a+}^{\alpha}g(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}g(t)dt, x > a$$
$$J_{b-}^{\alpha}g(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}g(t)dt, x < b$$

Here $\Gamma(\alpha)$ is the Gamma function and when $\alpha = 1$ the fractional integral reduces to classical integral. A function $g : I = [a, b] \longrightarrow R$ is said to be $\eta-$convex if

$$g(tx + (1-t)y) \leqslant g(y) + t\eta(g(x), g(y))$$

for all $x, y \in [a, b]$ and $t \in [0, 1]$ and the function η is definded by $\eta : g([a, b]) \times g([a, b]) \longrightarrow R$. In the above definition when we take $\eta(x, y) = x - y$, we recapture the basic definition of convex function. A function $f : [0, b] \longrightarrow R$ is called m-convex, $0 \leq m \leq 1$, if $\forall x, y \in [0, b], t \in [0, 1]$, we have

$$g(tx + m(1-t)y) \leq tg(x) + m(1-t)g(y)$$

Theorem 1.1. *[13] Let $I = [a, b]$ and $g : [a, b] \longrightarrow R$ a function differentiable I°. If $|\, g' \,|$ is $\eta-$ convex on the interval $[a, b]$, then the following inequality hold for the fractional integrals*

$$\left|\frac{g(a)+g(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha}g(b) + J_{b-}^{\alpha}g(a)\right]\right| \leqslant \frac{b-a}{2(\alpha+1)}\left(1 - \frac{1}{2^{\alpha}}\right)\left[|g'(a)| + |g'(b)|\right]$$

Lemma 1.2. *[5] Consider a function $g : [a, mb] \longrightarrow R$ be a mapping which is differentiable on the interval (a,mb).If $g' \in L[a, mb]$ then the following equality for fractional integrals holds*

$$\frac{g(a)+g(mb)}{2} - \frac{\Gamma(\alpha+1)}{2(mb-a)^{\alpha}}\left[J_{a+}^{\alpha}g(mb) + J_{mb-}^{\alpha}g(a)\right] = \frac{mb-a}{2}\int_0^1 \left[(1-t)^{\alpha} - t^{\alpha}\right]g'(ta + m(1-t)b)dt$$

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Theorem 1.3. *[12] Consider* $g : [a, b] \longrightarrow R$ *is a differentiable function on the interval* (a, b) *with* $a < b$. *If* $| g' |$ *is* $\eta-$ *convex on the interval [a,b],then the following inequality holds for the fractional integrals*

$$\Big|\frac{g(a)+g(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\big[J^{\alpha}_{a+}g(b)+J^{\alpha}_{b-}g(a)\big]\Big| \leqslant \frac{b-a}{2(\alpha+1)}\Big(1-\frac{1}{2^{\alpha}}\Big)\big[2|g'(b)|+\eta\big(|g'(a)|,|g'(b)|\big)\big]$$

2 Main Results

We introduce $\eta - m-$ convex functions first.

Definition 2.1. A function $g : I = [a, mb] \longrightarrow R$ is said to be $\eta - m-$ convex if

$$g(tx + m(1-t)y) \leqslant mg(y) + t\eta(g(x), g(y))$$

for all $x, y \in [a, mb], t \in [0, 1]$ and the function η is defined by $\eta : g([a, mb]) \times g([a, mb]) \longrightarrow R$. In the above definition when we take $\eta(x, y) = x - my$. we can directly obtain the basic definition of convex function.

Theorem 2.2. *Consider* $g : [a, mb] \longrightarrow R$ *is a differentiable function on the interval* (a, mb) *with* $a < mb$.*If* $| g' |$ *is* $\eta-m-$ *convex on the interval [a,mb],then the following inequality hold for the fractional integrals*

$$\Big|\frac{g(a)+g(mb)}{2}-\frac{\Gamma(\alpha+1)}{2(mb-a)^{\alpha}}\big[J^{\alpha}_{a+}g(mb)+J^{\alpha}_{mb-}g(a)\big]\Big|$$

$$\leqslant \frac{mb-a}{2(\alpha+1)}\Big(1-\frac{1}{2^{\alpha}}\Big)\big[2m|g'(b)|+\eta\big(|g'(a)|,|g'(b)|\big)\big]$$

Proof. Using lemma 1.2 and basic property of absolute value of real numbers we get

$$\Big| \frac{g(a)+g(mb)}{2}-\frac{\Gamma(\alpha+1)}{2(mb-a)^{\alpha}}\big[J^{\alpha}_{a+}g(mb)+J^{\alpha}_{mb-}g(a)\big] \Big|$$

$$\leqslant \frac{mb-a}{2}\int_0^1 | [(1-t)^{\alpha}-t^{\alpha}] \, | g' | \, (ta+m(1-t)b) \, | dt$$

Now by $\eta - m-$ convexity of $| g' |$

$$\leqslant \frac{mb-a}{2}\int_0^1 | [(1-t)^{\alpha}-t^{\alpha}] \, | [m \, | g'(b) | + t\eta(| g'(a) |, | g'(b) |)]dt$$

$$= \frac{mb-a}{2}\Big[\int_0^{\frac{1}{2}}[(1-t)^{\alpha}-t^{\alpha}][m \, | g'(b) | + t\eta(| g'(a) |, | g'(b) |)]dt$$

$$+\int_{\frac{1}{2}}^1[t^{\alpha}-(1-t)^{\alpha}][m \, | g'(b) | + t\eta(| g'(a) |, | g'(b) |)]dt\Big]$$

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$= \frac{mb-a}{2}[m \mid g'(b) \mid (\int_0^{\frac{1}{2}}[(1-t)^\alpha - t^\alpha]dt) + \eta(\mid g'(a) \mid, \mid g'(b) \mid)\ [\int_0^{\frac{1}{2}} t[(1-t)^\alpha -$ $t^\alpha]dt] + m \mid g'(b) \mid [\int_{\frac{1}{2}}^1 [t^\alpha - (1-t)^\alpha]dt]\ + \eta(\mid g'(a) \mid, \mid g'(b) \mid)[\int_{\frac{1}{2}}^1 t[t^\alpha - (1-t)^\alpha]dt]$
After further simplification we get

$$= \frac{mb-a}{2}[\frac{2m \mid g'(b) \mid}{\alpha+1}\{1 - \frac{1}{2^\alpha}\} + \eta(\mid g'(a) \mid, \mid g'(b) \mid)\{\frac{1}{\alpha+1} - \frac{1}{2^\alpha(\alpha+1)}\}]$$

Therefore, $\mid \frac{g(a)+g(mb)}{2} - \frac{\Gamma(\alpha+1)}{2(mb-a)^\alpha}[J^\alpha_{a+}g(mb) + J^\alpha_{mb-}g(a)] \mid$

$$\leqslant \frac{mb-a}{2(\alpha+1)}(1 - \frac{1}{2^\alpha})[2m \mid g'(b) \mid + \eta(\mid g'(a) \mid, \mid g'(b) \mid)]$$

□

Corollary 2.3. *If we consider* $\mid g' \mid$ *is* $\eta - m-$ *convex with respect to* η *defined by* $\eta(x,y) = x - my$
$\mid \frac{g(a)+g(mb)}{2} - \frac{\Gamma(\alpha+1)}{2(mb-a)^\alpha}[J^\alpha_{a+}g(mb) + J^\alpha_{mb-}g(a)] \mid$

$$\leqslant \frac{mb-a}{2(\alpha+1)}g(1 - \frac{1}{2^\alpha})[\mid g'(a) \mid + m \mid g'(b) \mid]$$

which is the result proved in [5].

Remark 2.4. *If we take m=1 in theorem 2.2, we obtain theorem 1.1 proved in [13].*

Remark 2.5. *If we take m=1 along with* $\mid g' \mid$ *is* $\eta-$ *convex with respect to* η *defined by* $\eta(x,y) = x - y$*, then theorem 2.3 reduces to the theorem 1.1 which is proved in [12].*

Lemma 2.6. *Let* $I = [a, mb]$ *and* $g : I \longrightarrow R$ *be a function differentiable function on the interior of* I*. If* $g' \in L[a, mb]$*, then this identity for Riemann-Liouville fractional integrals holds*
$\frac{mb-a}{2}\sum_{k=1}^4 I_k = \frac{m}{2}g\Big(\frac{a+mb}{2m}\Big) + \frac{1}{2}g\Big(\frac{a+mb}{2}\Big) - \frac{\Gamma(\alpha+1)}{2(mb-a)^\alpha}\Big[J^\alpha_{a+}g(mb) + (m^{\alpha+1})J^\alpha_{b-}g\Big(\frac{a}{m}\Big)\Big]$
where
$I_1 = \int_o^{\frac{1}{2}} t^\alpha g'\Big(tb + \frac{(1-t)a}{m}\Big)dt$
$I_2 = \int_o^{\frac{1}{2}}(-t^\alpha)g'(ta + m(1-t)b)dt$
$I_3 = \int_{\frac{1}{2}}^1 (t^\alpha - 1)g'\Big(tb + \frac{(1-t)a}{m}\Big)dt$
$I_4 = \int_{\frac{1}{2}}^1 (1 - t^\alpha)g'(ta + m(1-t)b)dt$

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Proof. Applying integration by parts we obtain

$$I_1 = \frac{2^{-\alpha}}{b-\frac{a}{m}} g\Big(\frac{a+mb}{2m}\Big) - \frac{\alpha}{b-\frac{a}{m}} \int_o^{\frac{1}{2}} t^{\alpha-1} g\Big(tb + \frac{(1-t)a}{m}\Big)$$

$$I_2 = \frac{2^{-\alpha}}{mb-a} g\Big(\frac{a+mb}{2}\Big) - \frac{\alpha}{mb-a} \int_o^{\frac{1}{2}} t^{\alpha-1} g(ta+m(1-t)b)dt$$

$$I_3 = \frac{(1-2^{-\alpha})}{b-\frac{a}{m}} g\Big(\frac{a+mb}{2m}\Big) - \frac{\alpha}{b-\frac{a}{m}} \int_{\frac{1}{2}}^1 t^{\alpha-1} g\Big(tb + \frac{(1-t)a}{m}\Big)dt$$

$$I_4 = \frac{(1-2^{\alpha})}{mb-a} g\Big(\frac{a+mb}{2}\Big) - \frac{\alpha}{mb-a} \int_{\frac{1}{2}}^1 t^{\alpha-1} g(ta+m(1-t)b)dt$$

Adding the last four equalities

$$\sum_{k=1}^4 I_k = \frac{1}{(b-\frac{a}{m})} g(\frac{a+mb}{2m}) + \frac{1}{mb-a} g(\frac{a+mb}{2})$$

$$-\frac{\alpha}{b-\frac{a}{m}}[\int_0^1 t^{\alpha-1} g(tb + \frac{(1-t)a}{m}\Big)dt\Big] - \frac{\alpha}{mb-a}[\int_0^1 t^{\alpha-1} g(ta+m(1-t)b)dt]$$

$$\sum_{k=1}^4 I_k = \frac{1}{b-\frac{a}{m}} g\Big(\frac{a+mb}{2m}\Big) + \frac{1}{mb-a} g\Big(\frac{a+mb}{2}\Big)$$

$$-\frac{\alpha}{b-\frac{a}{m}}\Big[\int_0^1 t^{\alpha-1} g\Big(tb + \frac{(1-t)a}{m}\Big)dt\Big] - \frac{\alpha}{mb-a}[\int_0^1 t^{\alpha-1} g(ta+m(1-t)b)dt]$$

and hence

$$\frac{mb-a}{2} \sum_{k=1}^4 I_k = \frac{m}{2} g\Big(\frac{a+mb}{2m}\Big) + \frac{1}{2} g(\frac{a+mb}{2}) - \frac{\Gamma(\alpha+1)}{2(mb-a)^\alpha}\Big[J_{a+}^\alpha g(mb) + (m^{\alpha+1}) J_{b-}^\alpha g\Big(\frac{a}{m}\Big)\Big]$$

□

Corollary 2.7. *If we take* $m = 1$ *in lemma2.6 , we get lemma 3.4 of [12].*

Theorem 2.8. *Let* $I = [a, mb]$ *and* $g : [a, mb] \longrightarrow R$ *be a function differentiable on* I°*. If* $|g'|$ *is* $\eta - m-$ *convex on the interval [a,mb] with* $0 < \alpha \leqslant 1$*, then the following inequality for Riemann-Liouville fractional integrals holds*

$$\Big| \frac{m}{2} g(\frac{a+mb}{2m}) + \frac{1}{2} g(\frac{a+mb}{2}) - \frac{\Gamma(\alpha+1)}{2(mb-a)^\alpha}[J_{a+}^\alpha g(mb) + (m^{\alpha+1}) J_{b-}^\alpha g(\frac{a}{m})] \Big|$$

$$\leqslant \frac{mb-a}{2^{\alpha+1}(\alpha+1)}[\frac{|g'(a)|}{m} + m|g'(b)| + \eta(|g'(a)|, |g'(b)|) + \eta(|g'(b)|, |g'(a)|)]$$

Proof. In view of lemma 2.6, using subadditivity of the modulus, we have

$$\Big| \frac{m}{2} g(\frac{a+mb}{2m}) + \frac{1}{2} g(\frac{a+mb}{2}) - \frac{\Gamma(\alpha+1)}{2(mb-a)^\alpha}[J_{a+}^\alpha g(mb) + (m^{\alpha+1}) J_{b-}^\alpha g(\frac{a}{m})] \Big|$$

$$\leqslant \frac{mb-a}{2} \sum_{k=1}^4 |I_k|$$

By the $\eta - m-$ convexity of $|g'|$, we obtain

$$|I_1| \leqslant \int_o^{\frac{1}{2}} t^\alpha |g'(tb + \frac{(1-t)a}{m})| dt \leqslant \int_o^{\frac{1}{2}} t^\alpha \Big[\frac{|g'(a)|}{m} + t\eta(|g'(b)|, |g'(a)|)]dt$$

and therefore

$$| I_1 | \leqslant \frac{| g'(a) |}{2^{\alpha+1} m(\alpha+1)} + \cdot \frac{1}{2^{\alpha+2}(\alpha+2)} \eta(| g'(b) |, | g'(a) |)$$

Similarly

$$| I_2 | \leqslant \frac{m | g'(b) |}{2^{\alpha+1}(\alpha+1)} + \frac{1}{2^{\alpha+2}(\alpha+2)} \eta(| g'(a) |, | g'(b) |)$$

In view of the fact that $| t_1^\alpha - t_2^\alpha | \leqslant | t_1 - t_2 |^\alpha$, for all $\alpha \in (0,1)$ and $t_1, t_2 \in [0,1]$, by $\eta - m-$ convexity of $| g' |$, we get

$$| I_3 | \leqslant \frac{| g'(a) |}{2^{\alpha+1} m(\alpha+1)} + \cdot \frac{\alpha+3}{2^{\alpha+2}(\alpha+2)(\alpha+1)} \eta(| g'(b) |, | g'(a) |)$$

Similarly

$$| I_4 | \leqslant \frac{m | g'(b) |}{2^{\alpha+1}(\alpha+1)} + \frac{\alpha+3}{2^{\alpha+2}(\alpha+2)(\alpha+1)} \eta(| g'(b) |, | g'(a) |)$$

The addition of above inequalities yields

$$| \frac{m}{2} g(\frac{a+mb}{2m}) + \frac{1}{2} g(\frac{a+mb}{2}) - \frac{\Gamma(\alpha+1)}{2(mb-a)^\alpha} \left[J_{a+}^\alpha g(mb) + (m^{\alpha+1}) J_{b-}^\alpha g(\frac{a}{m}) \right] |$$

$$\leqslant \frac{mb-a}{2^{\alpha+1}(\alpha+1)} [\frac{| g'(a) |}{m} + m | g'(b) | + \eta(| g'(a) |, | g'(b) |) + \eta(| g'(b) |, | g'(a) |)]$$

□

Remark 2.9. *If we take* $\eta(x,y) = x - my$ *in theorem 2.8, then*

$$| \frac{m}{2} g(\frac{a+mb}{2m}) + \frac{1}{2} g(\frac{a+mb}{2}) - \frac{\Gamma(\alpha+1)}{2(mb-a)^\alpha} [J_{a+}^\alpha g(mb) + (m^{\alpha+1}) J_{b-}^\alpha g(\frac{a}{m})] |$$

$$\leqslant \frac{mb-a}{2^{\alpha+1}(\alpha+1)} [\frac{| g'(a) |}{m} + m | g'(b) | + (1-m) | g'(a) | + (1-m) | g'(b) |]$$

which is the corresponding result for $m-$convexity.

Corollary 2.10. *If we take* $m = 1$ *then, we obtain the theorem 3.5 of [12].*

Remark 2.11. *If we take m=1 with respect to* η *defined by* $\eta(x,y) = x - y$ *then theorem 2.8 becomes the result of [12].*

Theorem 2.12. *Let* $g : [a, mb] \longrightarrow R$ *be a function differentiable on* (a, mb) *with* $a < mb$. *If* $| g' |^q$ $(q = \frac{p}{p-1})$ *is* $\eta - m-$*convex on the interval* $[a, mb]$ *for some fixed* $p > 1$ *with* $0 < \alpha \leqslant 1$. *Then the following inequality for Riemann-Liouville fractional integrals holds*

$$| \frac{m}{2} g(\frac{a+mb}{2m}) + \frac{1}{2} g(\frac{a+mb}{2}) - \frac{\Gamma(\alpha+1)}{2(mb-a)^\alpha} [J_{a+}^\alpha g(mb) + (m^{\alpha+1}) J_{b-}^\alpha g(\frac{a}{m})] |$$

$$\leq (\frac{mb-a}{2}) [[\frac{1}{2^{p\alpha+1}(p\alpha+1)}]^{\frac{1}{p}} [\frac{| g'(a) |^q}{2m} + \frac{1}{8} \eta(| g'(b) |^q, | g'(a) |^q)]^{\frac{1}{q}}$$
$$+ [\frac{1}{2^{p\alpha+1}(p\alpha+1)}]^{\frac{1}{p}} [\frac{m | g'(b) |^q}{2} + \frac{1}{8} \eta(| g'(a) |^q, | g'(b) |^q)]^{\frac{1}{q}}$$

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$$+[\frac{1}{2^{p\alpha+1}(p\alpha+1)}]^{\frac{1}{p}}[\frac{|g'(b)|^q}{2}+\frac{1}{8m}\eta(|g'(a)|^q,|g'(b)|^q)]^{\frac{1}{q}}$$
$$+\frac{1}{2^{p\alpha+1}(p\alpha+1)}]^{\frac{1}{p}}[\frac{|g'(a)|^q}{2}+\frac{m}{8}\eta(|g'(b)|^q,|g'(a)|^q)]^{\frac{1}{q}}]$$

Proof. In view of lemma 2.6 using Holder inequalities
$$|\frac{m}{2}g(\frac{a+mb}{2m})+\frac{1}{2}g(\frac{a+mb}{2})-\frac{\Gamma(\alpha+1)}{2(mb-a)^\alpha}[J^\alpha_{a+}g(mb)+(m^{\alpha+1})J^\alpha_{b-}g(\frac{a}{m})]|$$
$$\leqslant\frac{mb-a}{2}\sum_{k=1}^4|I_k|$$
But by $\eta-m$ convexity, we have
$$|I_1|\leqslant[\int_0^{\frac{1}{2}}t^{p\alpha}dt]^{\frac{1}{p}}[\int_0^{\frac{1}{2}}|g'(tb+\frac{(1-t)a}{m})|^q dt]^{\frac{1}{q}}$$
$$\leqslant[\frac{1}{2^{p\alpha+1}(p\alpha+1)}]^{\frac{1}{p}}[\frac{1}{m}\int_0^{\frac{1}{2}}|g'(a)|^q dt+\int_0^{\frac{1}{2}}t\eta(|g'(b)|^q,|g'(a)|^q)dt]^{\frac{1}{q}}$$
and therefore
$$|I_1|\leqslant[\frac{1}{2^{p\alpha+1}(p\alpha+1)}]^{\frac{1}{p}}[\frac{|g'(a)|^q}{2m}+\frac{1}{8}\eta(|g'(b)|^q,|g'(a)|^q)]^{\frac{1}{q}}$$
Similarly
$$|I_2|\leqslant[\frac{1}{2^{p\alpha+1}(p\alpha+1)}]^{\frac{1}{p}}[\frac{m|g'(b)|^q}{2}+\frac{1}{8}\eta(|g'(a)|^q,|g'(b)|^q)]^{\frac{1}{q}}$$
Now we have
$$|I_3|\leqslant[\int_{\frac{1}{2}}^1(1-t^\alpha)^p dt]^{\frac{1}{p}}[\int_{\frac{1}{2}}^1|g'(tb+\frac{(1-t)a}{m})|^q dt]^{\frac{1}{q}}$$
As for $\alpha\in(0,1]$ and $t_1,t_2\in[0,1]$, we have $|t_1^\alpha-t_2^\alpha|\leqslant|t_1-t_2|^\alpha$, so
$$\int_{\frac{1}{2}}^1(1-t^\alpha)^p dt\leqslant\int_{\frac{1}{2}}^1(1-t)^{p\alpha}dt=\frac{1}{2^{p\alpha+1}(p\alpha+1)}$$
and therefore
$$|I_3|\leqslant[\frac{1}{2^{p\alpha+1}(p\alpha+1)}]^{\frac{1}{p}}[\frac{|g'(b)|^q}{2}+\frac{1}{8m}\eta(|g'(a)|^q,|g'(b)|^q)]^{\frac{1}{q}}$$
In the similar fashion, we obtain
$$|I_4|\leqslant[\frac{1}{2^{p\alpha+1}(p\alpha+1)}]^{\frac{1}{p}}[\frac{|g'(a)|^q}{2}+\frac{m}{8}\eta(|g'(b)|^q,|g'(a)|^q)]^{\frac{1}{q}}$$
Adding the above expressions for I_1,I_2,I_3 and I_4, we get the required result. □

Corollary 2.13. *If we put m=1 in theorem 2.12, we will get*
$$|g(\frac{a+b}{2})-\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}[J^\alpha_{a+}g(b)+J^\alpha_{b-}g(a)]|$$
$$\leqslant(\frac{b-a}{[2^{p\alpha+1}(p\alpha+1)]^{\frac{1}{p}}})[[\frac{4|g'(a)|^q+\eta(|g'(b)|^q,|g'(a)|^q)}{4}]^{\frac{1}{q}}+[\frac{4|g'(b)|^q+\eta(|g'(a)|^q,|g'(b)|^q)}{4}]^{\frac{1}{q}}]$$

Remark 2.14. *If we define $\eta(x,y)=x-my$ along with m=1 then theorem 2.12 becomes a result for m−convexity proved in [10].*

Theorem 2.15. *Let $g:[a,mb]\longrightarrow R$ ba a function differentiable on the interval (a,mb) with $a<mb$. If $|\,g'\,|^q\left(q=\dfrac{p}{p-1}\right)$ is $\eta-m-$ convex on the interval [a,mb] on some fixed $p>1$ with $0<\alpha\leqslant 1$. Then the following inequality for Riemann-Liouville fractional integrals holds*

$$|\,\frac{m}{2}g(\frac{a+mb}{2m})+\frac{1}{2}g(\frac{a+mb}{2})-\frac{\Gamma(\alpha+1)}{2(mb-a)^{\alpha}}[J^{\alpha}_{a+}g(mb)+(m^{\alpha+1})J^{\alpha}_{b-}g(\frac{a}{m})]\,|$$
$$\leqslant\frac{mb-a}{2^{\alpha+1}(\alpha+1)}[(\frac{2(\alpha+2)\,|\,g'(a)\,|^q+m(\alpha+1)\eta(|\,g'(b)\,|^q,|\,g'(a)\,|^q)}{2(\alpha+2)})^{\frac{1}{q}}$$
$$+(\frac{2m(\alpha+2)\,|\,g'(b)\,|^q+(\alpha+1)\eta(|\,g'(a)\,|^q,|\,g'(b)\,|^q)}{2(\alpha+2)})^{\frac{1}{q}}$$
$$+(\frac{2(\alpha+2)\,|\,g'(a)\,|^q+m(\alpha+3)\eta(|\,g'(b)\,|^q,|\,g'(a)\,|^q)}{2(\alpha+2)m})^{\frac{1}{q}}$$
$$+(\frac{2m(\alpha+2)\,|\,g'(b)\,|^q+(\alpha+3)\eta(|\,g'(a)\,|^q,|\,g'(b)\,|^q)}{2(\alpha+2)})^{\frac{1}{q}}]$$

Proof. In view of lemma 2.6, using $\eta-m-$convexity and Holder's inequality, we obtain

$$|\,\frac{m}{2}g(\frac{a+mb}{2m})+\frac{1}{2}g(\frac{a+mb}{2})-\frac{\Gamma(\alpha+1)}{2(mb-a)^{\alpha}}[J^{\alpha}_{a+}g(mb)+(m^{\alpha+1})J^{\alpha}_{b-}g(\frac{a}{m})]\,|$$
$$\leqslant\frac{mb-a}{2}\sum_{k=1}^{4}|\,I_k\,|$$

We have $|\,I_1\,|\leqslant(\int_0^{\frac{1}{2}}t^{\alpha}dt)^{1-\frac{1}{q}}(\int_0^{\frac{1}{2}}t^{\alpha}\,|\,g'(tb+\frac{(1-t)a}{m})\,|^q\,dt)^{\frac{1}{q}}$

$$\leqslant(\frac{1}{2^{\alpha+1}(\alpha+1)})^{1-\frac{1}{q}}[\int_0^{\frac{1}{2}}t^{\alpha}\frac{|\,g'(a)\,|^q}{m}dt+\int_0^{\frac{1}{2}}t^{\alpha+1}\eta(|\,g'(b)\,|^q,|\,g'(a)\,|^q)dt]^{\frac{1}{q}}$$

and therefore

$$|\,I_1\,|\leqslant(\frac{1}{2^{\alpha+1}(\alpha+1)})(\frac{2(\alpha+2)\,|\,g'(a)\,|^q+m(\alpha+1)\eta(|\,g'(b)\,|^q,|\,g'(a)\,|^q)}{2m(\alpha+2)})^{\frac{1}{q}}$$

Similarly

$$|\,I_2\,|\leqslant(\frac{1}{2^{\alpha+1}(\alpha+1)})(\frac{2m(\alpha+2)\,|\,g'(b)\,|^q+(\alpha+1)\eta(|\,g'(a)\,|^q,|\,g'(b)\,|^q)}{2(\alpha+2)})^{\frac{1}{q}}$$

and

$$|\,I_3\,|\leqslant(\frac{1}{2^{\alpha+1}(\alpha+1)})(\frac{2(\alpha+2)\,|\,g'(a)\,|^q+m(\alpha+3)\eta(|\,g'(b)\,|^q,|\,g'(a)\,|^q)}{2(\alpha+2)m}\Big)^{\frac{1}{q}}$$

and analogously

$$|\,I_4\,|\leqslant(\frac{1}{2^{\alpha+1}(\alpha+1)})(\frac{2m(\alpha+2)\,|\,g'(b)\,|^q+(\alpha+3)\eta(|\,g'(a)\,|^q,|\,g'(b)\,|^q)}{2(\alpha+2)})^{\frac{1}{q}}$$

By adding the above inequalities for I_1,I_2,I_3 and I_4, we reach the conclusion. □

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Remark 2.16. ***(1)*** *If we take m=1 , the theorem 2.15 gives the result of [12] for $\eta-$convexity.*
(2) *If we take m=1 along with $\eta(x,y) = x - y$, then it become the result of [10].*

Theorem 2.17. *Consider g is a $\eta - m$convex function, where $g : [a, mb] \longrightarrow R$ is bounded from above by M_n Then for $\alpha > 0$*

$$g(\frac{a+mb}{2}) - \frac{1}{2}M_n \leqslant \frac{1}{mb-a}\int_a^{mb} g(x)dx \leqslant \frac{m}{2}[g(a)+g(b)]$$

$$+\frac{1}{4}[\eta(g(a),g(b)) + \eta(g(b),g(a))] \leqslant \frac{m}{2}[g(a)+g(b)] + \frac{1}{2}M_n$$

Proof. To construct right part of the required inequality, we consider an arbitrary point.For this we take $x = ta + m(1-t)b$ with $t \in [0,1]$. So, $g(x) \leq mg(b) + t\eta(g(a), g(b))$. It follows that

$$\frac{1}{mb-a}\int_a^{mb} g(x)dx \leq \frac{1}{mb-a}\int_a^{mb}[mg(b) + \frac{x-mb}{a-mb}\eta(g(a),g(b))]dx$$

$$= [\frac{1}{mb-a}mg(b)(mb-a) + \frac{\eta(g(a),g(b))}{2(mb-a)^2}.(mb-a)^2]$$

and therefore

$$\frac{1}{mb-a}\int_a^{mb} g(x)dx \leq mg(b) + \frac{1}{2}\eta(g(a),g(b)) \quad (1)$$

We also have

$$\frac{1}{mb-a}\int_a^{mb} g(x)dx \leq mg(a) + \frac{1}{2}\eta(g(b),g(a)) \quad (2)$$

Adding (1) and (2), we get

$$\frac{1}{mb-a}\int_a^{mb} g(x)dx \leq \frac{m}{2}[g(a)+g(b)] + \frac{1}{4}[\eta(g(a),g(b)) + \eta(g(b),g(a))] \quad (3)$$

From (1) and (2), we also have

$$\frac{1}{mb-a}\int_a^{mb} g(x)dx \leq \min[mg(b) + \frac{1}{2}\eta(g(a),g(b)), mg(a) + \frac{1}{2}\eta(g(b),g(a))] \quad (4)$$

From (3) and (4), we have

$$\frac{1}{mb-a}\int_a^{mb} g(x)dx \leq \frac{1}{2}[mg(a)+mg(b)] + \frac{1}{4}[\eta(g(a),g(b)) + \eta(g(b),g(a))]$$

and therefore

$$\frac{1}{mb-a}\int_a^{mb} g(x)dx \leq \frac{m}{2}[g(a)+g(b)] + \frac{1}{2}M_n \quad (5)$$

Next we construct the left part of the required inequality. We can easily write
$g(\frac{a+mb}{2}) = g(\frac{a+mb}{4} - \frac{t(mb-a)}{4} + \frac{a+mb}{4} + \frac{t(mb-a)}{4})$
By the definition of $\eta - m-$convexity, we have
$g(\frac{1}{2}(\frac{a+mb-t(mb-a)}{2}) + \frac{1}{2}(\frac{a+mb+t(mb-a)}{2}))$

$$\leq g(\frac{a+mb+t(mb-a)}{2}) + \frac{1}{2}\eta(g(\frac{a+mb-t(mb-a)}{2}), g(\frac{a+mb+t(mb-a)}{2}))$$

and therefore

$$g(\frac{a+mb}{2}) \leq g\Big(\frac{a+mb+t(mb-a)}{2}) + \frac{1}{2}M_n$$

This means

$$g(\frac{a+mb}{2}) - \frac{1}{2}M_n \leq g(\frac{a+mb+t(mb-a)}{2})$$

Using the same argument, we get
$g(\frac{a+mb}{2}) - \frac{1}{2}M_n \leq g(\frac{a+b+t(mb-a)}{2})$
and changing of variables gives

$$\frac{1}{mb-a}\int_a^{mb} g(x)dx = \frac{1}{mb-a}[\int_a^{\frac{a+mb}{2}} g(x)dx + \int_{\frac{a+mb}{2}}^{b} g(x)dx]$$

$$= \frac{1}{2}\int_0^1 [g(\frac{a+mb-t(mb-a)}{2}) + g(\frac{a+mb+t(mb-a)}{2})dt]$$

$$\geq \frac{1}{2}\int_0^1 [2g(\frac{a+mb}{2}) - M_n]dt$$

that is $\frac{1}{mb-a}\int_a^{mb} g(x)dx \geq \frac{1}{2}\int_0^1 [2g(\frac{a+mb}{2}) - M_n]dt$ Therefore

$$g(\frac{a+mb}{2}) - \frac{1}{2}M_n \leq \frac{1}{mb-a}\int_a^{mb} g(x)dx \qquad (6)$$

From (3),(5) and(6)

$$g(\frac{a+mb}{2}) - \frac{1}{2}M_n \leqslant \frac{1}{mb-a}\int_a^{mb} g(x)dx \leqslant \frac{m}{2}[g(a)+g(b)]$$

$$+\frac{1}{4}[\eta(g(a),g(b)) + \eta(g(b),g(a))] \leq \frac{m}{2}[g(a)+g(b)] + \frac{1}{2}M_n$$

□

Remark 2.18. ***(1).*** *If we take* $\eta(x,my) = x - my$*, then*

$$g(\frac{a+mb}{2}) \leq \frac{1}{mb-a}\int_a^{mb} g(x)dx \leq \frac{m}{2}[g(a)+g(b)] + \frac{1}{4}[(1-m)g(a)+(1-m)g(b)] \leq \frac{m}{2}[g(a)+g(b)] \qquad (7)$$

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If we take m =1, then $g(\frac{a+b}{2}) \leq \frac{1}{b-a}\int_a^b g(x)dx \leq \frac{1}{2}[g(a)+g(b)]$ *which is the Hermite-Hadamard Inequality for the Fractional Integrals.*
(2). *If we take m=1 in theorem 2.18 the we obtain the corresponding result for* $\eta-$*convexity proved in [9]*

Theorem 2.19. *Suppose that a function* $g:[a,mb]\longrightarrow R$ *is an* $\eta-m-$*convex function such that* η *is bounded above by* M_n. *Then for* $\alpha>0$ *the following inequality for fractional integrals holds.*

$$g(\frac{a+mb}{2})-M_n \leqslant \frac{m\Gamma(\alpha+1)}{2(mb-a)^\alpha}[J^\alpha_{a+}g(mb)+(m^\alpha)J^\alpha_{b-}g(\tfrac{a}{m})]$$

$$\leqslant \frac{g(a)+m^2g(b)}{2}+\frac{\alpha m\Big[\eta(g(a),g(b))+\eta(g(b),g(a))\Big]}{2(\alpha+1)}$$

$$\leqslant \frac{g(a)+m^2g(b)}{2}+\frac{\alpha M_n}{\alpha+1}$$

Proof. As g is $\eta-m-$convex such that η is bounded above by M_n,

$$g(\frac{x+my}{2})-\frac{1}{2}M_n \leqslant \frac{m}{2}[g(x)+g(y)]+\frac{1}{2}M_n$$

where $x,y\in[a,mb]$. Consider $x=ta+m(1-t)b, y=tb+\frac{(1-t)a}{m}$. Then we have

$$g(\frac{a+mb}{2})-\frac{1}{2}M_n \leqslant \frac{m}{2}[g(ta+m(1-t)b)+g(tb+\frac{(1-t)a}{m})]+\frac{1}{2}M_n$$

$$2g(\frac{a+mb}{2})-M_n \leqslant m[g(ta+m(1-t)b)+g(tb+\frac{(1-t)a}{m})]+M_n$$

Multiplying the above inequality by $t^{\alpha-1}$ and then integrating the resulting inequality with respect to t over the interval $[0,1]$

$$2g(\frac{a+mb}{2})\int_0^1 t^{\alpha-1}dt-\int_0^1 t^{\alpha-1}M_n dt$$

$$\leqslant m\int_0^1 t^{\alpha-1}[g(ta+m(1-t)b)+g(tb+\frac{(1-t)a}{m})]dt+\int_0^1 t^{\alpha-1}M_n dt$$

and therefore

$$\frac{2}{\alpha}g\Big(\frac{a+mb}{2}\Big)-\frac{M_n}{\alpha} \leqslant m\int_0^1 t^{\alpha-1}[g(ta+m(1-t)b)+g(tb+\frac{(1-t)a}{m})]dt+\frac{M_n}{\alpha} \quad (8)$$

Suppose that $u=ta+m(1-t)b, v=tb+\frac{(1-t)a}{m}$. Then

$$m\int_0^1 t^{\alpha-1}[g(ta+m(1-t)b)+g(tb+\frac{(1-t)a}{m})]dt+\frac{M_n}{\alpha}$$

$$= m\int_0^1 t^{\alpha-1}g(ta+m(1-t)b)dt + m\int_0^1 t^{\alpha-1}g(tb+\frac{(1-t)a}{m})dt$$
$$= m\int_{mb}^{a}(\frac{mb-u}{mb-a})^{\alpha-1}g(u)\frac{du}{a-mb} + m\int_{\frac{a}{m}}^{b}(\frac{v-\frac{a}{m}}{b-\frac{a}{m}})^{\alpha-1}g(v)\frac{dv}{b-\frac{a}{m}}$$
$$= \frac{m\Gamma(\alpha)}{(mb-a)^{\alpha}}\Big[J_{a+}^{\alpha}g(mb) + (m^{\alpha})J_{b-}^{\alpha}g(\tfrac{a}{m})\Big]$$

Therefore inequality (7) becomes

$$\frac{2}{\alpha}g(\frac{a+mb}{2}\Big) - \frac{M_n}{\alpha} \leqslant \frac{m\Gamma(\alpha)}{(mb-a)^{\alpha}}[J_{a+}^{\alpha}g(mb) + (m^{\alpha})J_{b-}^{\alpha}g(\frac{a}{m})] + \frac{M_n}{\alpha}$$

After the rearrangement of terms, we have

$$g(\frac{a+mb}{2}\Big) - M_n \leq \frac{m\Gamma(\alpha+1`)}{2(mb-a)^{\alpha}}[J_{a+}^{\alpha}g(mb) + (m^{\alpha})J_{b-}^{\alpha}g(\frac{a}{m})] \tag{9}$$

which is the first part of the required inequality. Next we construct the second part. Again from $\eta - m-$convexity, we have

$$mg(ta+m(1-t)b) \leq m[mg(b) + t\eta(g(a), g(b))] \tag{10}$$

and

$$mg(tb+\frac{(1-t)a}{m}) \leq m[\frac{g(a)}{m} + t\eta(g(b), g(a))] \tag{11}$$

Adding (9) and (10), and multiplying resulting inequality by $t^{\alpha-1}$ and integrating with respect to t over [0,1], we obtain

$$m[\int_0^1 t^{\alpha-1}[g(ta+m(1-t)b) + g(tb+\frac{(1-t)a}{m})]dt]$$
$$\leqslant [m^2g(b)+g(a)]\int_0^1 t^{\alpha-1}dt + [\eta(g(a),g(b)) + \eta(g(b),g(a))]\int_0^1 t^{\alpha}dt$$

By simplifiying, we have

$$\frac{m\Gamma(\alpha)}{(mb-a)^{\alpha}}[J_{a+}^{\alpha}g(mb)+(m^{\alpha})J_{b-}^{\alpha}g(\frac{a}{m})] \leqslant \frac{g(a)+m^2g(b)}{\alpha} + \frac{m[\eta(g(a),g(b)) + \eta(g(b),g(a))]}{\alpha+1}$$

and therefore

$$\frac{m\Gamma(\alpha+1)}{2(mb-a)^{\alpha}}[J_{a+}^{\alpha}g(mb)+(m^{\alpha})J_{b-}^{\alpha}g(\frac{a}{m})] \leqslant \frac{g(a)+m^2g(b)}{2} + \frac{\alpha m[\eta(g(a),g(b)) + \eta(g(b),g(a))]}{2(\alpha+1)} \tag{12}$$

From (9) and (12)

$$g(\frac{a+mb}{2}) - M_n \leqslant \frac{m\Gamma(\alpha+1)}{2(mb-a)^{\alpha}}[J_{a+}^{\alpha}g(mb) + (m^{\alpha})J_{b-}^{\alpha}g(\frac{a}{m})]$$
$$\leqslant \frac{g(a)+m^2g(b)}{2} + \frac{\alpha m[\eta(g(a),g(b)) + \eta(g(b),g(a))]}{2(\alpha+1)}$$

As η is bounded above by M_n, therefore we obtain

$$g(\frac{a+mb}{2}) - M_n \leqslant \frac{m\Gamma(\alpha+1)}{2(mb-a)^{\alpha}}[J_{a+}^{\alpha}g(mb) + (m^{\alpha})J_{b-}^{\alpha}g(\frac{a}{m})]$$

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$$\leqslant \frac{g(a)+m^2g(b)}{2}+\frac{\alpha m[\eta(g(a),g(b))+\eta(g(b),g(a))]}{2(\alpha+1)}$$

$$\leqslant \frac{g(a)+m^2g(b)}{2}+\frac{\alpha M_n}{\alpha+1}$$

□

Remark 2.20. *If we take m=1 we get theorem 2.1 of [12] proved for* $\eta-$*convexity.*

$$g(\frac{a+b}{2})-M_n \leqslant \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}[J^\alpha_{a+}g(b)+J^\alpha_{b-}g(a)]$$

$$\leqslant \frac{g(a)+g(b)}{2}+\frac{\alpha[\eta(g(a),g(b))+\eta(g(b),g(a))]}{2(\alpha+1)}$$

$$\leqslant \frac{g(a)+g(b)}{2}+\frac{\alpha M_n}{\alpha+1}$$

3 Concluding Remarks

In the present paper we introduced the notion of $\eta-m-$convex functions which is the generalization of $m-$convex and $\eta-$convex functions. We established a few results in this regard for Riemann-Liouville fractional integrals and generalized some Hermite-Hadamard and Fejer type inequalities. This generalization of convex functions indeed pave new paths of research in the field of inequalities. There is much more to be discussed with this $\eta-m-$convex functions like Chebyshev type inequalities, Ostrowski type inequalities and generalized trapezoidal inequalities etc. Therefore this article would be interesting for the researchers in this field.

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