Global automotive induction for the structure of numbers by Chevalley's theorem

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Summary - We increase the dimension of the set of P^N hyper surfaces whose intersection with an established fixed projective variety is not integral. The increases obtained are optimal. As an application, where possible, hyper surfaces are constructed whose intersections with all the varieties of a family of positive projective varieties are intact. The degree of constructed hyper surfaces is explicit.

Abstract (Bertini's theorem in family). We give upper bounds for the dimension of the set of hyper surfaces of P^N which intersection with a fixed integral projective is not integral. Our upper bounds are optimal. As an application, we construct, when possible, hypersurfaces whose intersections with all types of integral integral projective are integral. The degree of the hyper surfaces we construct is explicit.

1.

We fix K a body of any characteristic, which will be systematically implied. For example, $P^N = P^N_K$. By P^N sub-variety, we will hear a geometrically closed closed sub scheme of P^N. We notice $H_e = H_{e,N} = P(H^0(P^N, O(e)))$ the space of the hyper surfaces of degree e of P^N.

If $X \subset P^N$ is a sub manifold, we denote $F_e^{int}(X)$ (resp. $F_e^{igr}(X)$) the subset of H_e consists of hyper surfaces whose intersection with X is not geometrically integral with co dimension 1 in X (or is not geometrically irreducible and generically reduced by co dimension 1 in X). Bertini's theorem (see for example [9] I 6.10) shows that if dim (X) ≥ 2, $F_e^{int}(X)$ is strictly included in H_e.

For the questions we are going to study, the property "irreducible and generically reduced" behaves better than integrity. On the other hand, to make deformation arguments work; we will need to know that these bad places are closed. This is why we will use the following variant of Bertini's theorem, which is the subject of the first part of this article.

Theorem 1.1. - Let X is a P^N sub variety of dimension ≥ 2. Then $Fe^{int}(X)$ and $F_e^{igr}(X)$ are strict enclosures of H_e.

When K is infinite, a consequence of this theorem is that there are hyper surfaces in the complement of F_e^{int} (X) (resp. F_e^{igr} (X)). We seek in this article to obtain a "family" version of this theorem. We ask ourselves more precisely the following question:

Question 1.2. - Given a family of sub varieties of dimension ≥ 2 of P^N, can we find a hyper surface whose intersections with all the varieties of this family are geometrically intact (or geometrically irreducible and generically reduced)? If the answer is positive, can we find such a hyper surface of small degree?

In fact, this amount to controlling, for each variety X of the family, the co dimension of Fe ^{int} (X) (respectively F_e ^{igr} (X)) in H_e. Specifically, the following question must be answered:

Question 1.3. - The co dimension of F_e ^{int} (X) (resp. F_e ^{igr} (X)) in H_e tend towards infinity with e?

In the second part of this article, we obtain optimal co dim ratios $_{H\ e}$ $(F_e^{igr}(X))$ depending on e and dim (X). In particular, the answer to question 1.3 for $F_e^{igr}(X)$ is positive. The statement is as follows:

Theorem 1.4. __ Let X is a subvariety of P^N of dimension n ≥ 2. Then:

$$\mathrm{codim}_{\mathcal{H}_e}(\mathcal{F}_e^{\mathrm{igr}}(X)) \geq \begin{cases} n-1 & si \quad e=1 \\ \binom{e+n-1}{e} - n & si \quad e \geq 2. \end{cases}$$

In addition, these terminals are optimal. They are reached for a cone on a curve when e = 1 and for a linear space when e ≥ 2.

In the third part, we derive analogous F_e^{int}(X). The following statement shows that the answer to question 1.3 for F_e ^{int} (X) is positive if and only if X does not have a closed point of depth 1.

Theorem 1.5. ---- Let X is a sub variety of P^N of dimension n ≥ 2. Then:

(i) If X has no point of depth 1 and co dimension > 1,

$$\mathrm{codim}_{\mathcal{H}_e}(\mathcal{F}_e^{\mathrm{int}}(X)) \geq \begin{cases} n-1 & si \quad e=1 \\ \binom{e+n-1}{e} - n & si \quad e \geq 2. \end{cases}$$

(ii) If X does not have a closed point of depth 1,

$$\mathrm{codim}_{\mathcal{H}_e}(\mathcal{F}_e^{\mathrm{int}}(X)) \geq \begin{cases} e+1 & si \quad n \geq 3 \\ e-1 & si \quad n=2 \ et\ e \geq 2 \\ 1 & si \quad n=2 \ et\ e=1. \end{cases}$$

(iii) If X has a closed point of depth 1,

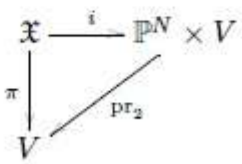
Co dim $_{He}$ (F_e ^{int} (X)) = 1 pour tout e.

In addition, these terminals are optimal.

Note that the condition "not to have closed point of depth 1 and co dimension> 1" is the condition S2 of Serre. It is particularly verified for normal varieties and varieties of Cohen-Macaulay.

Finally, in a fourth part, we obtain the following theorem answering the question 1.2. The restriction on the fibers of the family, in the case of integrity, is necessary in view of the previous theorem.

Theorem 1.6. --- Let there be a flat family of sub varieties of dimension n ≥ 2 of P^N, that is to say a commutative diagram of K-schemes of finite type:



Where π is flat with geometrically intact fibers of dimension n ≥ 2 and i is a closed immersion. So:

(i) Let e such that $\binom{e+n-1}{e} - n - 1 \geq$ dim (V). The set of hypersurface H of P^N of degree dim (V) + 2 such that for all v ∈ V, Xv ∩ H is geometrically irreducible and generically reduced co dimension 1 in X_v contains a non-empty open.

(ii) Let e ≥ dim (V) + 2. Suppose the X_v do not have closed points of depth 1. Then the set of hyper surfaces H of P^N of degree e such that for all v ∈ V, X_v ∩ H is geometrically integral of co dimension 1 in X_v contains a non-empty open.

If V is clean, these sets are non-empty open ones. Finally, when the body K is infinite, we can find such a hyper surface H defined on K.

These questions were motivated by the constructions of [1] varieties whose cotangent bundle is ample. In particular, Theorem 1.5 corrects and specifies Lemma 12 of loc. cit., wrong.

When K is finite, the methods used here to prove Theorem 1.6 do not make it possible to construct a hyper surface defined on K. The analog of this question for the smooth version of Bertini's theorem was studied and solved by Poonen in [10].

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2. Bertini's theorem

This part will prove Theorem 1.1.

2.1. Opening of the property "irreducible and generically reduced". --- For this, we will begin by determining conditions on a family of schemes under which the set of fibers that are irreducible and generically reduced is open: this is the role of Proposition 2.1. It is a very close proposition of statements of [6], and the demonstrations are modeled on those found there. Therefore, we will multiply the references to this book.

Proposal 2.1. - Let f: X → S be a morphism of schemas proper, flat and of finite presentation. It is assumed that the irreducible components of the fibers of f are all of the same size n. Then the set E of s ∈ S such that X_s is geometrically irreducible and generically reduced is open in S.

Evidence. - We split the proof in several steps.

Step 1. Under the assumptions of the statement, if S is the specter of a discrete dot-point valuation ring of s' and closed-point s, if moreover the irreducible components of X_S' are geometrically irreducible, then if s ∈ E, one also s ' ∈E.

Let's show first that X_S' is geometrically genetically reduced. By [6] 12.1.1 (vii), the set U of x ∈ X such that $X_{f\ (x)}$ is geometrically reduced in x is opened in X. Its complementary F is thus closed. By hypothesis, F_s is of dimension < n. Now, by the cleanliness of f, the dimension of the fibers of f: F →S is semi-continuous higher, hence dim (F_S') < n, and X_S' is geometrically generically reduced.

Suppose then by the absurdity that X_s' is not irreducible, and be η1 and η2 two distinct maximum points of X_S'. By cleanliness, the fiber dimension of f: $\overline{\{\eta i\}}$ → S is semi-continuous superiorly. We can therefore choose a maximum point z_i of $\overline{\{\eta i\}}_S$ whose adhesion is of dimension ≥ n. Since the irreducible components of X_S are of dimension n by hypothesis, z_i is necessarily a maximum point of X_S.

By [5] 2.3.4, which is applied by flatness, we see that the maximum points of X are exactly those of XS'. This allows us to test the hypotheses of [5] 3.4.1.1 and to show that if Z_1 and Z_2 coincide, we would have

$$\text{long}((O_{X_S})\ z_1) \geq \text{long}((O_X)_{\eta 1}) + \text{long}((O_X)_{\eta 2}) \geq 2,$$

This is impossible because X_S is generally reduced. So, $z_1 = z_2$, so X_S is not irreducible, which is absurd.

Step 2. Under the assumptions of the statement, if S is affine northerner integer of generic point s', if moreover the irreducible components of X_s' Are geometrically irreducible, and if E is not empty, then s' ∈ E.

Suppose that s ∈ E is different from s'. By [2] 7.1.7, which is applied by noetherian, one can find a scheme T, spectrum of a ring of discrete valuation of closed point and generated do t', and a morphism g: T → S such that g (t) = s and g (t') = s'. We can then apply step 1 after base change per g, which show s' ∈ E.

Step 3. Under the assumptions of the statement, if S = Spec (A) is a noetherian affine, E is stable by generation.

We give ourselves s ∈ E and s' a generation of s: we want to show that s' ∈ E. Quit to replace S by $\{\overline{\eta i}\}$, we can assume S generic point integer s'.

By [5] 4.6.8, there exists a finite extension ˜K of K = Frac (A) such that the irreducible components of $(X_S')\,\tilde{}_K$ are geometrically irreducible. Since there is a basis of ˜K on K formed of integer elements on A, the ring ˜A generated by these elements is finite on A and is of fractional bodies ˜K. We pose S~ = Spec (A˜), so that the morphism g: S˜ → S is subjective because finite and dominant.

Step 2 can then be applied after changing base by g. note ˜E the obvious set. We have ˜E = g^{-1} (E), so E = g (˜E) by subjectivity of g. Since ˜E contains the generic point of ˜ S, E contains the generic point of S, which concludes.

Step 4. Under the assumptions of the statement, E is open.

Since the proposition is local on S, we can suppose that S = Spec (A) is affine.

When S is Noetherian, we will show that E is open by applying criterion [3] 0.9.2.5. Let's check the assumptions. On the one hand, E is stable by generation in step 3. On the other hand, E is locally constructible by [6] 9.7.7 and 9.8.7.

If A is not Noetherian, we write A as the inductive limit of its finite-type sub-rings on Z. Then, by [6] 8.9.1, 8.10.5 (xii) and 11.2.6, we can find a sub- Noetherian ring A_0 of A and a morphism f_0: X_0→ Spec (A_0) satisfying the same properties as f, and such that f is obtained from f_0 by extension of the scalars. If g: Spec (A) → Spec (A_0) is the base change and if E_0 is the obvious set, we have E = g^{-1} (E_0). Since E_0 is open by the Noetherian case, E is open. Hence the result.

2.2. Demonstration of the theorem. Let us now demonstrate the version we will need of Bertini's theorem that is Theorem 1.1.

Proof of TheoreШШШШШm 1.1. - By [9] I 6.10 applied to the inclusion of X in P^N, F_e^{int} (X) is strictly included in H_e. Like $F_e^{igr}(X) \subset F_e^{int}(X)$, just show that F_e^{int} (X) and F_e^{igr}(X) are closed in H_e. We prove for F_e^{igr}(X); the closing of F_e^{int}(X) is shown in the same way using [6] 12.2.4 (viii) instead of proposal 2.1.
We introduce $Z \subset P^N \times H_e$ the family setting the parameters $Z_H = X \cap H$ and $q: Z \to H_e$ the canonical projection. We note U the open of H_e or $X \subset H$.

As the Hilbert polynomial of its fibers is constant, by applying the criterion III 9.9 of [8], we see that $q_U: q^{-1}(U) \to U$ is flat. Let U' be the subset of U constituted by H such that $X \cap H$ is geometrically irreducible and generically reduced. It is opened by Proposition 2.1. As a result, F_e^{igr}(X), which is the complement of U' in H_e, is closed.

3. Mino ration of the co dimension of F_e^{igr}(X)

This part contains the proof of Theorem 1.4. As this one is insensitive to the extension of the scalars, one will suppose that K is algebraically closed.

The idea of using a degeneracy of X to a meeting of linear spaces to show this theorem is due to Zak.

3.1. ---- Optimality of the memorizers. We show in this paragraph that the lower bounds of Theorem 1.4 are optimal.

Proposal 3.1. - Let $n \geq 2$ and $e \geq 1$. Then we can find a sub variety X of a projective space P^N for which the inequality of Theorem 1.4 is an equality.

Evidence. -- If $e \geq 2$, Lemma 3.2 below shows that we can take for X a linear subspace of dimension n of P^N.

If e = 1, we choose for X a basic cone an integral curve of degree ≥ 2 and vertex a linear subspace L of dimension n - 2 of P^N. Let G denote the set of elements of H_1 containing L. If $H \in G$, $X \cap H$ is the union of linear subspaces of P^N and cannot therefore be irreducible and generically reduced by degree. Therefore, $G \subset F_1^{igr}(X)$. On the other hand, Lemma 3.3 below, $\text{codim}_{H1}(G) = n-1$. We deduce $\text{co dim}_{H1}(F_1^{igr}(X)) \leq n - 1$ and therefore that the inequality of 1.4 is an equality.

Lemma 3.2. - Let X is a linear subspace of dimension n of P^N, and $e \geq 2$. Then:

$$\text{codim}_{\mathscr{H}_e}(\mathscr{F}_e^{\text{igr}}(X)) = \binom{e+n-1}{e} - n$$

Demonstration. - We are immediately reduced to the case where N = n and X = P^n. The closed points of $F_e{}^{igr}(P^n)$ correspond to the hyper surfaces of which an equation is not irreducible, so is a union of subsets corresponding to the degrees k and e-k of the two factors of a decomposition. This allows calculating:

$$\begin{aligned}\operatorname{codim}_{\mathscr{H}_e}(\mathscr{F}_e^{\mathrm{igr}}(\mathbb{P}^n)) &= \min_{1\leq k\leq e-1}\left(\binom{e+n}{n}-\binom{k+n}{n}-\binom{e-k+n}{n}+1\right)\\ &= \min_{k\in\{1,e-1\}}\left(\binom{e+n}{n}-\binom{k+n}{n}-\binom{e-k+n}{n}+1\right)\\ &= \binom{e+n}{n}-n-1-\binom{e-1+n}{n}+1\\ &= \binom{e+n-1}{e}-n.\end{aligned}$$

Lemma 3.3. - Let X is a subvariety of P^N of dimension n. Let G denote the set of hyper surfaces of degree e containing X. We have

$$\operatorname{codim}_{\mathscr{H}_e}(\mathscr{G}) \geq \binom{e+n}{e}.$$

Moreover, when X is a linear subspace of P^N, we have equality.

Evidence. Let L be a linear subspace of dimension N - n - 1 of P^N not meeting X. The fibers of the projection π_L: X → P^n from L is all non-empty finite. This shows that G and the set C of the vertex cones L do not do not intersect in H_e. We can then apply the theorem of the projective intersection:

$$\operatorname{codim}_{\mathscr{H}_e}(\mathscr{G}) \geq \dim(\mathscr{C})+1 = \binom{e+n}{e}.$$

Finally, when X is a linear subspace of P^N, X and L are additional in P^N, and it is easy to see that G and C are additional linear subspaces in H_e. We then

$$\operatorname{codim}_{\mathscr{H}_e}(\mathscr{G}) = \dim(\mathscr{C})+1 = \binom{e+n}{e}.$$

3.2. Reduction in the case of a hypersurface which is a cone on a plane curve. –

In this section, the proof of Theorem 1.4 is prepared by performing a number of reductions. We begin by reducing ourselves by projection in the case where X is a hyper surface

Proposal 3.4. - Theorem 1.4 is deduced from the particular case where X is a hyper surface.

Evidence. - If $X = P^N$, it is lemma 3.2 when $e \geq 2$ or lemma 3.3 when $e = 1$. If X is a hyper surface, there is nothing to prove.

If X is of co dimension ≥ 2 in P^N, one chooses, by the theorem of Bertini smooth, a linear subspace L1 of dimension N - n of P^N whose intersection with X consists of reduced points P_i. Let L_2 be a hyper plane of L_1 containing P_1 and none of P_i, I > 1, and L a hyper plane of L_2 not containing P_1. We denote π: $P^N \rightarrow P^{n+1}$ the projection since L and $Q = \pi (P_1)$. The variety $Y = \pi (X)$ with its reduced structure is a hyper surface of P^{n+1}; we consider $\pi \mid X: X \rightarrow Y$. By construction, $\pi \mid X^{-1}(Q)$ is the reduced point P_1. Consequently, since $\pi \mid X$ is proper, $\pi \mid X$ is generically finite of degree 1 that is, birational. We denote E the strict bound of Y above which $\pi \mid X$ is not an isomorphism.

Let C be the closed off $H_{e,N}$ made up of the vertex cones L. By theorem of the projective intersection,

$$\operatorname{codim}_{\mathscr{H}_{e,N}}(\mathscr{F}_e^{\mathrm{igr}}(X)) \geq \operatorname{codim}_{\mathscr{C}}(\mathscr{F}_e^{\mathrm{igr}}(X) \cap \mathscr{C}).$$

We compare by comparing the Cartier divisors that $\pi * O_{P_{n+1}}(e) = O_{PN \setminus \{L\}}(e)$, and that pulling back the global sections induces a bisection π * between $H_{e,\, n+1}$ and C. With this identification between $H_{e,\, n+1}$ and C, we will show that

$$(\mathscr{F}_e^{\mathrm{igr}}(X) \cap \mathscr{C}) \subset \mathscr{F}_e^{\mathrm{igr}}(Y) \cup \mathscr{G},$$

Where G denotes the set of hyper surfaces containing an irreducible component of E of co dimension 1 in Y. We can then conclude by applying the hypothesis to Y on the one hand, and Lemma 3.3 on the other hand.

For this, let $H \,/\!\in F_e^{\,igr}(Y) \cup G$. As $H \,/\!\in G$, $(Y \cap H) \setminus E$ is dense in $(Y \cap H)$, and $\pi \mid X$ is subjective, $(X \cap \pi {*}H) \setminus \pi \mid^{-1} X(E)$ is dense in $(X \cap \pi * H)$. As $S \,/\!\in F_e^{\,igr}(Y)$, $(Y \cap H) \setminus E$ is irreducible and generically reduced. This is also the case of $(X \cap \pi * H) \setminus \pi \mid^{-1} X (E)$ which is isomorphic to it, and of $(X \cap \pi * H)$ by density. So we have $H \,/\!\in (F_e^{igr}(X) \cap C)$.

An argument of deformation then makes it possible to carry out the following reduction:

Proposal 3.5. - Theorem 1.4 is deduced from the particular case where X is a hyper surface which is a cone on a plane curve.

Evidence. - By proposition 3.4, we can suppose that X is a hyper surface. By Bertini, one chooses coordinates in which it is of equation $G (x_0, ..., x_N) = 0$ so that the curve $G (x_0, x_1, x_2, 0, ..., 0) = 0$ is integrated.

Let X be the sub variety of $P^N \times A^1$ of equation G ($x_0, x_1, x_2, tx_3, \ldots, t_{xN}$) = 0. This is a flat family of P^N hyper surfaces, trivial of fiber X to above $A^1 \setminus \{0\}$, and such that X_0 is a cone on a plane smooth curve.

$A^1 \setminus \{0\}$, and such that X0 is a cone on a plane smooth curve. Let F be the subset of $A^1 \times H_e$ consisting of (t, H) such that $X_t \cap H$ is not irreducible and generically reduced by co dimension 1 in X_t. As in the proof in 2.2 of Theorem 1.1, we show that F is closed in $A^1 \times H_e$.

The fibers of p_1: $F \rightarrow A^1$ are the $p_1^{-1}(t) = F_e^{igr}(X_t)$. By closing F, p1 is clean, so the dimension of its fibers is semi-continuous higher. Thus, there exists t = 0 such that dim (F_e^{igr} (X_t)) ≤ dim ($F_e{}^{igr}(X_0)$).

Since X_t is protectively equivalent to X and we know how to increase the dimension of $F_e{}^{igr}$ (X_0) by hypothesis, we can conclude.

3.3. End of the demonstration. - Let's prove the theorem 1.4. We will adopt the same strategy as for the proof of proposition 3.5, now using a deformation of X in a meeting of hyper planes.

Proof of Theorem 1.4. - Step 1. Construction of a deformation.

Proposition 3.5, assumes that X is a hyper surface of equation G (x_0, x_1, x_2) = 0, which is a vertex cone the linear space S of equations $x_0 = x_1 = x_2 = 0$ on a curve plane integrates C of degree d. By Bertini, we choose our coordinates so that G (x_0, x_1, 0) = 0 consists of d reduced points. Let X be the sub variety of $P^N \times A^1$ with equation G (x_0, x_1, t_{x2}) = 0.

It is a flat P^N hyper surfaces family, trivial of X fiber over $A^1 \setminus \{0\}$, and such that X_0 is reduced meeting of d distinct hyper planes $L_1, \ldots, L_d$ intersecting along a common linear space L: $x_0 = x_1 = 0$.

We denote $Z \subset P^N \times A^1 \times H_e$ the family setting the parameters $Z_{t,H} = X_t \cap H$ and q : Z → $A^1 \times H_e$ the canonical projection. Let F be the subset of $A^1 \times H_e$ consisting of (t, H) such that $Z_{t,H}$ is not irreducible and generically reduced co dimension 1 in X_t. We denote U the open of $A^1 \times H_e$ where $X_t \cap H$ is of co dimension 1 in X_t. As the Hilbert polynomial of its fibers is constant, by applying criterion III 9.9 of [8], we see that q_U: q^{-1} (U) → U is flat. Let U' be the subset of U made up of (t, H) such that $X_t \cap H$ is irreducible and generally reduced. It is opened by Proposition 2.1. Therefore, F, which is the complement of U' in H_e, is closed. Let F' be the union of irreducible components of F dominating A^1. Since X is trivial above $A^1 \setminus \{0\}$, so are F; thus, F and F' coincide above $A^1 \setminus \{0\}$.

We will show that $F'_0 \subset G_1 \cup G_2 \cup U_i F_e^{igr}(L_i)$, or G_1 is the set of hyper surfaces containing L and G_2 the set of hyper surfaces whose intersection with L is eS. Let's first admit this inclusion. When $e \geq 2$, co dim H_e (F_e^{igr} (Li)) = (e^{e+n-1}) − n by the lemma 3.2, and co dim $_{He}(G_1 \cup G_2)$ = (e^{e+n-1}) – 1. So, co dim $_{He}$ (F'_0) ≥ (e^{e+n-1}) – n. By cleanliness of P_1: F' → A_1, the dimension of its fibers is semi-continuous superiorly, and exists t = 0 Such as co dim $_{He}$ (F'_t) ≥ (e^{e+n-1}) – n. But, $F'_t = F_t = F_e^{igr}$ (X_t), which allows to conclude because X_t is protectively equivalent to X.

When e = 1, co dim $_{H1}(F_1^{igr}(L_i))$ = n + 1 and codim $_{H1}$ (G1 ∪ G2) = n-1. We conclude by reasoning identically.

Step 2. Basic change.

It remains to prove the inclusion allowed above. We reason with the absurd by choosing H ∈ F'_0 such that H /∈ $G_1 \cup G_2 \cup_i F_e^{igr}$ (L_i). This implies that $X_0 \cap H$ is reduced and has exactly irreducible components distinct $H_i = L_i \cap H$ (there are no submerged points because $X_0 \cap H$ is a complete intersection).

On the one hand, H belongs to an irreducible component of F that dominates A^1, and on the other hand, like H / ∈ G_1, (0, H) ∈ U. We can therefore find an integral curve B, a morphism f: B → F ∩ U such that $p_1 \circ f: B \to A^1$ is dominant, and a point b ∈ B such that f (b) = (0, H). We denote $Z_B \subset X_B$ pulled back from $Z \subset X \times H_e$ by f: B → $A^1 \times H_e$. Since f has values in U, by base change, q_B: Z_B → B is clean and flat. Since $Z_b = X_0 \cap H$ is reduced, applying [6] 12.2.4 (v), and leaving constricting B, we can assume the geometrically reduced fibers of q_B. Since q is at values in F, the fibers of q_B are not geometrically intact. They are therefore necessarily geometrically reducible. After replacing B with a coating, it can then be assumed that the generic fiber of q_B is reducible, i.e., Z_B is reducible. Finally, by normalizing, we see that we can choose B smooth.
In the rest of the demonstration, we will obtain a contradiction by showing the irreducibility of Z_B.

Step 3. Z_B indissoluble.

Like **H /∈ G2**, H contain a closed point P ∈ L \ S. Thus P ∈ Z_b = X0∩H. Let Z be an irreducible component of Z_B containing P. per dimension, q_B: Z →B is dominant, therefore flat. Suppose for a moment that Z contains the irreducible components $H_1, \ldots, H_d$ from Z_b. So, if b' ∈ B,

Deg (Z'_b) = deg (Z_b) by flatness

=deg (Z_b) because $Z_b = Z_b$

= deg ($Z_{b'}$) by flatness.

Since $Z_{b'}$ and $Z_{b'}$ are of the same size, and we have an inclusion, this implies $Z_{b'} = Z_{b'}$, that is Z = Z. This contradicts the reduction of Z.

It remains to prove the assertion admitted above: we must show that $H_k \subset Z_b$ for $1 \leq k \leq d$. This is what we will obtain in the following, as a consequence of Raman jam-

Samuel's theorem. In the next step, we introduce the varieties to which we can apply this theorem.

Step 4. Normalization.

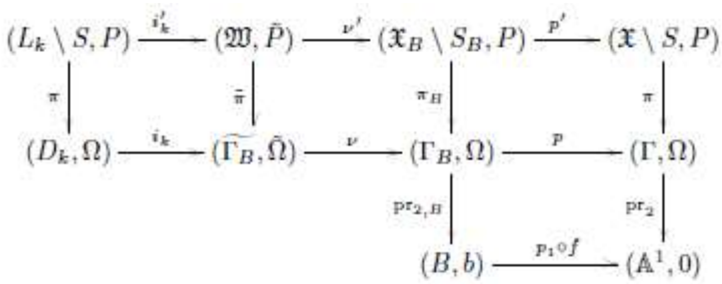
Consider the sub variety Γ of $P^2 \times A^1$ of equation G $(x_0, x_1, tx_2) = 0$. The projection from S induces π: X \ S → Γ, smooth because of fibers of affine spaces A^{N-2}. The projection pr_2: Γ → A^1 is trivial of fiber C over $A^1 \setminus \{0\}$. Above 0, it is smooth except at the point Ω = [0: 0: 1]: indeed, $pr_2^{-1}(0)$ consists of d straight lines $D_1, . . . , D_d$ intersecting in Ω. Note that $\pi^{-1}(D_k) = L_k \setminus S$.

By dragging X \ S $^{\pi}$→Γ ^{pr2}→ A^1 by $p_1 \circ f$: B → A^1, we obtain $X_B \setminus S_B$ $^{\pi B}$→ Γ_B $^{pr2, B}$→ B. We identify Ω and D_k with the corresponding sub manifolds of (Γ_B) b = Γ_0, and P and $L_k \setminus S$ with the corresponding sub manifolds of $(X_B \setminus S_B)_b = (X \setminus S)_0$.

Note ν: $\sim\Gamma_B \to \Gamma_B$ the normalization of Γ_B. Since $D_k \setminus \Omega$ is in the smooth place of Γ_B, above which ν is an isomorphism, we can consider the strict transform of D_k in $\sim\Gamma_B$: we note it again D_k and we note i_k: $D_k \to \sim\Gamma_B$ the inclusion . Let's show that Ω has a unique antecedent ~Ω in $\sim\Gamma_B$. On the one hand, the irreducible components of $(\sim\Gamma_B)_b$ are exactly the $i_k (D_k)_{1 \leq k \leq d}$, so that $\nu^{-1}(\Omega)$ consists of $i_k(\Omega)_{1 \leq k \leq d}$. On the other hand, like the generic fiber of pr_2, B $^{\circ\nu}$ is the normalization of C that is connected, [6] 15.5.9 (ii) shows that $(\sim\Gamma_B)_b$ is connected. This is only possible if $i_k(\Omega)$ does not depend on k: we note this point $\tilde{\Omega}$.

One draws back ν by π_B to obtain Ш equipped with two projections $\tilde{\pi}$ and ν' respectively on $\sim\Gamma_B$ and $X_B \setminus S_B$. Pulling back i_k by ~π, we get i'_k: $L_k \setminus S$ → Ш, which is the strict transform of $L_k \setminus S$. As Ω has a unique antecedent ν, P has a unique antecedent ν', equal to $i'_k(P)$ for everything k, we note ~P.

The Cartesian diagram of varieties pointed out below summarizes the constructions carried out:



Step5. Application of the Raman jam-Samuel theorem.

Like H ∊ G_1, Z_b does not contain L. Thus, Z is not included in the place where ν' is not an isomorphism, and we can consider its strict transform W in W. By cleanliness of ν', ν'(W) = Z, and so we have $\tilde{P}$ ∊ W.

Note that since Z_b does not contain L, W does not contain $\tilde{\pi}^{-1}(\tilde{\Omega})$. We can therefore apply the Raman jam-Samuel theorem in its form [7] 21.14.3 (i) to the normal base smooth morphism $\tilde{\pi}$: Ш → $\tilde{\Gamma}_B$ and to the divisor W of Ш in the point ~P. Thus, W is

Cartier in Ш en ˜P. In particular, as W meets $L_k \setminus S$ in the point ˜P, W meets $L_k \setminus S$ in a divisor of $L_k \setminus S$. This means that Z met L_k in a divisor of L_k, necessarily equal to H_k. That's what we wanted to show.

4. Mino ration of the co dimension of F_e^{int} (X)

The purpose of this part is to deduce Theorem 1.5 from Theorem 1.4. Since these statements are insensitive to scalar extension, we will assume that K is algebraically closed.

It is necessary to control the submerged points which appear when one intersects X with a hyper surface

Lemma 4.1. - Let X is a K-scheme of reduced finite type. Then X contains only a finite number of points of depth 1 and co dimension ≥ 2 in X.

Evidence. - The function coprof ($_x$) = dim ($O_{X,x}$) -prof ($O_{X,x}$) is semi-continuous superiorly over X by [6] 12.1.1 (v).

Let F_n be the closed {x ∈ X | coprof (x) ≥ n}. By positivity of the depth, the points of F_n are codimension ≥ n. If n ≥ 1, F_n does not contain co dimension points n. Indeed, such a point would check prof(x) = 0, and [4] ch. 0, 16.4.6 (i) shows that it would be a submerged point of X.

Thus, for n ≥ 1, the points of F_n of height n + 1 are generic points of irreducible components of F_n, and they are therefore in finite number. The points of X of depth 1 and co dimension ≥ 2 are exactly the union of all these points for n ≥ 1, and are therefore in finite number.

Lemma 4.2. - Let X ⊂ P^N be a subvariety of dimension ≥ 2. Let x_1,. . . x_r, its co dimension points ≥ 2 and depth 1, which are finitely numbered by Lemma 4.1.

So if H ∈ H_e does not contain X, the submerged points of X ∩ H are exactly the x_i belonging to H.

Evidence. - Indeed, by [4] chap. 0, 16.4.6 (i), the immersed points are exactly those of depth 0 and codimension ≥ 1. It is then enough to notice that during a nontrivial section by a hypersurface, codimension and depth both fall by 1.

Let's now show Theorem 1.5:

Proof of Theorem 1.5. - We consider the points of X of depth 1 and codimension ≥ 2. Let y_1, . . .y_2 and those who are closed, and z_1,. . . ,z_2 and the others. Let F_i be the set of hyper surfaces containing y_i and G_i the set of hyper surfaces containing z_i.

By Lemma 4.2, $F_e^{int}(X) = F_e^{igr}(X) \cup U_i F_i$. Now, in He, Fi is a linear subspace of co dimension 1, Gi is of co dimension ≥ e + 1 by Lemma 3.3, and we minore the co dimension of F_e^{igr} (X) using Theorem 1.4. We deduce the desired minorizations.

Finally, these memorizations are optimal as a consequence of the proof and the fact that the lower bounds of Theorem 1.4 are optimal. More precisely, here are varieties realizing the cases of equality:

(i) Case where X has no point of depth 1 and co dimension> 1:

(a) When e = 1, a cone on a plane curve integrates.

(b) When e ≥ 2, a linear space.

(ii) Case where X does not have a closed point of depth 1:

(a) when n ≥ 3, a variety containing a point of depth 1 of which adhesion is a straight line, but no closed point of depth 1.

(b) When n = 2 and e ≥ 2, a plane.

(c) When n = 2 and e = 1, a cone on a plane curve integrates.

(iii) Case where X has a closed point of depth 1:

(a) Any variety containing a closed point of depth 1.

5. Bertini's theorem as a family

As application of the previous results, let's prove Theorem 1.6.

Proof of Theorem 1.6. - Let's show (ii): we proceed in the same way for (i) using Theorem 1.4 instead of Theorem 1.5 (ii).

Let F be the subset of $V \times H_e$ consisting of (v, H) such that $X_v \cap H$ is not geometrically integrity of co dimension 1 in X_v. As in the proof in 2.2 of Theorem 1.1, and using [6] 12.2.4 (viii) in place of Proposition 2.1, we show that F is closed in $V \times H_e$.

The fibers of p_1: F → V are the $p_1^{-1}(v) = F_e^{int}(X_v)$ and are therefore of co dimension ≥ e-1 in H_e by Theorem 1.5 (ii). The co dimension of F in $V \times H_e$ is therefore ≥ e - 1 = dim (V) + 1. Thus, p_2: F → H_e is not dominant.

The set that interests us is complementary to the image of p_2; since it is constructible by Chivalry's theorem, it contains a non-empty open. When V is clean, F is also clean, and its image by p_2 is closed, thus complementary to an open. Finally, if the field K is infinite, every non-empty manifold has a K-point, hence the existence of the hyper surface defined on K sought.

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