Relative Defects of Differential Monomials-An Approach of Integrated Moduli of Logarithmic Derivative of Entire Functions

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Abstract

The paper aims at the comparison between relative Valiron defect and relative Nevanlinna defect of differential monomials from the view point of integrated moduli of logarithmic derivative of entire functions. A few examples are provided here to validate the conclusion of the results obtained.

AMS Subject Classification(2010): 30D35, 30D30.
Keywords and Phrases: Transcendental entire function, differential monomials, Relative Nevanlinna defect, Relative Valiron defect, integrated moduli of the logarithmic derivative.

1 Introduction, Definitions and Notations.

Let f be a meromorphic function defined in the open complex plane $\mathbb{C}$. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $n(t,a;f)$ $(\bar{n}(t,a;f))$ the number of a-points (distinct a-points) of f in $|z| \leq t$, where an ∞ -point is a pole of f.We put

$$N(r,a;f) = \int_0^r \frac{n(t,a;f) - n(0,a;f)}{t} dt + n(0,a;f) \log r$$

and

$$\bar{N}(r,a;f) = \int_0^r \frac{\bar{n}(t,a;f) - \bar{n}(0,a;f)}{t} dt + \bar{n}(0,a;f) \log r.$$

The function $N(r, a; f)$ ($\bar{N}(r, a; f)$) are called the counting function of a-points (distinct a-points) of f. In many occasions $N(r, \infty; f)$ and $\bar{N}(r, \infty; f)$ are denoted by $N(r, f)$ and $\bar{N}(r, f)$ respectively.

We also put

$$m(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| f(re^{i\theta}) \right| d\theta,$$

where

$$\begin{aligned} \log^+ x &= \log x, \text{ if } x \geq 1 \\ &= 0, \text{ if } 0 \leq x < 1. \end{aligned}$$

For $a \in \mathbb{C}$ we denote by $m(r, \frac{1}{f-a})$ the function $m(r, a; f)$ and we mean by $m(r, \infty; f)$ the function $m(r, f)$, which is called the proximity function of f.

The function $T(r, f) = m(r, f) + N(r, f)$ is called the characteristic function of f. If $a \in \mathbb{C} \cup \{\infty\}$, the quantity

$$\delta(a; f) = 1 - \limsup_{r \to \infty} \frac{N(r, a; f)}{T(r, f)} = \liminf_{r \to \infty} \frac{m(r, a; f)}{T(r, f)}$$

is called the Nevanlinna deficiency of the value 'a'. From the second fundamental theorem it follows that the set of values of $a \in \mathbb{C} \cup \{\infty\}$ for which $\delta(a; f) > 0$ is countable and $\sum_{a \neq \infty} \delta(a; f) + \delta(\infty; f) \leq 2$ $\{p.43,\ [3]\}$. If in particular $\sum_{a \neq \infty} \delta(a; f) + \delta(\infty; f) = 2$, we say that f has the maximum deficiency sum.

Similarly, the Valiron deficiency $\Delta(a; f)$ of the value 'a' is defined as

$$\Delta(a; f) = 1 - \liminf_{r \to \infty} \frac{N(r, a; f)}{T(r, f)} = \limsup_{r \to \infty} \frac{m(r, a; f)}{T(r, f)}.$$

Milloux [5] introduced the concept of absolute defect of 'a' with respect to f'. Later Xiong [9] extended this definition. He introduced the term

$$\delta_R^{(k)}(a; f) = 1 - \limsup_{r \to \infty} \frac{N(r, a; f^{(k)})}{T(r, f)}, \quad \text{for } k = 1, 2, 3, ...$$

and called it the relative Nevanlinna defect of 'a' with respect to $f^{(k)}$. Xiong [9] has shown various relations between the usual defects and the relative defects for meromorphic functions. Singh [7] introduced the term relative defect for distinct zeros and poles and established various relations between it, relative defects and the usual defects.

Let f be a transcendental meromorphic function defined in the open complex plane $\mathbb{C}$. Also let $n_0, n_1, n_2, ..., n_k$ be non-negative integers such that $\sum_{i=0}^{k} n_i \geq 1$. We call $P[f] = b_0 f^{n_0} (f^{(1)})^{n_1} ... (f^{(k)})^{n_k}$ where $T(r, b_0) = S(r, f)$, to be a differential monomial generated by f. The numbers $\gamma_P = \sum_{i=0}^{k} n_i$ and $\Gamma_P = \sum_{i=0}^{k} (i+1) n_i$ are respectively called the degree and weight of $P[f]$ $\{cf.\ [2]\}$.

We may now recall the following definition.

If f be a meromorphic function in the complex plane. Then the integrated moduli of the logarithmic derivative $I(r, f)$ is defined by

$$I(r, f) = \frac{r}{\pi} \int_0^{2\pi} \left| \frac{f'(re^{i\theta})}{f(re^{i\theta})} \right| d\theta,$$

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for $0 < r < +\infty$ $\{cf.$ [8]$\}$

In this paper we call the following four terms by using the concept of $I(r, f)$

$$\delta_I(a; f) = 1 - \limsup_{r\to\infty} \frac{N(r, a; L(f))}{I(r, f)},$$

$$\Delta_I(a; f) = 1 - \liminf_{r\to\infty} \frac{N(r, a; f)}{I(r, f)},$$

$$\delta_I^P(a; f) = 1 - \limsup_{r\to\infty} \frac{N(r, a; P[f])}{I(r, f)}$$

and

$$\Delta_I^P(a; f) = 1 - \liminf_{r\to\infty} \frac{N(r, a; P[f])}{I(r, f)}.$$

The term $S(r, f)$ denotes any quantity satisfying $S(r, f) = o\{I(r, f)\}$ as $r \to \infty$ through all values of r if f is of finite order.

The following definitions are well known.

Definition 1.1 *[11] Let $f(z)$ be a meromorphic function and $a \in \mathbb{C} \cup \{\infty\}$. We denote by $n(r, a; f \mid= 1)$ the number of simple zeros of $f - a$ in $|z| \le r$. $N(r, a; f \mid= 1)$ is defined in terms of $n(r, a; f \mid= 1)$ in the usual way. Also we put*

$$\delta_1(a; f) = 1 - \limsup_{r\to\infty} \frac{N(r, a; f \mid= 1)}{T(r, f)}.$$

Yang [10] proved that there exists at most a denumerable number of complex numbers $a \in \mathbb{C} \cup \{\infty\}$ for which

$$\delta_1(a; f) > 0 \text{ and } \sum_{a\in\mathbb{C}\cup\{\infty\}} \delta_1(a; f) \le 4.$$

Definition 1.2 *The quantity $\Theta(\infty; f)$ is defined as*

$$\Theta(\infty; f) = 1 - \limsup_{r\to\infty} \frac{\overline{N}(r, f)}{T(r, f)}.$$

Definition 1.3 *The order ρ_f and lower order λ_f of a meromorphic function f are defined as follows*

$$\rho_f = \limsup_{r\to\infty} \frac{\log T(r, f)}{\log r} \quad \text{and} \quad \lambda_f = \liminf_{r\to\infty} \frac{\log T(r, f)}{\log r}.$$

2 Lemmas.

In this section we present some lemmas which will be needed in the sequel.

Lemma 2.1 *[4] Let f be a transcendental meromorphic function of finite order or of non-zero lower order and $\sum_{a\in\mathbb{C}\cup\{\infty\}} \delta_1(a; f) = 4$. Then*

$$\lim_{r\to\infty} \frac{T(r, P[f])}{T(r, f)} = \Gamma_P - (\Gamma_P - \gamma_P)\Theta(\infty; f).$$

Lemma 2.2 *[3] Let k be any positive integer and $\psi = \sum_{i=0}^{k} a_i f^{(i)}$, where a_i are meromorphic functions, such that $T(r, a_i) = S(r, f)$, for $i = 0, 1, 2, ..., k$. Then*

$$m(r, \frac{\psi}{f}) = S(r, f).$$

Lemma 2.3 *[1] If f be a transcendental meromorphic function of finite order or of non-zero lower order with $\sum_{a\in\mathbb{C}\cup\{\infty\}} \delta_1(a; f) = 4$ then for any α,*

$$\Delta_R^P(\alpha; f) = \{1 - \Gamma_P + (\Gamma_P - \gamma_P)\Theta(\infty; f)\} + \limsup_{r\to\infty} \frac{m(r, \alpha; P[f])}{T(r, f)}$$

and

$$\delta_R^P(\alpha; f) = \{1 - \Gamma_P + (\Gamma_P - \gamma_P)\Theta(\infty; f)\} + \liminf_{r\to\infty} \frac{m(r, \alpha; P[f])}{T(r, f)}.$$

Lemma 2.4 *[8] Let f be an entire function of finite order $'\rho'$ with no zeros in $\mathbb{C}$. Then*

$$\lim_{r\to\infty} \frac{I(r, f)}{T(r, f)} = \pi\rho.$$

Lemma 2.5 *Let f be a transcendental entire function of non-zero finite order $'\rho'$ having no zeros in $\mathbb{C}$. Then*

$$\delta_I(a; f) = \left(1 - \frac{1}{\pi\rho}\right) + \liminf_{r\to\infty} \frac{m(r, a; f)}{I(r, f)}$$

and

$$\Delta_I(a; f) = \left(1 - \frac{1}{\pi\rho}\right) + \limsup_{r\to\infty} \frac{m(r, a; f)}{I(r, f)}.$$

Proof. We know that

$$\begin{aligned}
\delta_I(a; f) &= 1 - \limsup_{r\to\infty} \frac{N(r, a; f)}{I(r, f)} \\
&= 1 - \limsup_{r\to\infty} \left\{ \frac{N(r, a; f)}{T(r, f)} \cdot \frac{T(r, f)}{I(r, f)} \right\} \\
&= 1 - \limsup_{r\to\infty} \frac{N(r, a; f)}{T(r, f)} \cdot \lim_{r\to\infty} \frac{T(r, f)}{I(r, f)} \\
&= 1 - \limsup_{r\to\infty} \frac{N(r, a; f)}{T(r, f)} \cdot \frac{1}{\pi\rho} \\
&= \frac{1}{\pi\rho} \left\{ 1 - \limsup_{r\to\infty} \frac{N(r, a; f)}{T(r, f)} \right\} + \left(1 - \frac{1}{\pi\rho}\right) \\
&= \frac{1}{\pi\rho} \left\{ \liminf_{r\to\infty} \frac{m(r, a; f)}{T(r, f)} \right\} + \left(1 - \frac{1}{\pi\rho}\right) \\
&= \frac{1}{\pi\rho} \left\{ \liminf_{r\to\infty} \frac{m(r, a; f)}{I(r, f)} \cdot \frac{I(r, f)}{T(r, f)} \right\} + \left(1 - \frac{1}{\pi\rho}\right) \\
&= \frac{1}{\pi\rho} \left\{ \liminf_{r\to\infty} \frac{m(r, a; f)}{I(r, f)} \cdot \pi\rho \right\} + \left(1 - \frac{1}{\pi\rho}\right) \\
&= \left(1 - \frac{1}{\pi\rho}\right) + \liminf_{r\to\infty} \frac{m(r, a; f)}{I(r, f)}.
\end{aligned}$$

This proves the first part of Lemma 2.5.
Similarly, we can prove the second part of the lemma. ■

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Lemma 2.6 *If f be a transcendental entire function function of finite order $'\rho'$ having no zeros in $\mathbb{C}$ with $\sum_{a\in\mathbb{C}\cup\{\infty\}} \delta_1(a;f) = 4$ then for any α,*

$$\delta_I^P(\alpha;f) = \{1-\Gamma_P + (\Gamma_P-\gamma_P)\,\Theta\,(\infty,f)\} + \liminf_{r\to\infty} \frac{m(r,\alpha;P[f])}{I(r,f)}$$

and

$$\Delta_I^P(\alpha;f) = \{1-\Gamma_P + (\Gamma_P-\gamma_P)\,\Theta\,(\infty,f)\} + \limsup_{r\to\infty} \frac{m(r,\alpha;P[f])}{I(r,f)}.$$

Proof. We know that

$$\begin{aligned}
\delta_I^P(\alpha;f) &= 1-\lim_{r\to\infty}\sup \frac{N(r,a;P[f])}{I(r,f)} \\
&= 1-\lim_{r\to\infty}\sup \left(\frac{N(r,a;P[f])}{T(r,f)}\cdot\frac{T(r,f)}{I(r,f)}\right) \\
&= 1-\lim_{r\to\infty}\sup \frac{N(r,a;P[f])}{T(r,f)}\cdot\lim_{r\to\infty}\frac{T(r,f)}{I(r,f)} \\
&= 1-\lim_{r\to\infty}\sup \frac{N(r,a;P[f])}{T(r,f)}\cdot\frac{1}{\pi\rho} \\
&= \left(1-\frac{1}{\pi\rho}\right)+\frac{1}{\pi\rho}\left\{1-\limsup_{r\to\infty}\frac{N(r,\alpha;P[f])}{T(r,f)}\right\} \\
&= \left(1-\frac{1}{\pi\rho}\right)+\frac{1}{\pi\rho}\left\{1-\Gamma_P+(\Gamma_P-\gamma_P)\,\Theta\,(\infty,f)+\liminf_{r\to\infty}\frac{m(r,\alpha;P[f])}{T(r,f)}\right\} \\
&= \left(1-\frac{1}{\pi\rho}\right)+\frac{1}{\pi\rho}\left\{1-\Gamma_P+(\Gamma_P-\gamma_P)\,\Theta\,(\infty,f)+\liminf_{r\to\infty}\left(\frac{m(r,\alpha;P[f])}{I(r,f)}\cdot\frac{I(r,f)}{T(r,f)}\right)\right\} \\
&= \left(1-\frac{1}{\pi\rho}\right)+\frac{1}{\pi\rho}\left\{1-\Gamma_P+(\Gamma_P-\gamma_P)\,\Theta\,(\infty,f)+\liminf_{r\to\infty}\frac{m(r,\alpha;P[f])}{I(r,f)}\cdot\lim_{r\to\infty}\frac{I(r,f)}{T(r,f)}\right\} \\
&= \left(1-\frac{1}{\pi\rho}\right)+\frac{1}{\pi\rho}\left\{1-\Gamma_P+(\Gamma_P-\gamma_P)\,\Theta\,(\infty,f)+\liminf_{r\to\infty}\frac{m(r,\alpha;P[f])}{I(r,f)}\cdot\pi\rho\right\} \\
&= \{1-\Gamma_P+(\Gamma_P-\gamma_P)\,\Theta\,(\infty,f)\}+\liminf_{r\to\infty}\frac{m(r,\alpha;P[f])}{I(r,f)}.
\end{aligned}$$

This completes the proof of the the first part of Lemma 2.6.
Similarly, we can prove the second part of the lemma. ■

3 Theorems.

In this section we present the main results of the paper.

Theorem 3.1 *Let f be a transcendental entire function of non-zero finite order $'\rho'$ having no zeros in $\mathbb{C}$ and 'a' be any non-zero finite complex number. Then*

$$\delta_I(0;f)+\Delta_I^P(\infty;f)+\delta_I(a;f)\le(2\gamma_P-1)\Delta_I(\infty;f)+\Delta_I^P(0;f)+\left(1-\frac{1}{\pi\rho}\right)\{3-2\gamma_P\}.$$

Proof. Let us consider the following identity

$$\frac{a}{f} = 1-\frac{f-a}{P[f]}\cdot\frac{P[f]}{f}.$$

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Since $m\left(r,\frac{1}{f}\right) \leq m\left(r,\frac{a}{f}\right)+O(1)$. In view of Lemma 2.2, we get from the above identity that

$$m\left(r,\frac{1}{f}\right) \leq m\left(r,\frac{f-a}{P[f]}\right)+m\left(r,\frac{P[f]}{f^{\gamma_P}}\cdot f^{\gamma_P-1}\right)$$

$$i.e., m\left(r,\frac{1}{f}\right) \leq m\left(r,\frac{f-a}{P[f]}\right)+(\gamma_P-1)\,m(r,f)+S(r,f). \tag{1}$$

Now by using the relation $T(r,f)=T\left(r,\frac{1}{f}\right)+O(1)$ and in view of Lemma 2.2, it follows from Equation (1) that

$$m\left(r,\frac{1}{f}\right) \leq T\left(r,\frac{f-a}{P[f]}\right)-N\left(r,\frac{f-a}{P[f]}\right)+(\gamma_P-1)\,m(r,f)+S(r,f)$$

$$i.e., m\left(r,\frac{1}{f}\right) \leq T\left(r,\frac{P[f]}{f-a}\right)-N\left(r,\frac{f-a}{P[f]}\right)+(\gamma_P-1)\,m(r,f)+S(r,f)+O(1)$$

$$\begin{aligned} i.e., m\left(r,\frac{1}{f}\right) \quad \leq \quad & N\left(r,\frac{P[f]}{f-a}\right)+m\left(r,\frac{P[f]}{f-a}\right)+(\gamma_P-1)\,m(r,f-a) \\ & -N\left(r,\frac{f-a}{P[f]}\right)+(\gamma_P-1)\,m(r,f)+S(r,f)+O(1) \end{aligned}$$

$$i.e., m\left(r,\frac{1}{f}\right) \leq N\left(r,\frac{P[f]}{f-a}\right)-N\left(r,\frac{f-a}{P[f]}\right)+2(\gamma_P-1)\,m(r,f)+S(r,f)+O(1). \tag{2}$$

In view of {p.34, [3]}, it follows from Equation (2) that

$$\begin{aligned} m\left(r,\frac{1}{f}\right) \quad \leq \quad & N(r,P[f])+N\left(r,\frac{1}{f-a}\right)-N(r,f-a) \\ & -N\left(r,\frac{1}{P[f]}\right)+2(\gamma_P-1)\,m(r,f)+S(r,f)+O(1) \end{aligned}$$

$$\begin{aligned} i.e., \lim_{r\to\infty}\inf \frac{m\left(r,\frac{1}{f}\right)}{I(r,f)} \quad \leq \quad & \lim_{r\to\infty}\inf\left\{\frac{N(r,P[f])}{I(r,f)}-\frac{N(r,f)}{I(r,f)}-\frac{N\left(r,\frac{1}{P[f]}\right)}{I(r,f)}\right\} \\ & +\lim_{r\to\infty}\sup\frac{N\left(r,\frac{1}{f-a}\right)}{I(r,f)}+2(\gamma_P-1)\lim_{r\to\infty}\sup\frac{m(r,f)}{I(r,f)} \end{aligned}$$

$$\begin{aligned} i.e., \lim_{r\to\infty}\inf \frac{m\left(r,\frac{1}{f}\right)}{I(r,f)} \quad \leq \quad & \lim_{r\to\infty}\inf\frac{N(r,P[f])}{I(r,f)}-\lim_{r\to\infty}\inf\frac{N(r,f)}{I(r,f)}-\lim_{r\to\infty}\inf\frac{N\left(r,\frac{1}{P[f]}\right)}{I(r,f)} \\ & +\lim_{r\to\infty}\sup\frac{N\left(r,\frac{1}{f-a}\right)}{I(r,f)}+2(\gamma_P-1)\lim_{r\to\infty}\sup\frac{m(r,f)}{I(r,f)} \end{aligned}$$

$$\begin{aligned} i.e., \delta_I(0;f)-\left(1-\frac{1}{\pi\rho}\right) \quad \leq \quad & \left\{1-\Delta_I^p(\infty;f)\right\}-\left\{1-\Delta_I(\infty;f)\right\}-\left\{1-\Delta_I^P(0;f)\right\} \\ & +\left\{1-\delta_I(a;f)\right\}+2(\gamma_P-1)\left\{\Delta_I(\infty;f)-\left(1-\frac{1}{\pi\rho}\right)\right\} \end{aligned}$$

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$$i.e., \delta_I(0;f) + \Delta_I^P(\infty;f) + \delta_I(a;f) \leq \Delta_I(\infty;f) + \Delta_I^P(0;f)$$
$$+2\gamma_P \cdot \Delta_I(\infty;f) - 2 \cdot \Delta_I(\infty;f) - 2\left(\gamma_P - 1\right)\left(1 - \frac{1}{\pi\rho}\right) + \left(1 - \frac{1}{\pi\rho}\right).$$

$$\begin{aligned} i.e., \delta_I(0;f) + \Delta_I^P(\infty;f) + \delta_I(a;f) &\leq (2\gamma_P - 1)\Delta_I(\infty;f) + \Delta_I^P(0;f) \\ &\quad + \left(1 - \frac{1}{\pi\rho}\right)\{1 - 2\left(\gamma_P - 1\right)\} \end{aligned}$$

$$i.e., \delta_I(0;f) + \Delta_I^P(\infty;f) + \delta_I(a;f) \leq (2\gamma_P - 1)\Delta_I(\infty;f) + \Delta_I^P(0;f) + \left(1 - \frac{1}{\pi\rho}\right)\{3 - 2\gamma_P\}.$$

This proves the theorem. ■

Remark 3.1 *The sign '$\leq$' in Theorem 3.1 cannot be replaced by '$<$' only. This is evident from the following example.*

Example 1 *Let $f = \exp z$, $n_0 = 1$, $n_1 = ... = n_k = 0$ and $b_0 = 1$. Then we see that $N(r,f) = 0$ and $P[f] = f$.*

So,

$$\begin{aligned} T(r,f) &= m(r,f) = \frac{1}{2\Pi}\int_0^{2\Pi} \log^+ \left|f(re^{i\theta})\right| d\theta \\ &= \frac{1}{2\Pi}\int_0^{2\Pi} \log^+ \left|e^{re^{i\theta}}\right| = \frac{1}{2\Pi}\int_0^{2\Pi} \log^+ \left|e^{r\cos\theta} \cdot e^{ri\sin\theta}\right| d\theta \\ &= \frac{1}{2\Pi}\int_0^{2\Pi} \log^+(e^{r\cos\theta})d\theta = \frac{1}{2\Pi}\int_{-\frac{\Pi}{2}}^{\frac{\Pi}{2}} r\cos\theta d\theta = \frac{r}{\Pi}, \end{aligned}$$

$$\begin{aligned} I(r,f) &= \frac{r}{2\pi}\int_0^{2\pi} \left|\frac{f'\left(re^{i\theta}\right)}{f\left(re^{i\theta}\right)}\right| d\theta \\ &= \frac{r}{2\pi}\int_0^{2\pi} \left|\frac{e^{re^{i\theta}} \cdot re^{i\theta} \cdot i}{e^{re^{i\theta}}}\right| d\theta \\ &= \frac{r}{2\pi}\int_0^{2\pi} \left|re^{i\theta} \cdot i\right| d\theta = \frac{r}{2\pi}\int_0^{2\pi} \left|r\left(\cos\theta + i\sin\theta\right) \cdot i\right| d\theta \\ &= \frac{r}{2\pi}\int_0^{2\pi} (r)\, d\theta = \frac{r^2}{2\pi}\int_0^{2\pi} d\theta = \frac{r^2}{2\pi} \cdot 2\pi = r^2 \neq 0 \end{aligned}$$

and

$$\begin{aligned} \rho &= \limsup_{r\to\infty} \frac{\log^+ T\left(r;f\right)}{\log r} = \limsup_{r\to\infty} \frac{\log \frac{r}{\pi}}{\log r} \\ &= \limsup_{r\to\infty} \frac{\log r - \log\pi}{\log r} = \limsup_{r\to\infty}\left[1 - \frac{\log\pi}{\log r}\right] = 1. \end{aligned}$$

Thus

$$\delta_I(0;f) = \Delta_I(\infty;f) = \Delta_I^P(0;f) = \Delta_I^P(\infty;f) = 1,$$

$$\delta_I(a;f) = \left(1-\frac{1}{\pi\rho}\right)+\liminf_{r\to\infty}\frac{m(r,a;f)}{I(r,f)} = \left(1-\frac{1}{\pi}\right)+\liminf_{r\to\infty}\left(\frac{r}{\pi}\cdot\frac{1}{r^2}\right)$$
$$= \left(1-\frac{1}{\pi}\right)+\liminf_{r\to\infty}\frac{1}{\pi r} = \left(1-\frac{1}{\pi}\right)$$

and

$$\gamma_P = 1.$$

Hence,

$$\delta_I(0;f)+\Delta_R^P(\infty;f)+\delta_I(a;f) = 3-\frac{1}{\pi} = (2\gamma_P-1)\Delta_I(\infty;f)+\Delta_I^P(0;f)+\left(1-\frac{1}{\pi\rho}\right)\{3-2\gamma_P\}.$$

Theorem 3.2 *If f be a transcendental entire function of finite non-zero order $'\rho'$ such that f has no zeros in $\mathbb{C}$ with $\delta_I(\infty;f)=1$ and $\sum\limits_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;f)=4$, then*

$$\delta_I(0;f)+\frac{2-\gamma_P}{\pi\rho} \le \Delta_I^P(0;f)+\{\Gamma_P-(\Gamma_P-\gamma_P)\Theta(\infty,f)\}.$$

Proof. Since

$$f = P[f]\cdot\frac{f}{P[f]}. \tag{3}$$

In view of Lemma 2.2 and by using the relation $T(r,f) = T\left(r,\frac{1}{f}\right)+O(1)$, we obtain from Equation (3) that

$$m(r,f) \le m(r,P[f])+m\left(r,\frac{f}{P[f]}\right)$$
$$i.e., m(r,f) \le m(r,P[f])+T\left(r,\frac{f}{P[f]}\right)-N\left(r,\frac{f}{P[f]}\right)$$
$$i.e., m(r,f) \le m(r,P[f])+T\left(r,\frac{P[f]}{f}\right)-N\left(r,\frac{f}{P[f]}\right)+O(1)$$

$$i.e., m(r,f) \le m(r,P[f])+N\left(r,\frac{P[f]}{f}\right)+m\left(r,\frac{P[f]}{f}\right)$$
$$+(\gamma_P-1)m(r,f)-N\left(r,\frac{f}{P[f]}\right)+O(1). \tag{4}$$

Now in view of {p.34, [3] }, it follows from Equation (4) that

$$m(r,f) \le m(r,P[f])+N(r,P[f])+N\left(r,\frac{1}{f}\right)-N(r,f)$$
$$-N\left(r,\frac{f}{P[f]}\right)+(\gamma_P-1)m(r,f)+S(r,f)+O(1)$$

$$i.e., \liminf_{r\to\infty}\frac{m(r,f)}{I(r,f)} \le \liminf_{r\to\infty}\left\{\frac{N(r,P[f])}{I(r,f)}-\frac{N(r,f)}{I(r,f)}-\frac{N\left(r,\frac{f}{P[f]}\right)}{I(r,f)}\right\}$$
$$+\limsup_{r\to\infty}\left\{\frac{m(r,P[f])}{I(r,f)}+\frac{N\left(r,\frac{1}{f}\right)}{I(r,f)}\right\}+\limsup_{r\to\infty}\left\{(\gamma_P-1)\frac{m(r,f)}{I(r,f)}\right\}$$

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$$+ \lim_{r\to\infty} \sup \frac{m(r, P[f])}{I(r,f)} + \lim_{r\to\infty} \sup \frac{N\left(r, \frac{1}{f}\right)}{I(r,f)} + (\gamma_P - 1) \lim_{r\to\infty} \sup \frac{m(r,f)}{I(r,f)}$$

$$i.e., \lim_{r\to\infty} \inf \frac{m(r,f)}{I(r,f)} \leq \lim_{r\to\infty} \inf \frac{N(r, P[f])}{I(r,f)} - \lim_{r\to\infty} \inf \frac{N(r,f)}{I(r,f)} - \lim_{r\to\infty} \inf \frac{N\left(r, \frac{f}{P[f]}\right)}{I(r,f)}. \quad (5)$$

Since $\delta_I(\infty; f) = 1$. Then $\Delta_I(\infty; f) = 1$. In view of Lemma 2.1, Lemma 2.5, & Lemma 2.6 and by using the Equation (5) we obtain that

$$\begin{aligned} \delta_I(\infty; f) - \left(1 - \frac{1}{\pi\rho}\right) \leq & \left\{1 - \Delta_I^P(\infty; f)\right\} - \{1 - \Delta_I(\infty; f)\} - \left\{1 - \Delta_I^P(0; f)\right\} \\ & + \left[\Delta_I^P(\infty; f) - \{1 - \Gamma_P + (\Gamma_P - \gamma_P)\Theta(\infty, f)\}\right] \\ & + \{1 - \delta_I(0; f)\} + (\gamma_P - 1)\left\{\Delta_I(\infty; f) - \left(1 - \frac{1}{\pi\rho}\right)\right\} \end{aligned}$$

$$i.e., 1 - \left(1 - \frac{1}{\pi\rho}\right) \leq \Delta_I^P(0; f) + \{\Gamma_P - (\Gamma_P - \gamma_P)\Theta(\infty, f)\} - \delta_I(0; f) + (\gamma_P - 1)\left[1 - \left(1 - \frac{1}{\pi\rho}\right)\right]$$

$$i.e., \frac{1}{\pi\rho} + \delta_I(0; f) \leq \Delta_I^P(0; f) + \{\Gamma_P - (\Gamma_P - \gamma_P)\Theta(\infty, f)\} + \frac{(\gamma_P - 1)}{\pi\rho}$$

$$i..e., \frac{1}{\pi\rho}\{1 - (\gamma_P - 1)\} + \delta_I(0; f) \leq \Delta_I^P(0; f) + \{\Gamma_P - (\Gamma_P - \gamma_P)\Theta(\infty, f)\}$$

$$i.e., \delta_I(0; f) + \frac{2 - \gamma_P}{\pi\rho} \leq \Delta_I^P(0; f) + \{\Gamma_P - (\Gamma_P - \gamma_P)\Theta(\infty, f)\}.$$

This completes the proof of the theorem. ■

Remark 3.2 *If we omit the condition $\delta_I(\infty, f) = 1$ of Theorem 3.2 and the other conditions remaining the same, using the first part of Lemma 2.2 we may establish the next theorem.*

Theorem 3.3 *Let f be a transcendental entire function of non-zero finite order $'\rho'$ having no zeros in $\mathbb{C}$ with $\sum\limits_{a\in\mathbb{C}\cup\{\infty\}} \delta_1(a; f) = 4$. Then*

$$\begin{aligned} & \delta_I(\infty, f) + \delta_I(0; f) + (1 - \gamma_P)\Delta_I(\infty; f) \\ \leq \; & \Delta_I^P(0; f) + (2 - \gamma_P)\left(1 - \frac{1}{\pi\rho}\right) + \{\Gamma_P - (\Gamma_P - \gamma_P)\Theta(\infty, f)\}. \end{aligned}$$

Proof. In view of Lemma 2.4, Lemma 2.5, & Lemma 2.6 and by using Equation (5) it follows that

$$\begin{aligned} \delta_I(\infty; f) - \left(1 - \frac{1}{\pi\rho}\right) \leq & \left\{1 - \Delta_I^P(\infty; f)\right\} - \{1 - \Delta_I(\infty; f)\} - \left\{1 - \Delta_I^P(0; f)\right\} \\ & + \left[\Delta_I^P(\infty; f) - \{1 - \Gamma_P + (\Gamma_P - \gamma_P)\Theta(\infty, f)\}\right] \\ & + \{1 - \delta_I(0; f)\} + (\gamma_P - 1)\left\{\Delta_I(\infty; f) - \left(1 - \frac{1}{\pi\rho}\right)\right\} \end{aligned}$$

$$\begin{aligned} & i.e., \delta_I(\infty, f) + \delta_I(0; f) + (1 - \gamma_P)\Delta_I(\infty; f) \\ \leq \; & \Delta_I^P(0; f) + (2 - \gamma_P)\left(1 - \frac{1}{\pi\rho}\right) + \{\Gamma_P - (\Gamma_P - \gamma_P)\Theta(\infty, f)\}. \end{aligned}$$

Thus the theorem is established. ■

Theorem 3.4 *Let f be any transcendental entire function of non-zero finite order $'\rho'$ with no zeros in $\mathbb{C}$. Also let $a, b \neq 0, \infty$ be any two distinct complex numbers. Then*

$$\begin{aligned} & 2\delta_I(a;f) + 2\Delta_I^P(\infty;f) + \delta_I(b;f) \\ \leq\ & 2\Delta_I^P(0;f) + \Delta_I(\infty;f)(3\gamma_P - 1) + \left(1 - \frac{1}{\pi\rho}\right)(4 - 3\gamma_P). \end{aligned}$$

Proof. Considering the identity

$$\frac{b-a}{f-a} = \frac{P[f]}{f-a}\left\{\frac{f-a}{P[f]} - \frac{f-b}{P[f]}\right\}$$

we obtain in view of Milloux's theorem $\{p.55, [3]\}$ that

$$m\left(r, \frac{b-a}{f-a}\right) \leq m\left(r, \frac{f-a}{P[f]}\right) + m\left(r, \frac{f-b}{P[f]}\right) + m\left(r, \frac{P[f]}{(f-a)^{\gamma_P}}\right) + (\gamma_P - 1)\, m(r, f-a)$$

$$\begin{aligned} \text{i.e.,} m\left(r, \frac{b-a}{f-a}\right) \leq\ & T\left(r, \frac{f-a}{P[f]}\right) - N\left(r, \frac{f-a}{P[f]}\right) \\ & + T\left(r, \frac{f-b}{P[f]}\right) - N\left(r, \frac{f-b}{P[f]}\right) + (\gamma_P - 1)\, m(r,f) + S(r,f). \end{aligned} \quad (6)$$

Since $m\left(r, \frac{1}{f-a}\right) \leq m\left(r, \frac{b-a}{f-a}\right) + O(1)$ and $T(r,f) = T\left(r, \frac{1}{f}\right) + O(1)$, it follows from Equation (6) that

$$\begin{aligned} m\left(r, \frac{1}{f-a}\right) \leq\ & T\left(r, \frac{P[f]}{f-a}\right) - N\left(r, \frac{f-a}{P[f]}\right) + T\left(r, \frac{P[f]}{f-b}\right) \\ & - N\left(r, \frac{f-b}{P[f]}\right) + (\gamma_P - 1)\, m(r,f) + S(r,f) + O(1). \end{aligned} \quad (7)$$

$$\begin{aligned} i.e., m\left(r, \frac{1}{f-a}\right) \leq\ & N\left(r, \frac{P[f]}{f-a}\right) + m\left(r, \frac{P[f]}{(f-a)^{\gamma_P}}\right) \\ & + (\gamma_P - 1)\, m(r, f-a) - N\left(r, \frac{f-a}{P[f]}\right) \\ & + N\left(r, \frac{P[f]}{f-b}\right) + m\left(r, \frac{P[f]}{(f-b)^{\gamma_P}}\right) \\ & + (\gamma_P - 1)\, m(r, f-b) - N\left(r, \frac{f-b}{P[f]}\right) \\ & + (\gamma_P - 1)\, m(r,f) + S(r,f) + O(1) \end{aligned}$$

$$\begin{aligned} i.e., m\left(r, \frac{1}{f-a}\right) \leq\ & N\left(r, \frac{P[f]}{f-a}\right) - N\left(r, \frac{f-a}{P[f]}\right) \\ & + N\left(r, \frac{P[f]}{f-b}\right) - N\left(r, \frac{f-b}{P[f]}\right) \\ & + 3(\gamma_P - 1)\, m(r,f) + S(r,f) + O(1). \end{aligned} \quad (8)$$

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In view of $\{p.34, [3]\}$, we get from Equation (8) that

$$\begin{aligned} m\left(r, \frac{1}{f-a}\right) &\leq N(r, P[f]) + N\left(r, \frac{1}{f-a}\right) - N(r, f-a) - N\left(r, \frac{1}{P[f]}\right) \\ &+N(r, P[f]) + N\left(r, \frac{1}{f-b}\right) - N(r, f-b) - N\left(r, \frac{1}{P[f]}\right) \\ &+3(\gamma_P - 1) m(r, f) + S(r, f) + O(1) \end{aligned}$$

$$\begin{aligned} i.e., m\left(r, \frac{1}{f-a}\right) &\leq 2N(r, P[f]) - 2N(r, f) - 2N\left(r, \frac{1}{P[f]}\right) \\ &+N\left(r, \frac{1}{f-a}\right) + N\left(r, \frac{1}{f-b}\right) \\ &+3(\gamma_P - 1) m(r, f) + S(r, f) + O(1) \end{aligned}$$

$$\begin{aligned} i.e., \lim_{r\to\infty} \inf \frac{m\left(r, \frac{1}{f-a}\right)}{I(r, f)} &\leq 2 \lim_{r\to\infty} \inf \left\{ \frac{N(r, P[f])}{I(r, f)} - \frac{N(r, f)}{I(r, f)} - \frac{N\left(r, \frac{1}{P[f]}\right)}{I(r, f)} \right\} \\ &+ \lim_{r\to\infty} \sup \left\{ \frac{N\left(r, \frac{1}{f-a}\right)}{I(r, f)} + \frac{N\left(r, \frac{1}{f-b}\right)}{I(r, f)} \right\} \\ &+ \lim_{r\to\infty} \sup \left\{ [3(\gamma_P - 1)] \frac{m(r, f)}{I(r, f)} \right\} \end{aligned}$$

$$\begin{aligned} i.e., \lim_{r\to\infty} \inf \frac{m\left(r, \frac{1}{f-a}\right)}{I(r, f)} &\leq 2 \left\{ \lim_{r\to\infty} \inf \frac{N(r, P[f])}{I(r, f)} - \lim_{r\to\infty} \inf \frac{N(r, f)}{I(r, f)} - \lim_{r\to\infty} \inf \frac{N\left(r, \frac{1}{P[f]}\right)}{I(r, f)} \right\} \\ &+ \lim_{r\to\infty} \sup \frac{N\left(r, \frac{1}{f-a}\right)}{I(r, f)} + \lim_{r\to\infty} \sup \frac{N\left(r, \frac{1}{f-b}\right)}{I(r, f)} \\ &+3(\gamma_P - 1) \lim_{r\to\infty} \sup \frac{m(r, f)}{I(r, f)} \end{aligned}$$

$$\begin{aligned} i.e., \delta_I(a; f) - \left(1 - \frac{1}{\pi\rho}\right) &\leq 2\left\{1 - \Delta_I^P(\infty; f)\right\} - 2\{1 - \Delta_I(\infty; f)\} - 2\left\{1 - \Delta_I^P(0; f)\right\} \\ &+ \{1 - \delta_I(a; f)\} + \{1 - \delta_I(b; f)\} + 3(\gamma_P - 1) \left\{ \Delta_I(\infty; f) - \left(1 - \frac{1}{\pi\rho}\right) \right\} \end{aligned}$$

$$\begin{aligned} & i.e., 2\delta_I(a; f) + 2\Delta_I^P(\infty; f) + \delta_I(b; f) - \left(1 - \frac{1}{\pi\rho}\right) \\ \leq\ & 2\Delta_I^P(0; f) + 2\Delta_I(\infty; f) + 3(\gamma_P - 1)\Delta_I(\infty; f) - 3(\gamma_P - 1)\left(1 - \frac{1}{\pi\rho}\right) \end{aligned}$$

$$i.e., 2\delta_I(a; f) + 2\Delta_I^P(\infty; f) + \delta_I(b; f) \leq 2\Delta_I^P(0; f) + \Delta_I(\infty; f)\{2 + 3\gamma_P - 3\} + \left(1 - \frac{1}{\pi\rho}\right)\{1 - 3\gamma_P + 3\}$$

$$i.e., 2\delta_I(a; f) + 2\Delta_I^P(\infty; f) + \delta_I(b; f) \leq 2\Delta_I^P(0; f) + \Delta_I(\infty; f)(3\gamma_P - 1) + \left(1 - \frac{1}{\pi\rho}\right)(4 - 3\gamma_P).$$

Thus the theorem is proved. ■

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Remark 3.3 *The condition $a, b \neq 0, \infty$ in Theorem 3.4 is essential as we see in the following examples.*

Example 2 *Let $f = \exp(z^2)$, $n_0 = 1$, $n_1 = ... = n_k = 0$ and $b_0 = 1$. Also let $a = 0$ and $b = 0$. Then we get that $N(r, f) = 0$, $P[f] = \exp(z^2)$ and $\gamma_P = 1$.*

So,

$$\begin{aligned} T(r,f) &= m(r,f) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left| f\left(re^{i\theta}\right)\right| d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left| e^{r^2 e^{2i\theta}} \right| d\theta = \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left| e^{r^2(\cos 2\theta + i\sin 2\theta)} \right| d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left(e^{r^2 \cos 2\theta} \right) d\theta = \frac{1}{2\pi}\int_0^{2\pi} r^2 \cos 2\theta d\theta = \frac{r^2}{\pi}, \end{aligned}$$

$$\rho = \limsup_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log r} = \limsup_{r\to\infty} \frac{\log^{[2]} e^{r^2}}{\log r} = \limsup_{r\to\infty} \frac{2\log r}{\log r} = 2$$

and

$$\begin{aligned} I(r,f) &= \frac{r}{2\pi}\int_0^{2\pi} \left| \frac{f'\left(re^{i\theta}\right)}{f\left(re^{i\theta}\right)} \right| d\theta = \frac{r}{2\pi}\int_0^{2\pi} \frac{\left|e^{r^2 e^{2i\theta}}\right| \cdot \left|2ir^2 e^{2i\theta}\right|}{\left|e^{r^2 e^{2i\theta}}\right|} d\theta \\ &= \frac{r}{2\pi} \cdot 2r^2 \int_0^{2\pi} \frac{e^{r^2\cos 2\theta} \cdot e^{c\cos 2\theta}}{e^{r^2 \cos 2\theta}} d\theta = \frac{r^3}{\pi}\int_0^{2\pi} e^{\cos 2\theta} d\theta \\ &= \frac{r^3}{\pi} \cdot \frac{1}{2}\int_0^{4\pi} e^{\cos\eta} d\eta = \frac{r^3}{2\pi} \cdot 4\pi I_0(1) = \frac{2r^3}{\pi} \cdot I_0(1) \neq 0, \end{aligned}$$

where $I_n(z)$ is the Modified Bessel Function of the first kind such that

$$I_n(z) = \frac{1}{\pi}\int_0^{\pi} e^{z\cos\theta} \cdot \cos n\theta d\theta.$$

Thus,

$$\Delta_I^P(\infty; f) = \delta_I(0; f) = \Delta_I^P(0; f) = \Delta_I(\infty; f) = 1.$$

Hence,

$$2\delta_I(0; f) + 2\Delta_I^P(\infty; f) + \delta_I(0; f) = 2 + 2 + 1 = 5$$

and

$$2\Delta_I^P(0; f) + \Delta_I(\infty; f)(3\gamma_P - 1) + \left(1 - \frac{1}{\pi\rho}\right)(4 - 3\gamma_P) = 2 + 2 + 1 - \frac{1}{2\pi} = 5 - \frac{1}{2\pi},$$

which is contrary to Theorem 3.4.

Example 3 *Let $f = \exp z$, $n_0 = 1$, $n_1 = ... = n_k = 0$ and $b_0 = 1$. Also let $a = 0$ and $b = \infty$. Then we see that $N(r, f) = 0$, $P[f] = \exp z$ and $\gamma_P = 1$.*

So,

$$T(r,f) = \frac{r^2}{\pi},\ I(r,f) = \frac{2r^3}{\pi} \cdot I_0(1) \neq 0 \text{ and } \rho = 2.$$

Thus,

$$\Delta_I^P(\infty; f) = \delta_I(0; f) = \delta_I(\infty; f) = \Delta_I^P(0; f) = \Delta_I(\infty; f) = 1.$$

Hence,

$$2\delta_I(0;f)+2\Delta_I^P(\infty;f)+\delta_I(\infty;f)=2+2+1=5$$

and

$$2\Delta_I^P(0;f)+\Delta_I(\infty;f)(3\gamma_P-1)+\left(1-\frac{1}{\pi\rho}\right)(4-3\gamma_P)=2+2+1-\frac{1}{2\pi}=5-\frac{1}{2\pi}.$$

So, we arrive at a contradiction.

Example 4 *Let* $f=\exp\left(z^2\right)$, $n_0=1$, $n_1=...=n_k=0$ *and* $b_0=1$. *Also let* $a=\infty$ *and* $b=0$. *Then we get that* $N(r,f)=0$, $P[f]=\exp\left(z^2\right)$ *and* $\gamma_P=1$.

Also,

$$T(r,f)=\frac{r}{\pi},\ I(r,f)=r^2\neq 0 \text{ and } \rho=2.$$

Therefore,

$$\Delta_I^P(\infty;f)=\delta_I(0;f)=\delta_I(\infty;f)=\Delta_I^P(0;f)=\Delta_I(\infty;f)=1.$$

Thus,

$$2\delta_I(\infty;f)+2\Delta_I^P(\infty;f)+\delta_I(0;f)=2+2+1=5$$

and

$$2\Delta_I^P(0;f)+\Delta_I(\infty;f)(3\gamma_P-1)+\left(1-\frac{1}{\pi\rho}\right)(4-3\gamma_P)=2+2+1-\frac{1}{2\pi}=5-\frac{1}{2\pi},$$

*which is contrary to the conclusion of Theorem **3.4**.*

Theorem 3.5 *If f be any transcendental entire function of non-zero finite order $'\rho'$ having no zeros in $\mathbb{C}$ with* $\sum_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;f)=4$, *then*

$$\begin{aligned}&\delta_I(a;f)+\delta_I^P(b;f)+\delta_I^P(c;f)+2\\ \leq(\gamma_P-1)\Delta_I(\infty;f)\quad&+\left(1-\frac{1}{\pi\rho}\right)\left[(2-\gamma_P)-2\left\{\Gamma_P-(\Gamma_P-\gamma_P)\Theta(\infty;f)\right\}\right],\end{aligned}$$

where 'a' is a finite complex number and $'b'$, $'c'$ are two distinct non-zero complex numbers.

Proof. Since

$$\frac{1}{f-a}=\frac{P[f]}{f-a}\cdot\frac{1}{P[f]}.$$

In view of Lemma 2.3, we obtain from the identity that

$$m\left(r,\frac{1}{f-a}\right)\leq m\left(r,\frac{1}{P[f]}\right)+m\left(\frac{P[f]}{(f-a)^{\gamma_P}}\right)+(\gamma_P-1)\,m(r,f)+S(r,f)$$

$$i.e., m\left(r,\frac{1}{f-a}\right)\leq m\left(r,\frac{1}{P[f]}\right)+(\gamma_P-1)\,m(r,f)+S(r,f). \tag{9}$$

Applying $T(r,f)=T\left(r,\frac{1}{f}\right)+O(1)$, we get from Equation (9) that

$$m\left(r,\frac{1}{f-a}\right)\leq T(r,P[f])-N\left(r,\frac{1}{P[f]}\right)+(\gamma_P-1)\,m(r,f)+S(r,f)+O(1). \tag{10}$$

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Now by Nevanlinna's second fundamental theorem, it follows from Equation (10) that

$$\begin{aligned} i.e., m\left(r,\frac{1}{f-a}\right) &\leq \overline{N}\left(r,\frac{1}{P[f]}\right)+\overline{N}\left(r,\frac{1}{P[f]-b}\right)+\overline{N}\left(r,\frac{1}{P[f]-c}\right) \\ &\quad -N\left(r,\frac{1}{P[f]}\right)+(\gamma_P-1)\,m(r,f)+S(r,f)+O(1). \end{aligned}$$

Since

$$\overline{N}\left(r,\frac{1}{P[f]}\right)-N\left(r,\frac{1}{P[f]}\right)\leq 0.$$

In view of Lemma 2.1, Lemma 2.5 and 2.6, we obtain from Equation (10) that

$$\begin{aligned} m\left(r,\frac{1}{f-a}\right) &\leq \overline{N}\left(r,\frac{1}{P[f]-b}\right)+\overline{N}\left(r,\frac{1}{P[f]-c}\right) \\ &\quad +(\gamma_P-1)\,m(r,f)+S(r,f)+O(1) \end{aligned}$$

$$\begin{aligned} i.e., m\left(r,\frac{1}{f-a}\right) &\leq N\left(r,\frac{1}{P[f]-b}\right)+N\left(r,\frac{1}{P[f]-c}\right) \\ &\quad +(\gamma_P-1)\,m(r,f)+S(r,f)+O(1) \end{aligned}$$

$$\begin{aligned} i.e., m\left(r,\frac{1}{f-a}\right) &\leq T\left(r,\frac{1}{P[f]-b}\right)+T\left(r,\frac{1}{P[f]-c}\right) \\ &\quad -m\left(r,\frac{1}{P[f]-b}\right)-m\left(r,\frac{1}{P[f]-c}\right) \\ &\quad +(\gamma_P-1)\,m(r,f)+S(r,f)+O(1) \end{aligned}$$

$$i.e., m\left(r,\frac{1}{f-a}\right)$$

$$\leq 2T(r,P[f])-m\left(r,\frac{1}{P[f]-b}\right) -m\left(r,\frac{1}{P[f]-c}\right)+(\gamma_P-1)\,m(r,f)+S(r,f)+O(1)$$

$$\begin{aligned} & i.e., \lim\inf_{r\to\infty}\frac{m\left(r,\frac{1}{f-a}\right)}{I(r,f)} \\ \leq\ & 2\lim\inf_{r\to\infty}\frac{T(r,P[f])}{I(r,f)}-\lim\inf_{r\to\infty}\frac{m\left(r,\frac{1}{P[f]-b}\right)}{I(r,f)}-\lim\inf_{r\to\infty}\frac{m\left(r,\frac{1}{P[f]-c}\right)}{I(r,f)}+(\gamma_P-1)\lim\sup_{r\to\infty}\frac{m(r,f)}{I(r,f)} \end{aligned}$$

$$\begin{aligned} & i.e., \delta_I(a;f)-\left(1-\frac{1}{\pi\rho}\right) \\ \leq\ & 2\left\{\frac{\Gamma_P-(\Gamma_P-\gamma_P)\,\Theta(\infty;f)}{\pi\rho}\right\}-\left[\delta_I^P(b;f)-\{1-\Gamma_P+(\Gamma_P-\gamma_P)\,\Theta(\infty,f)\}\right] \\ & -\left[\delta_I^P(c;f)-\{1-\Gamma_P+(\Gamma_P-\gamma_P)\,\Theta(\infty,f)\}\right]+(\gamma_P-1)\left[\Delta_I(\infty;f)-\left(1-\frac{1}{\pi\rho}\right)\right] \end{aligned}$$

$$\begin{aligned} & i.e., \delta_I(a;f)+\delta_I^P(b;f)+\delta_I^P(c;f)-\left(1-\frac{1}{\pi\rho}\right) \\ \leq\ & \frac{2}{\pi\rho}\left[\Gamma_P-(\Gamma_P-\gamma_P)\,\Theta(\infty;f)\right]+2\{1-\Gamma_P+(\Gamma_P-\gamma_P)\,\Theta(\infty,f)\} \\ & +(\gamma_P-1)\,\Delta_I(\infty;f)-(\gamma_P-1)\left(1-\frac{1}{\pi\rho}\right) \end{aligned}$$

$$i.e., \delta_I(a;f)+\delta_I^P(b;f)+\delta_I^P(c;f) \leq \frac{2}{\pi\rho}\left[\Gamma_P-(\Gamma_P-\gamma_P)\Theta(\infty;f)\right]-2\left[\Gamma_P-(\Gamma_P-\gamma_P)\Theta(\infty,f)\right]$$
$$+(\gamma_P-1)\Delta_I(\infty;f)-(\gamma_P-1)\left(1-\frac{1}{\pi\rho}\right)+\left(1-\frac{1}{\pi\rho}\right)-2$$

$$i.e., \delta_I(a;f)+\delta_I^P(b;f)+\delta_I^P(c;f)+2 \leq (\gamma_P-1)\Delta_I(\infty;f)$$
$$+\left(1-\frac{1}{\pi\rho}\right)\left[(2-\gamma_P)-2\left\{\Gamma_P-(\Gamma_P-\gamma_P)\Theta(\infty;f)\right\}\right].$$

This proves the theorem. ■

Future Prospect : In the line of the works as carried out in the paper one may think of finding out relative deficiencies of higher index in case of meromorphic functions with respect to another one on the basis of sharing of values of them and this treatment can be done under the flavour of bicomplex analysis. As a consequence, the derivation of relevant results is still open to the future workers of this branch.

Acknowledgement

The first author sincerely acknowledges the financial support rendered by DST-FIST 2019-2020 running at the Department of Mathematics, University of Kalyani, P.O.: Kalyani, Dist: Nadia, Pin: 741235, West Bengal, India.

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