Some result's in the theory of fractional integral equations of mixed Volterra-Fridlhom types

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Abstract- The aim of this work is finding some result's in the existence, uniqueness and stability solutions of Fractional integral equations of mixed Volterra-Fridlhom types, by using Picard approximation method. Furthermore the study leads us to improve and extend the above method and the study become more general and detailed than those introduced by Butris results.

Keywords-Existence, uniqueness and stability solution, Fractional integral equations of Volterra-Fridlhom types ,Picard approximation method.

I.Introdaction

Many results about the existence , uniqueness and stability solutions of Fractional integral equations have been obtain by Picard approximation method that were proposed by many studies [2,3,5,7,9]. They are many subjects in physics and technology using mathematical methods that depends on the linear and nonlinear integral equations, and it became clear that the existence of solutions and it is algorithm structure from more important problems **in** many studies and researches [4,6,10,11,17,18] dedicates for treatment the autonomous and non-autonomous systems and specially with integral equations. Picard approximation method [11,12] owing to the great possibilities of exploiting computers are becoming versatile means of the finding and approximate construction of a solutions of integral equations. Rama [14]assumes the Picard approximation method to study the ac solutions for integral equations and it is algorithm structure and this method include uniformly sequences of a functions and the results of that study is using of the a solutions on wide range in the difference of new processes industry and technology as in the studies[1,2,3,8,13,14,15,16,17].

Butris [1]used both methods Picard approximation and Banach fixed point theorems for studying the existence and uniqueness solutions to the following Volterra integral equations:-

$$u(t) = f(t) + \int_a^t F(t,s)u(s)\,ds, \quad (t \in [0,a])$$

In this equation the functions $f(t)$ and $F(t,s)$ be continuous on (finite) interval $0 \le t \le a$ and the square region $0 \le t \le a$, $o \le s \le a$, respectively.

Our work is extend some results of Butris [**1**], by using the Picard approximation method only.

Consider the following fractional integrals equations of

(Volterra-Fridlhom) and (Fridlhom- Volterra) types:-

$$u(t) = f(t) + \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}F\big(t,s,u(s),w(s)\big)ds + \frac{1}{\Gamma(\beta)}\int_a^b (t-s)^{\beta-1}G(t,s,u(s),w(s))ds \qquad (V_\alpha + F_\beta)$$

and

$$w(t) = g(t) + \frac{1}{\Gamma(\beta)}\int_a^b (t-s)^{\beta-1}G(t,s,u(s),w(s))ds + \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}F\big(t,s,u(s),w(s)\big)ds \qquad (F_\beta + V_\alpha)$$

where

$u \in D_1 \subseteq R^n, w \in D_2 \subseteq R^n, D_1, D_2$ are close and bounded domains ,

$0 < \alpha < \beta < 1$.

The vector functions $f(t), g(t)$ and $F(t,u,w), G(t,u,w)$ are define and continuous on the domain:-

$$D = \{(t,s,u,w);\, t,s \in R^1, |t-s| \le h, u \in D_1, w \in D_2 \} \tag{1}$$

Suppose that the vector functions $F(t,u,w)$ and $G(t,u,w)$ satisfy the following inequalities

$$\left.\begin{array}{c} \|F(t,s,u,w)\| \le M_\alpha, \qquad \|G(t,s,u,w)\| \le M_\beta \\ \|F(t,s,u_1,w_1) - F(t,s,u_2,w_2)\| \le K_\alpha \, \| u_1 - u_2 \| + K_\beta \, \| w_1 - w_2 \|) \\ \|G(t,s,u_1,w_1) - G(t,s,u_2,w_2)\| \le L_\alpha \|u_1 - u_2\| + L_\beta \|w_1 - w_2\|) \end{array}\right\} \tag{2}$$

for all $t,s \in R^1$, $u, u_1, u_2 \in D_1, w, w_1, w_2 \in D_2$ where$M_\alpha, N_\alpha, K_\alpha$ and L_α are

positive constant and $\|.\| = \max_{t\in[0,h]}|.|$.

We define non-empty sets as follows:-

$$\left.\begin{array}{c} D_F = D_1 - (\rho_1 + q_1 + \| f(a) \|) \\ D_G = D_2 - (\rho_1 + q_2 + \| g(a) \|) \end{array}\right\} \tag{3}$$

where $\rho_1 = \left(\frac{h^\alpha M_\alpha}{\alpha\Gamma(\alpha)} + \frac{h^\beta M_\beta}{\beta\Gamma(\beta)}\right)$, $q_1 = \max_{t\in[0,h]}|f(t)|$ and $q_2 = \max_{t\in[0,h]}|g(t)|$

Furthermore, we assume that the largest Eigen-value of the matrix

$$\omega = \begin{pmatrix} K_\alpha \rho_1 & K_\beta \rho_1 \\ L_\alpha \rho_1 & L_\beta \rho_1 \end{pmatrix}$$

less than one ,i.e.

$$\varphi_{\max}(\omega) = \frac{1}{2}(\rho_1 (K_\alpha + K_\beta) + \sqrt{(\rho_1 (K_\alpha + K_\beta))^2 - 4\rho_1\left(K_\alpha L_\beta - K_\beta L_\alpha\right))} < 1. \tag{4}$$

Define a sequence of functions $\{u_m(t)\}_{m=0}^{\infty}$ and$\{w_m(t)\}_{m=0}^{\infty}$ by the following:-

$$u_m(t) = f(t) + \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1} F\big(t,s,u_{m-1}(s),w_{m-1}(s)\big)ds$$
$$+\frac{1}{\Gamma(\beta)}\int_a^b (t-s)^{\beta-1} H(t,s,u_{m-1}(s),w_{m-1}(s))\, ds \qquad (5)$$

with $u_0(a) = f(a)$, m =0,1,2,…,

$$w_m(t) = g(t) + \frac{1}{\Gamma(\beta)}\int_a^b (t-s)^{\beta-1} H(t,s,u_{m-1}(s),w_{m-1}(s))\, ds + \frac{1}{\Gamma(\alpha)}\int_a^b (t-s)^{\alpha-1} H\big(t,s,u_{m-1}(s),w_{m-1}(s)\big)\, ds \qquad (6)$$

with $w_0(a) = g(a), m = 0,1,2, \ldots$

Definition1 . [12]. Let f be a continuous function defined on a domain

$D = \{(t,u): a \le t \le b, c \le u \le d\}$. Then f is said to satisfy a Lipchitz condition in the variable u on D, provided that a constant $L > 0$exists with property that $|f(t,u_1) - f(t,u_2)| \le L|\, u_1 - u_2|$, for all $(t,u_1), (t,\, u_2) \in D$. the constant L is called a Lipchitz constant for f

. **Definition2**. [12].A solution $u(t)$ is said to be stable if for each $\varepsilon > 0$, there exists a $\delta > 0$ such that any solution $\check{u}(t)$ which satisfies $\| \check{u}(t_0) - u(t_0) \| < \delta$ for some t_0, also satisfies $\| \check{u}(t) - u(t) \| < \varepsilon$ for all $t \ge t_0$.

II. Existence solutions of fractional integral equations of $\left(V_\alpha + F_\beta\right)$ and $(F_\beta + V_\alpha)$.

The investigation of the existence solution of $\left(V_\alpha + F_\beta\right)$ and $(F_\beta + V_\alpha)$ will be the introduced by the following theorem:-

Theorem1. Let the vector functions $f(t), g(t)$and $F(t,u,w), G(t,u,w)$ defined and continuous on the domain (1) suppose these functions are satisfying the inequalities (2),(3) and the conditions (4), (5). Then there exist a sequence of vector functions (6) and (7) converges uniformly on the domains (4)

to the limit vector function $\begin{pmatrix} u(t) \\ w(t) \end{pmatrix}$ which is a continuous on the domain (1) and satisfies the following integral equations:-

$$\begin{pmatrix} u(t) \\ w(t) \end{pmatrix} = \begin{pmatrix} f(t) + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} F\big(t,s,u(s),w(s)\big) ds \\ + \frac{1}{\Gamma(\beta)} \int_a^b (t-s)^{\beta-1} G(t,s,u(s),w(s)) ds \\ g(t) + \frac{1}{\Gamma(\beta)} \int_a^b (t-s)^{\beta-1} G(t,s,u(s),w(s)) ds \\ + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} F\big(t,s,u(s),w(s)\big) ds \end{pmatrix} \tag{7}$$

and it is exist solution of$\left(V_\alpha + F_\beta\right)$ and $(F_\beta + V_\alpha)$.

Provide that:-

$$\begin{pmatrix} \|u_m(t) - f(a)\| \\ \|w_m(t) - g(a)\| \end{pmatrix} \le \begin{pmatrix} \rho_1 + q_1 + \| f(a) \| \\ \rho_1 + q_2 + \| g(a) \| \end{pmatrix} \tag{8}$$

and

$$\begin{pmatrix} \|u_{m+1}(t) - f(a)\| \\ \|w_{m+1}(t) - g(a)\| \end{pmatrix} \le \omega^m (E - \omega)^{-1} \emptyset_0 \tag{9}$$

for all $t \in [0,h]$ and $f(a) \in D_F$, $g(a) \in D_G$, $m = 0,1,2, \dots ....$

where

$$\emptyset_0 = \begin{pmatrix} \rho_1 + q_1 + \| f(a) \| \\ \rho_1 + q_2 + \| g(a) \| \end{pmatrix} \tag{10}$$

By mathematical indication we can prove that :-

$$\|u_m(t) - u_0\| \le \rho_1 + q_1 + \| f(a) \| \tag{11}$$

That is $u_m(t) \in D_F$, for all $t \in [0,h]$ and $f(a) \in D_F$.

Similarly, from the sequence of functions (6) and we obtain that

$$\|w_m(t) - g(a)\| \le \rho_1 + q_2 + \| g(a) \| \tag{12}$$

that is $w_m(t) \in D_G$, for all $t \in [0,h]$ and $g(a) \in D_G$.

Next we, shall prove that the sequences of functions (5) and (6) converge uniformly on the domain (8). Then by mathematics induction, we have

$$\|U_{m+1}(t) - u_m(t)\| \leq K_\alpha \rho_1 \parallel u_m(t) - u_{m-1}(t) \parallel +$$

$$K_\beta \rho_1 \parallel w_m(t) - w_{m-1}(t) \parallel \quad , \tag{13}$$

$$\|w_{m+1}(t) - w_m(t)\| \leq L_\alpha \rho_1 \parallel u_m(t) - u_{m-1}(t) \parallel +$$

$$L_\beta \rho_1 \quad \parallel w_m(t) - w_{m-1}(t) \parallel \tag{14}$$

for all m=1,2,3,…. .

Rewrite (13) and (14) in a vector form ,we get

$$\emptyset_{m+1} \leq \omega(t)\, \emptyset_m \tag{15}$$

where

$$\emptyset_{m+1} = \begin{pmatrix} \|u_{m+1}(t) - u_m(t)\| \\ \|w_{m+1}(t) - w_m(t)\| \end{pmatrix}$$

$$\emptyset_m = \begin{pmatrix} \|u_m(t) - u_{m-1}(t)\| \\ \|w_m(t) - w_{m-1}(t)\| \end{pmatrix},$$

and

$$\omega(t) = \begin{pmatrix} K_\alpha\left(\frac{(t-s)^\alpha}{\alpha\Gamma(\alpha)} + \frac{(t-s)^\beta}{\beta\Gamma(\beta)}\right) & K_\beta\left(\frac{(t-s)^\alpha}{\alpha\Gamma(\alpha)} + \frac{(t-s)^\beta}{\beta\Gamma(\beta)}\right) \\ L_\alpha\left(\frac{(t-s)^\alpha}{\alpha\Gamma(\alpha)} + \frac{(t-s)^\beta}{\beta\Gamma(\beta)}\right) & L_\beta\left(\frac{(t-s)^\alpha}{\alpha\Gamma(\alpha)} + \frac{(t-s)^\beta}{\beta\Gamma(\beta)}\right) \end{pmatrix}$$

Now we, take the maximum value for the both sides of the inequality (15),we have

$$\emptyset_{m+1} \leq \omega\, \emptyset_m \tag{16}$$

where $\omega = \max_{t\in[0,h]} \omega(t)$.

By repeating (16), we find that $\emptyset_{m+1} \leq \omega^m \emptyset_0$ and also we get

$$\sum_{i=1}^{m} \emptyset_i \le \sum_{i=1}^{\infty} \omega^{i-1} \emptyset_0 \qquad (17)$$

Using the condition (14) , thus the sequence of functions (5) and (6) are uniformly convergent, that is

$$\lim_{m\to\infty} \sum_{i=1}^{m} \omega^{m-1}\emptyset_0 = \sum_{i=1}^{\infty} \omega^{\,m-1}\emptyset_0 = (E-\omega)^{-1}\emptyset_0 \qquad (18)$$

Let

$$\lim_{m\to\infty} \begin{pmatrix} u_m(t) \\ w_m(t) \end{pmatrix} = \begin{pmatrix} u(t) \\ w(t) \end{pmatrix} \qquad (19)$$

Since the sequence of function (5) and (6) are defined and continuous in the domain (1), then the limit function $\begin{pmatrix} u(t) \\ w(t) \end{pmatrix}$ is also defined and continuous in the domain (1).

By using the conditions and inequalities of a theorem , we can prove that the inequalities (10) and (11) will be satisfied for all $t \in [0,h], f(a) \in D_F, g(a) \in D_G$, $m = 0,1,2\ldots.$.

III. Uniqueness solutions of fractional integral equations of $(\boldsymbol{V_\alpha} + \boldsymbol{F_\beta})$ and $(\boldsymbol{F_\beta} + \boldsymbol{V_\alpha})$.

The investigation of the uniqueness solutions of $(V_\alpha + F_\beta)$ and $(F_\beta + V_\alpha)$ will be introduced by:-

Thorem2. Let all assumptions and conditions of Theorem 3 be satisfied.

Then the solution $\begin{pmatrix} u(t) \\ w(t) \end{pmatrix}$ is a unique .

Proof. Let $\begin{pmatrix} \breve{u}(t) \\ \breve{w}(t) \end{pmatrix}$ be another solution of $(V_\alpha + F_\beta)$ and $(F_\beta + V_\alpha)$;that is

$$\breve{u}(t) = f(t) + \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1} F(t,s,\breve{u}(s),\breve{w}(s))ds + \frac{1}{\Gamma(\beta)}\int_a^b (t-s)^{\beta-1} G(t,s,\breve{u}(s),\breve{w}(s))ds \qquad (20)$$

and

$$\breve{w}(t) = g(t) + \frac{1}{\Gamma(\beta)}\int_a^b (t-s)^{\beta-1} G(t,s,\breve{u}(s),\breve{w}(s))ds + .$$

$$\frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1} F(t,s,\breve{u}(s),\breve{w}(s))ds \tag{21}$$

Assuming

$$\| u(t) - \breve{u}(t) \| \leq K_{\alpha}\rho_1 \ \| u(t) - \bar{u}(t) \| + K_{\beta}\rho_1 \| w(t) - \bar{w}(t) \| \tag{22}$$

and

$$\| w(t) - \breve{w}(t) \| \leq L_{\alpha}\rho_1 \ \| u(t) - \bar{u}(t) \| + L_{\beta}\rho_1 \| w(t) - \bar{w}(t) \| \tag{23}$$

Rewrite (22) and (23) in a vector form ,we find that

$$\begin{pmatrix} \|u(t) - \breve{u}(t)\| \\ \|w(t) - \breve{w}(t)\| \end{pmatrix} \leq \omega \begin{pmatrix} \| u(t) - \breve{u}(t) \| \\ \| w(t) - \breve{w}(t) \| \end{pmatrix} \tag{24}$$

By iterating the inequality (24) we have

$$\begin{pmatrix} \|u(t) - \breve{u}(t)\| \\ \|w(t) - \breve{w}(t)\| \end{pmatrix} \leq \omega^m \begin{pmatrix} \| u(t) - \breve{u}(t) \| \\ \| w(t) - \breve{w}(t) \| \end{pmatrix}$$

Then by the condition (15), we find that

$$\begin{pmatrix} \|u(t) - \breve{u}(t)\| \\ \|w(t) - \breve{w}(t)\| \end{pmatrix} \to \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Thus $\begin{pmatrix} u(t) \\ w(t) \end{pmatrix} = \begin{pmatrix} \breve{u}(t) \\ \breve{w}(t) \end{pmatrix}$.

Hence the solutions $\begin{pmatrix} u(t) \\ w(t) \end{pmatrix}$ of $(V_{\alpha} + F_{\beta})$ and $(F_{\beta} + V_{\alpha})$ is a unique on the domain (1).

V. Stability solutions of fractional integral equations of $(\boldsymbol{V_{\alpha}} + \boldsymbol{F_{\beta}})$ and $(\boldsymbol{F_{\beta}} + \boldsymbol{V_{\alpha}})$.

In this section, we can study the stability solutions of $\left(V_{\alpha}+F_{\beta}\right)$ and $(F_{\beta}+V_{\alpha})$. respectively.

Theorem3. Suppose that the functions $\mathrm{F(t,u,w)}$ and $\mathrm{f(t,u,w)}$ be continuous in the domain (1) and satisfy the inequalities (2) and (3). Then the solution (8) is stable for all $\mathrm{t}\geq 0$.

Proof. Taking

$$\| \mathrm{u(t)}-\bar{\mathrm{u}}(\mathrm{t}) \| \leq \| \mathrm{f(a)}-\bar{\mathrm{f}}(\mathrm{a}) \| + \mathrm{K}_{\alpha}\rho_1 \mathrm{u(t)}-\bar{\mathrm{u}}(\mathrm{t}) \| + \mathrm{K}_{\beta}\rho_1 \| \mathrm{w(t)}-\bar{\mathrm{w}}(\mathrm{t}) \| \tag{25}$$

and

$$\| \mathrm{w(t)}-\bar{\mathrm{w}}(\mathrm{t}) \| \leq \| \mathrm{g(a)}-\bar{\mathrm{g}}(\mathrm{a}) \| + \mathrm{L}_{\alpha}\rho_1 \mathrm{u(t)}-\bar{\mathrm{u}}(\mathrm{t}) \| + \mathrm{L}_{\beta}\rho_1 \| \mathrm{w(t)}-\bar{\mathrm{w}}(\mathrm{t}) \| \tag{26}$$

where

$$\bar{\mathrm{u}} = \bar{\mathrm{f}}(\mathrm{a}) + \frac{1}{\Gamma(\alpha)}\int_a^t (\mathrm{t}-\mathrm{s})^{\alpha-1}\, \mathrm{F}(\mathrm{t},\mathrm{s},\bar{\mathrm{u}}(\mathrm{s}),\bar{\mathrm{w}}(\mathrm{s}))\mathrm{ds}+$$
$$\frac{1}{\Gamma(\beta)}\int_a^b (\mathrm{t}-\mathrm{s})^{\beta-1}\mathrm{H}(\mathrm{t},\mathrm{s},\mathrm{u}\,(\mathrm{s}),\mathrm{w}\,(\mathrm{s}))\,\mathrm{ds}$$

and

$$\bar{\mathrm{w}} = \bar{\mathrm{g}}(\mathrm{a}) + \frac{1}{\Gamma(\beta)}\int_a^t (\mathrm{t}-\mathrm{s})^{\beta-1}\, \mathrm{F}(\mathrm{t},\mathrm{s},\bar{\mathrm{u}}(\mathrm{s}),\bar{\mathrm{w}}(\mathrm{s}))\mathrm{ds}+$$
$$\frac{1}{\Gamma(\alpha)}\int_a^b (\mathrm{t}-\mathrm{s})^{\alpha-1}\mathrm{H}(\mathrm{t},\mathrm{s},\mathrm{u}\,(\mathrm{s}),\mathrm{w}\,(\mathrm{s}))\,\mathrm{ds}$$

Rewrite (25) and (26) in a vector form that is

$$\begin{pmatrix} \|\mathrm{u(t)}-\bar{\mathrm{u}}(\mathrm{t})\| \\ \|\mathrm{w(t)}-\bar{\mathrm{w}}(\mathrm{t})\| \end{pmatrix} \leq \begin{pmatrix} \| \mathrm{f(a)}-\bar{\mathrm{f}}(\mathrm{a}) \| \\ \| \mathrm{g(a)}-\bar{\mathrm{g}}(\mathrm{a}) \| \end{pmatrix} + \omega \begin{pmatrix} \| \mathrm{u(t)}-\bar{\mathrm{u}}(\mathrm{t}) \| \\ \| \mathrm{w(t)}-\bar{\mathrm{w}}(\mathrm{t}) \| \end{pmatrix}$$

for $\left\|\mathrm{f(a)}-\overline{\bar{\mathrm{f}}(\mathrm{a})}\right\| \leq \delta_1, \| \mathrm{g(a)}-\bar{\mathrm{g}}(\mathrm{a}) \| \leq \delta_2$ then

$$\begin{pmatrix} \|u(t)-\bar{u}(t)\| \\ \|w(t)-\overline{w}(t)\| \end{pmatrix} \le \begin{pmatrix} \delta_1 \\ \delta_2 \end{pmatrix} + \omega \begin{pmatrix} \| u(t)-\bar{u}(t) \| \\ \| w(t)-\overline{w}(t) \| \end{pmatrix}$$

By using the condition (5) we have

$$\begin{pmatrix} \|u(t)-\bar{u}(t)\| \\ \|w(t)-\overline{w}(t)\| \end{pmatrix} \le \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \end{pmatrix}, \epsilon_1 , \epsilon_2 \ \ge 0.$$

Also by using the definition of the stability, we find that $\begin{pmatrix} \bar{u}(t) \\ \overline{w}(t) \end{pmatrix}$ is a stable solution for $t \ge 0$ of $(V_\alpha + F_\beta)$ and $(F_\beta + V_\alpha)$.

Similar results can be obtained for other class of fractional integral equations of Volterra-Fridlhom types. In particular, the fractional integral equations which has the form:-

$$u(t) = f(t) + \frac{1}{\Gamma(\alpha)} \int_{\varepsilon(a)}^{\varepsilon(t)} (t-s)^{\alpha-1} F\big(t,s,u(s),w(s)\big) ds$$
$$+ \frac{1}{\Gamma(\beta)} \int_{\varepsilon(a)}^{\varepsilon(b)} (t-s)^{\beta-1} G(t,s,u(s)\,,w(s)) ds \qquad (27)$$

and

$$w(t) = g(t) + \frac{1}{\Gamma(\beta)} \int_{\varepsilon(a)}^{\varepsilon(b)} (t-s)^{\beta-1} G(t,s,u(s)\,,w(s)) d$$
$$+ \frac{1}{\Gamma(\alpha)} \int_{\varepsilon(a)}^{\varepsilon(t)} (t-s)^{\alpha-1} F\big(t,s,u(s),w(s)\big) ds \qquad (28)$$

where

$u \in D_1 \subseteq R^n, w \in D_2 \subseteq R^n, D_1, D_2$ are close and bounded domains ,

$0 < \alpha \ < \beta < 1$.

The vector functions $f(t), g(t)$ and $F(t,u,w), G(t,u,w)$ are define and continuous on the domain:-

$$D^* = \left\{ (t,s,u,w); t,s \in R^1, \|\varepsilon(t)-\varepsilon(s)\| \le h_1, \|\varepsilon(b)-\varepsilon(a)\| \le h_2 , u \in D_1, w \in D_2 \right\} \quad (29)$$

Suppose that the vector functions $F(t,u,w)$ and $G(t,u,w)$ satisfy the following inequalities(2).

We define non-empty sets as follows:-

$$\left. \begin{array}{l} D_F^* = D_1 - (\rho_2 + q_1 + \| f(a) \|) \\ D_G^* = D_2 - (\rho_2 + q_2 + \| g(a) \|) \end{array} \right\} \quad (30)$$

where $\rho_{2=}(\frac{h_1^\alpha}{\alpha\Gamma(\alpha)} + \frac{h_2^\beta}{\beta\Gamma(\beta)})$

Furthermore, we assume that the largest eigen-value of the matrix

$$\omega^* = \begin{pmatrix} K_\alpha \rho_2 & K_\beta \rho_2 \\ L_\alpha \rho_2 & L_\beta \rho_2 \end{pmatrix}$$

less than one ,i.e.

$$\varphi_{\max}(\omega^*) = \frac{1}{2}(\rho_2 (K_\alpha + K_\beta) + \sqrt{(\rho_2(K_\alpha + K_\beta))^2 - 4\rho_2 (K_\alpha L_\beta - K_\beta L_\alpha)}) < 1 \ . \quad (31)$$

We can state and prove a similar theorem of a theorem1 by using the condition (31).

VII. Conclusion.

In this work, we study the solution for nonlinear system of Fractional integral equations of mixed Volterra-Fridlhom types. We prove some theorems in the existence, uniqueness and stability in closed and bounded domains, by using Picard approximation method. Furthermore the study leads us to improve and extend the above method and the study become more general and detailed than those introduced by Butris result.

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