Applications and Properties of Sanna Distribution

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Abstract

We developed new mixture model known as Sanna distribution which is obtained as proper mixture of one parameter Lindley distribution (θ) and Quasi Akash distribution $(2, \theta)$, where 2 is shape parameter and θ is scale parameter. We have computed expressions of some necessary statistical characteristics like moments, characteristic function, order statistics of Sanna distribution. The expressions for reliability measures of our proposed model are evolved and model parameters have been estimated by maximum likelihood estimation method. We have computed Likelihood ratio statistic for testing the significance of mixing parameter *p*. Finally, proposed model and its related models have been fitted to two real data sets for examining the significance of newly introduced model.

Keywords: Quasi Akash Distribution, Lindley Distribution, Mixture Models, Sanna Distribution, Mixing Parameter, Structural Properties, Maximum Likelihood Estimation, Applications.

Mathematics Subject Classification: 60E05, 62F10

1. Introduction

Data analysis is very important for decision making. The data generated from many important fields of real life is of random nature. Researchers over the years have fitted various probability models to the random data. There are many instances where data possesses characteristics of two or more probability distributions. Whenever super population (mixture model *F*) is a genuine mixture of *v* distinct populations $(F_{1,} F_2 ... F_v)$ we have to apply mixture models to such data. On many occasions real life data may come from v (more than 1) different populations with diverse proportions and main aim of data analyst is to estimate the mixing proportions $(p_1, p_2, ... p_v)$ in which in which they occur in the super population. Mixture models find greater applicability in diverse fields like cluster analysis, health, etc. Many researchers have developed different mixture models over the years. Everitt (1996) gave an introduction to finite mixture distributions. Shukla (2018) formulated Prakaamy distribution and studied its vital properties and applications. Shanker (2015) introduced Akash distribution with properties and applications in real life. Shanker & Shukla (2017) developed a Quasi Shanker distribution and obtained its applications. Rama Shankar (2016) introduced quasi Akash distribution and studied its applications and properties. Para & Jan (2018) developed three parameter weighted Pareto type II Distribution and applied it in medical sciences. Shanker, Fesshaye & Sharma (2016) proposed two parameter Lindley distribution with applications to lifetime data

A continuous random variable Z is said to have a mixture distribution if its p.d.f $f(z)$ is obtained as a mixture of v distinct populations having density functions $f_1(z), f_2(z),..., f_v(z)$ and with mixing proportions $p_1, p_2,...p_v$ respectively. Mathematically

$$f(z) = p_1 f_1(z) + p_2 f_2(z) + ... + p_v f_v(z)$$

Where $0 \le p_i \le 1$ and

$$\sum_{i=1}^{v} p_i = 1$$

In this paper we have obtained Sanna distribution as a mixture of Lindley distribution and Quasi Akash distribution.

2. Sanna Distribution:

A non-negative random variable Z is said to follow a Sanna Distribution (SD) if its probability density function $f(z)$ can be obtained as a mixture of one parameter Lindley (θ) distribution with p.d.f $f_1(z)$ and Quasi Akash $(2,\theta)$ distribution with shape parameter $\alpha = 2$ fixed and scale parameter θ with p.d.f $f_2(z)$. Mathematically

$$f(z) = pf_1(z) + (1-p) f_2(z) \tag{2.1}$$

Where p is a mixing parameter and

$$f_1(z) = \frac{\theta^2}{(\theta+1)}(1+z)e^{-\theta z} \qquad z > 0, \theta > 0 \tag{2.2}$$

is the p.d.f of one parameter Lindley (θ) distribution with the corresponding c.d.f $F_1(z)$ given below

$$F_1(z) = 1 - \left[\frac{\theta+1+\theta z}{(\theta+1)}\right]e^{-\theta z} \qquad z > 0, \theta > 0 \tag{2.3}$$

And

$$f_2(z) = \frac{\theta^2}{2(\theta+1)}\left(2+\theta z^2\right)e^{-\theta z} \qquad z > 0, \theta > 0 \tag{2.4}$$

is the p.d.f of Quasi Akash Distribution.

The cumulative distribution function of Quasi Akash Distribution is given by

$$F_2(z) = 1 - \left[1 + \frac{\theta z(\theta z + 2)}{2(\theta+1)}\right]e^{-\theta z} \qquad z > 0, \theta > 0 \tag{2.5}$$

Putting the values of $f_1(z)$ and $f_2(z)$ in (2.1) we obtain the p.d.f of Sanna distribution $f(z)$ as

$$f(z) = \frac{\theta^2 e^{-\theta z}}{2(\theta+1)}\left[2p(1+z)+(1-p)(2+\theta z^2)\right] \qquad z>0, \theta>0, 0\le p\le 1 \tag{2.6}$$

The probability density function graphs of Sanna distribution are given below

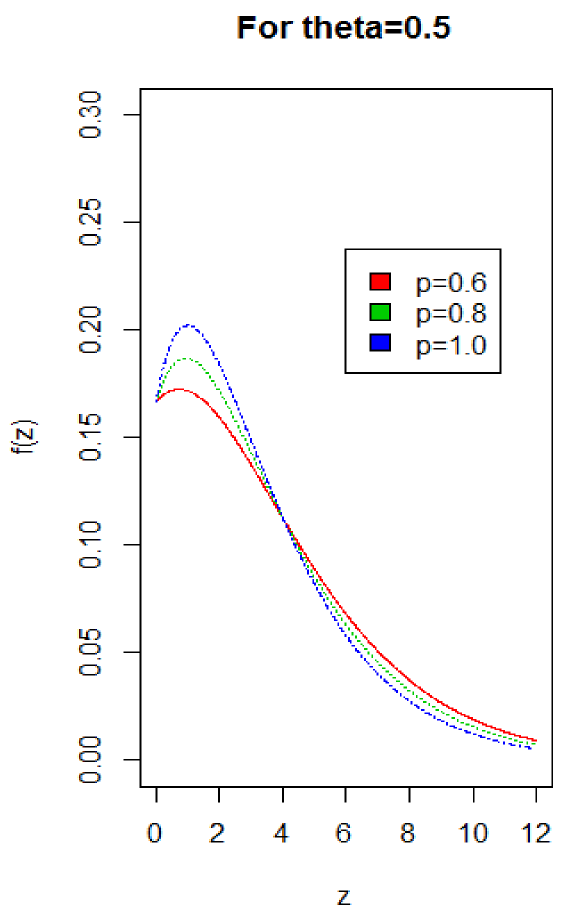

Figure 1 (a) Graph of density function

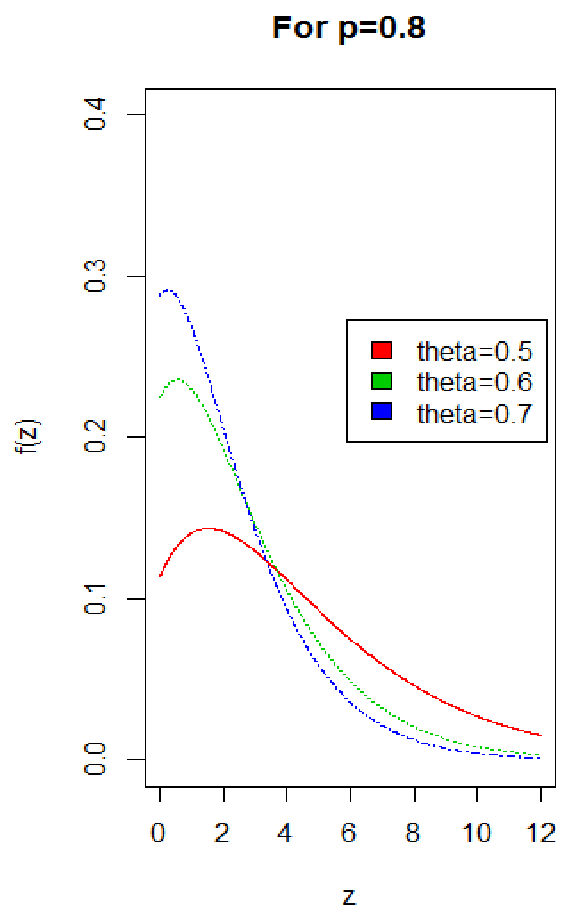

Figure 1 (b) Graph of density function

The corresponding c.d.f of Sanna distribution is obtained by using (2.3) and (2.5) as

$$F(z) = \left[p\left(1-\left[\frac{\theta+1+\theta z}{(\theta+1)}\right]e^{-\theta z}\right)+(1-p)\left[1-\left[1+\frac{\theta z(\theta z+2)}{2(\theta+1)}\right]e^{-\theta z}\right]\right] \tag{2.7}$$

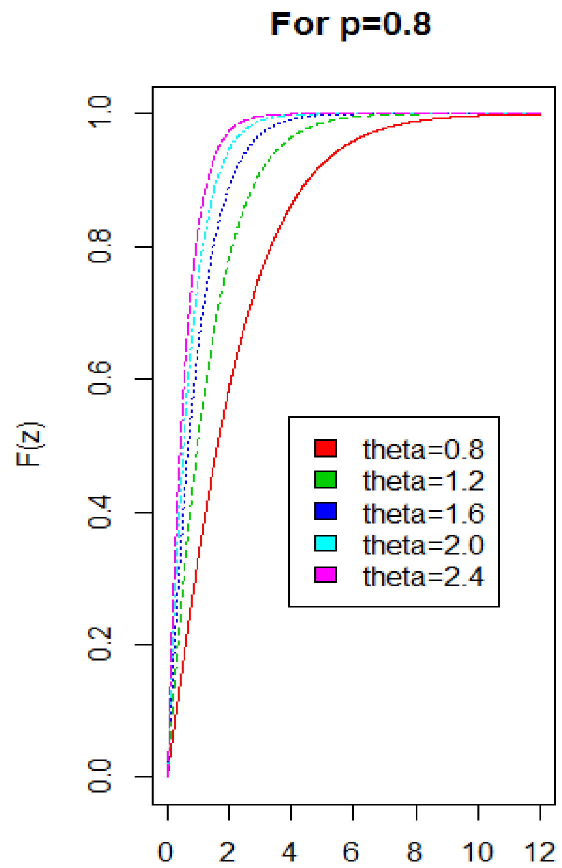

Figure 2(a) Graph of cumulative distribution function

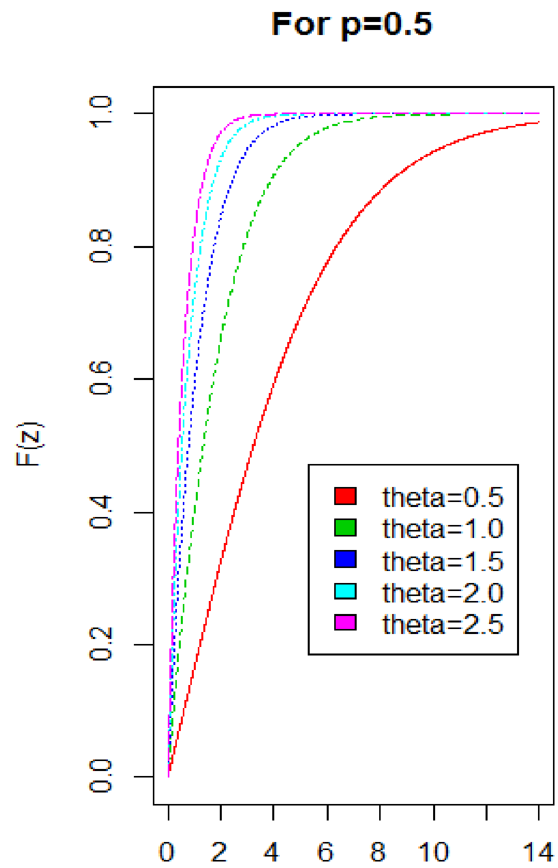

Figure 2(b) Graph of cumulative distribution function

The above graph represents the cumulative distribution function of Sanna distribution for different parameter values.

3. Reliability Analysis

This particular section of paper introduces survival function, hazard rate, reverse hazard rate of the proposed Sanna distribution for random variable Z , where Z denotes the lifetime of a system.

3.1 Reliability function R(z)

The reliability function or survival function $R(z,\theta,p)$ gives the numerical value of a probability of surviving of a system or living beings beyond a specified time (t) .

Mathematically

$R(z,\theta,p)=P(Z>t)=1-F(z)$

The reliability function or the survival function of Sanna distribution is calculated as:

$$R(z,\theta,p)=1-\left[\begin{array}{l} p\left(1-\left[\dfrac{\theta+1+\theta z}{(\theta+1)}\right]e^{-\theta z}\right) \\ +(1-p)\left[1-\left[1+\dfrac{\theta z(\theta z+2)}{2(\theta+1)}\right]e^{-\theta z}\right] \end{array}\right]$$

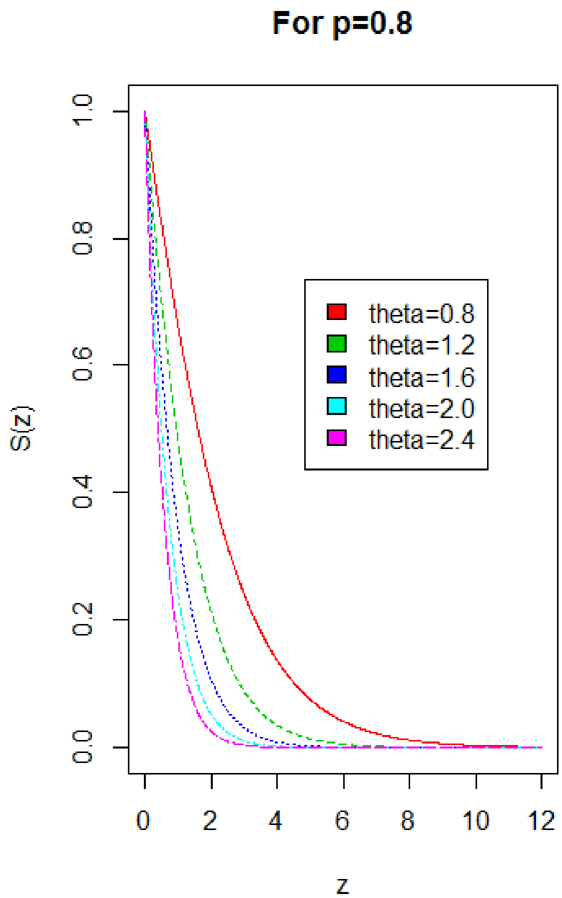

Figure 3(a) Graph of Survival function

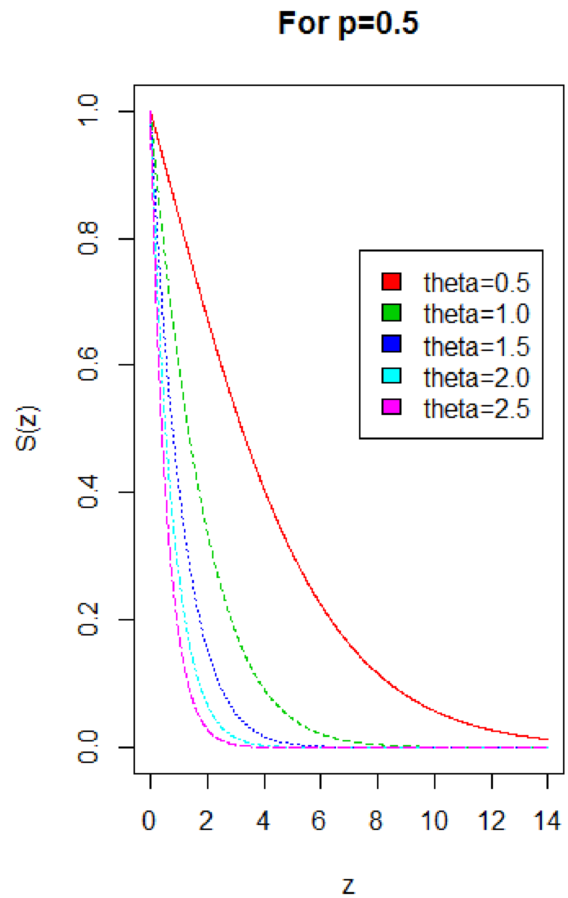

Figure 3(b) Graph of Survival function

The above graph represents the survival function of Sanna distribution for different parameter values.

3.2 Hazard Function:

The hazard function which is defined as chance that a system which is surviving up to time *"t"* will fail in the small time interval after *"t"* is obtained as:

$$H.R = h(z;\theta,p) = \frac{f(z,\theta,p)}{R(z,\theta,p)}$$

$$= \frac{\frac{\theta^2 e^{-\theta z}}{2(\theta+1)}\left[2p(1+z)+(1-p)(2+\theta z^2)\right]}{1-\left[\begin{array}{l} p\left(1-\left[\frac{\theta+1+\theta z}{(\theta+1)}\right]e^{-\theta z}\right) \\ +(1-p)\left[1-\left[1+\frac{\theta z(\theta z+2)}{2(\theta+1)}\right]e^{-\theta z}\right]\end{array}\right]}$$

3.3 Reverse Hazard Rate

The reverse hazard rate of the Sanna distribution is given as:

$$R.H.R = h(z,\theta,p) = \frac{f(z,\theta,p)}{F(z,\theta,p)}$$

$$= \frac{\frac{\theta^2 e^{-\theta z}}{2(\theta+1)}\left[2p(1+z)+(1-p)(2+\theta z^2)\right]}{\left[\begin{array}{l} p\left(1-\left[\frac{\theta+1+\theta z}{(\theta+1)}\right]e^{-\theta z}\right) \\ +(1-p)\left[1-\left[1+\frac{\theta z(\theta z+2)}{2(\theta+1)}\right]e^{-\theta z}\right]\end{array}\right]}$$

4. Statistical properties:

Moments, characteristic function, generating function, mean deviation characterizes probability models. Here we have obtained these statistical properties for our proposed Sanna distribution.

4.1 Moments

Assuming Z to be a random variable having Sanna distribution with parameters θ and p .Then the r^{th} moment about origin for a given probability distribution is given by

$$\mu_r{}' = E\left(Z^r\right) = \int_0^\infty z^r f(z)dz \qquad r=1,2,3\ldots$$

$$= \int_0^\infty z^r \left[\frac{\theta^2 e^{-\theta z}}{2(\theta+1)}\left[2p(1+z)+(1-p)(2+\theta z^2)\right]\right]dz$$

$$\mu_r{}' = \left[\frac{pr!(\theta+1+r)}{\theta^r(\theta+1)} + (1-p)\left(\frac{r!\left(2\theta+(r+1)(r+2)\right)}{2\theta^r(\theta+1)}\right)\right] \qquad (4.1.1)$$

Put r=1 in equation (4.1.1) we get

$$\mu_1{}' = \left[\frac{p(\theta+2)}{\theta(\theta+1)} + \frac{(1-p)(\theta+3)}{\theta(\theta+1)}\right]$$

Which is mean of the Sanna distribution

Put r=2 in equation (4.1.1) we get

$$\mu_2{}' = \left[\frac{2p(\theta+3)}{\theta^2(\theta+1)} + \frac{2(1-p)(\theta+6)}{\theta^2(\theta+1)}\right]$$

Put r=3 in equation (4.1.1) we get

$$\mu_3{}' = \left[\frac{6p(\theta+4)}{\theta^3(\theta+1)} + \frac{6(1-p)(\theta+10)}{\theta^3(\theta+1)}\right]$$

Put r=4 in equation (4.1.1) we get

$$\mu_4{}' = \left[\frac{24p(\theta+5)}{\theta^4(\theta+1)} + \frac{24(1-p)(\theta+15)}{\theta^4(\theta+1)}\right]$$

The moments about mean are given as

$$\mu_2 = \left[\frac{p(\theta^2+4\theta+2)}{\theta^2(\theta+1)^2} + \frac{(1-p)(\theta^2+8\theta+3)}{\theta^2(\theta+1)^2}\right]$$

Which is the variance of Sanna distribution.

$$\mu_3 = \left[\frac{2p\left(\theta^3+6\theta^2+6\theta+2\right)}{\theta^3(\theta+1)^3} + \frac{2(1-p)\left(\theta^3+15\theta^2+9\theta+3\right)}{\theta^3(\theta+1)^3}\right]$$

$$\mu_4 = \left[\frac{3p\left(3\theta^4+24\theta^3+44\theta^2+32\theta+8\right)}{\theta^4(\theta+1)^4} + \frac{3(1-p)\left(12\theta^4+1024\theta^3+1632\theta^2+1152\theta+240\right)}{\theta^4(\theta+1)^4}\right]$$

4.2 Coefficient of variation, skewness, kurtosis and Index of Dispersion of Sanna Distribution.

The coefficient of variation (C.V), coefficient of skewness ($\sqrt{\beta_1}$), coefficient of kurtosis (β_2) and index of dispersion (γ) of the SD are determined as

$$C.V = \frac{(\mu_2)^{\frac{1}{2}}}{\mu_1{}'} = \frac{\left\{p(\theta^4+4\theta+2)+(1-p)(\theta^2+8\theta+3)\right\}^{\frac{1}{2}}}{\left[p(\theta+2)+(1-p)(\theta+3)\right]}$$

$$\sqrt{\beta_1} = \frac{\mu_3}{\mu_2{}^{3/2}} = \left\{\frac{\left[2p\left(\theta^3+6\theta^2+6\theta+2\right)+2(1-p)\left(\theta^3+15\theta^2+9\theta+3\right)\right]}{\left\{p(\theta^4+4\theta+2)+(1-p)(\theta^2+8\theta+3)\right\}^{3/2}}\right\}$$

$$\beta_2 = \frac{\mu_4}{\mu_2{}^2}$$

$$= \left\{\frac{\left[3p\left(3\theta^4+24\theta^3+44\theta^2+32\theta+8\right)+3(1-p)\left(12\theta^4+1024\theta^3+1632\theta^2+1152\theta+240\right)\right]}{\left\{p(\theta^4+4\theta+2)+(1-p)(\theta^2+8\theta+3)\right\}^2}\right\}$$

$$\gamma = \frac{\mu_2}{\mu_1'} = \frac{\left\{p(\theta^4 + 4\theta + 2) + (1-p)(\theta^2 + 8\theta + 3)\right\}}{\left\{\theta(\theta+1)\left[p(\theta+2) + (1-p)(\theta+3)\right]\right\}}$$

4.3 Mean deviation about mean and median of Sanna distribution (SD)

We have derived the expressions for mean deviation about mean and median of SD in this section.

Theorem 1.1: If Z has the SD (θ, p), then the mean deviation about mean $(\delta_1(Z))$ and mean deviation about median $(\delta_2(Z))$ are given as:

$$\delta_1(Z) = \left[\begin{array}{l} 2\mu\left[\begin{array}{l} p\left(1 - \left[\dfrac{\theta+1+\theta\mu}{(\theta+1)}\right]e^{-\theta\mu}\right) \\ +(1-p)\left[1 - \left[1 + \dfrac{\theta\mu(\theta\mu+2)}{2(\theta+1)}\right]e^{-\theta\mu}\right]\end{array}\right] \\ -2\left[\dfrac{1}{2\theta(\theta+1)}\left\{\begin{array}{l} 2p\left(\theta(1-(\mu\theta+1)e^{-\theta\mu}) + (2 - e^{-\theta\mu}(\mu^2\theta^2 + 2\mu\theta + 2))\right) + \\ (1-p)\left(\begin{array}{l} 2\theta(1-(\mu\theta+1)e^{-\mu\theta}) + (6 - e^{-\theta\mu}(\mu^3\theta^3 + 3\mu^2\theta^2 \\ +6\mu\theta+6)) \end{array}\right)\end{array}\right\}\right]\end{array}\right]$$

And

$$\delta_2(Z) = \left[\mu - 2\left[\dfrac{1}{2\theta(\theta+1)}\left\{\begin{array}{l} 2p\left(\theta(1-(M\theta+1)e^{-\theta M}) + (2 - e^{-\theta M}(M^2\theta^2 + 2M\theta + 2))\right) + \\ (1-p)\left(\begin{array}{l} 2\theta(1-(M\theta+1)e^{-M\theta}) + (6 - e^{-\theta M}(M^3\theta^3 + 3M^2\theta^2 \\ +6M\theta+6)) \end{array}\right)\end{array}\right\}\right]\right]$$

respectively.

Proof: Mean deviation about mean and mean deviation about median are defined as

$$\delta_1(Z) = \int_0^\infty |z-\mu| f(z)dz$$

And $\delta_2(Z) = \int_0^\infty |z-M| f(z)dz$

respectively.

Where μ and *M* are mean and median respectively of random variable *Z*. The measures $\delta_1(Z)$ and $\delta_2(Z)$ can be calculated by using the simplified relationships.

$$\delta_1(Z) = \int_0^\mu (\mu-z) f(z)dz + \int_\mu^\infty (z-\mu) f(z)dz$$

$$\delta_1(Z) = 2\mu F(\mu) - 2\int_0^\mu zf(z)dz \qquad (4.3.1)$$

And

$$\delta_2(Z) = \int_0^M (M-z) f(z)dz + \int_M^\infty (z-M) f(z)dz$$

$$\delta_2(Z) = \mu - 2\int_0^M zf(z)dz \qquad (4.3.2)$$

Where $f(z) = \frac{\theta^2 e^{-\theta z}}{2(\theta+1)}\left[2p(1+z)+(1-p)(2+\theta z^2)\right]$

Now

$$\int_0^\mu zf(z)dz = \left[\frac{1}{2\theta(\theta+1)}\left\{\begin{matrix} 2p\left(\theta(1-(\mu\theta+1)e^{-\theta\mu}) + (2-e^{-\theta\mu}(\mu^2\theta^2+2\mu\theta+2))\right)+ \\ (1-p)\left(\begin{matrix} 2\theta(1-(\mu\theta+1)e^{-\mu\theta}) + (6-e^{-\theta\mu}(\mu^3\theta^3+3\mu^2\theta^2 \\ +6\mu\theta+6)) \end{matrix}\right) \end{matrix}\right\}\right] \qquad (4.3.3)$$

And

$$\int_0^M zf(z)dz = \left[\frac{1}{2\theta(\theta+1)}\left\{\begin{array}{l} 2p\left(\theta(1-(M\theta+1)e^{-\theta M}) + (2-e^{-\theta M}(M^2\theta^2+2M\theta+2))\right)+ \\ (1-p)\left(\begin{array}{l}2\theta(1-(M\theta+1)e^{-M\theta})+(6-e^{-\theta M}(M^3\theta^3+3M^2\theta^2 \\ +6M\theta+6))\end{array}\right)\end{array}\right\}\right] \qquad (4.3.4)$$

Using expressions (4.3.1), (4.3.2), (4.3.3) and (4.3.4) and expression for c.d.f (2.7) we obtain mean deviation about mean $(\delta_1(Z))$ and mean deviation about median $(\delta_2(Z))$

$$\delta_1(Z) = \left[\begin{array}{l} 2\mu\left[\begin{array}{l} p\left(1-\left[\frac{\theta+1+\theta\mu}{(\theta+1)}\right]e^{-\theta\mu}\right) \\ +(1-p)\left[1-\left[1+\frac{\theta\mu(\theta\mu+2)}{2(\theta+1)}\right]e^{-\theta\mu}\right]\end{array}\right] \\ -2\left[\frac{1}{2\theta(\theta+1)}\left\{\begin{array}{l} 2p\left(\theta(1-(\mu\theta+1)e^{-\theta\mu}) + (2-e^{-\theta\mu}(\mu^2\theta^2+2\mu\theta+2))\right)+ \\ (1-p)\left(\begin{array}{l}2\theta(1-(\mu\theta+1)e^{-\mu\theta})+(6-e^{-\theta\mu}(\mu^3\theta^3+3\mu^2\theta^2 \\ +6\mu\theta+6))\end{array}\right)\end{array}\right\}\right]\end{array}\right]$$

&

$$\delta_2(Z) = \left[\mu-2\left[\frac{1}{2\theta(\theta+1)}\left\{\begin{array}{l} 2p\left(\theta(1-(M\theta+1)e^{-\theta M}) + (2-e^{-\theta M}(M^2\theta^2+2M\theta+2))\right)+ \\ (1-p)\left(\begin{array}{l}2\theta(1-(M\theta+1)e^{-M\theta})+(6-e^{-\theta M}(M^3\theta^3+3M^2\theta^2 \\ +6M\theta+6))\end{array}\right)\end{array}\right\}\right]\right]$$

4.4 Moment generating function and Characteristic function of Sanna Distribution (SD)

We will derive moment generating function and characteristic function of SD in this segment of paper.

Theorem 1.1: If $Z \sim SD(\theta, p)$ then the moment generating function $M_Z(t)$ and characteristic generating function $\phi_Z(t)$ are

$$M_Z(t) = \left[\frac{\theta^2}{(\theta+1)}\left\{p\left(\frac{1}{(\theta-t)}+\frac{1}{(\theta-t)^2}\right)\right\}+(1-p)\left(\frac{1}{(\theta-t)}+\frac{\theta}{(\theta-t)^3}\right)\right]$$

And

$$\phi_Z(t) = \left[\frac{\theta^2}{(\theta+1)} \left\{ p \left(\frac{1}{(\theta - it)} + \frac{1}{(\theta - it)^2} \right) \right\} + (1-p) \left(\frac{1}{(\theta - it)} + \frac{\theta}{(\theta - it)^3} \right) \right]$$

respectively.

Proof: We begin with the well-known definition of the moment generating function given by

$$M_Z(t) = E\left(e^{tZ}\right) = \int_0^\infty e^{tz} f(z)\,dz$$

$$= \int_0^\infty e^{tz} \left[\frac{\theta^2 e^{-\theta z}}{2(\theta+1)} \left[2p(1+z) + (1-p)(2+\theta z^2) \right] \right] dz$$

$$M_Z(t) = \left[\frac{\theta^2}{(\theta+1)} \left\{ p \left(\frac{1}{(\theta - t)} + \frac{1}{(\theta - t)^2} \right) \right\} + (1-p) \left(\frac{1}{(\theta - t)} + \frac{\theta}{(\theta - t)^3} \right) \right] \quad (4.4.1)$$

Which is the m.g.f of Sanna distribution.

Also we know that $\phi_Z(t) = M_Z(it)$

Therefore,

$$\phi_Z(t) = \left[\frac{\theta^2}{(\theta+1)} \left\{ p \left(\frac{1}{(\theta - it)} + \frac{1}{(\theta - it)^2} \right) \right\} + (1-p) \left(\frac{1}{(\theta - it)} + \frac{\theta}{(\theta - it)^3} \right) \right] \quad (4.4.2)$$

Which is the characteristic function of Sanna distribution.

5. Order Statistics of Sanna Distribution

Consider $Z_{(1)}, Z_{(2)}, Z_{(3)}, \ldots, Z_{(n)}$ to be the ordered statistics of the random sample $z_1, z_2, z_3, \ldots z_n$ obtained from the Sanna distribution with cumulative distribution function $F(z)$ and probability density function $f(z)$, then the probability density function of v^{th} order statistics $Z_{(v)}$ is given by:

$$f_v(z, \theta, P) = \frac{n!}{(v-1)!(n-v)!} f(z)\left[F(z)\right]^{v-1}\left[1 - F(z)\right]^{n-v} \qquad v=1, 2, 3\ldots n$$

Using the equations (2.6) and (2.7), the probability density function of v^{th} order statistics of Sanna distribution is given by:

$$f_{(v)}(z,\theta,p)=\left[\begin{array}{l}\frac{n!}{(v-1)!(n-v)!}\left[\frac{\theta^2 e^{-\theta z}}{2(\theta+1)}\left[2p(1+z)+(1-p)(2+\theta z^2)\right]\right]\\ \left[\left[p\left(1-\left[\frac{\theta+1+\theta z}{(\theta+1)}\right]e^{-\theta z}\right)+(1-p)\left[1-\left[1+\frac{\theta z(\theta z+2)}{2(\theta+1)}\right]e^{-\theta z}\right]\right]\right]^{v-1}\\ \left[1-\left[p\left(1-\left[\frac{\theta+1+\theta z}{(\theta+1)}\right]e^{-\theta z}\right)+(1-p)\left[1-\left[1+\frac{\theta z(\theta z+2)}{2(\theta+1)}\right]e^{-\theta z}\right]\right]\right]^{n-v}\end{array}\right].$$

Then, the p.d.f of first order statistic $X_{(1)}$ of Sanna distribution is given by:

$$f_{(1)}(z,\theta,p)=\left[\begin{array}{l}n\left[\frac{\theta^2 e^{-\theta z}}{2(\theta+1)}\left[2p(1+z)+(1-p)(2+\theta z^2)\right]\right]\\ \left[1-\left[p\left(1-\left[\frac{\theta+1+\theta z}{(\theta+1)}\right]e^{-\theta z}\right)+(1-p)\left[1-\left[1+\frac{\theta z(\theta z+2)}{2(\theta+1)}\right]e^{-\theta z}\right]\right]\right]^{n-1}\end{array}\right].$$

and the pdf of n^{th} order statistic $X_{(n)}$ of Sanna distribution is given as:

$$f_{(n)}(z,\theta,p)=\left[\begin{array}{l}n\left[\frac{\theta^2 e^{-\theta z}}{2(\theta+1)}\left[2p(1+z)+(1-p)(2+\theta z^2)\right]\right]\\ \left[\left[p\left(1-\left[\frac{\theta+1+\theta z}{(\theta+1)}\right]e^{-\theta z}\right)+(1-p)\left[1-\left[1+\frac{\theta z(\theta z+2)}{2(\theta+1)}\right]e^{-\theta z}\right]\right]\right]^{n-1}\end{array}\right].$$

6. Method of Maximum Likelihood Estimation of Sanna distribution.

Considering $Z_1, Z_2, Z_3....Z_n$ to be the random sample of size n drawn from Sanna distribution having density function given by (2.6), then the likelihood function of Sanna distribution is given as:

$$L(z\mid\theta,p)=\prod_{i=1}^{n}\left[\frac{\theta^2 e^{-\theta z_i}}{2(\theta+1)}\left[2p(1+z_i)+(1-p)(2+\theta z_i^{\,2})\right]\right]$$

Taking log on both sides of likelihood function we get log likelihood function as:

$$\log L=\left\{2n\log\theta-n\log 2-n\log(\theta+1)-\theta\sum_{i=1}^{n}z_i+\sum_{i=1}^{n}\log\left(2p(1+z_i)+(1-p)(2+\theta z_i^{\,2})\right)\right\} \qquad (6.1)$$

Differentiating the log-likelihood function with respect to θ & p.This is done by partially differentiate (6.1) with respect to θ & p and equating the result to zero, we obtain the following normal equations,

$$\frac{\partial\log L}{\partial\theta}=\left[\frac{2n}{\theta}-\frac{n}{(\theta+1))}-\sum_{i=1}^{n}x_i+\sum_{i=1}^{n}\left[\frac{\{(1-p)z_i^{\,2}\}}{\left(2p(1+z_i)+(1-p)(2+\theta z_i^{\,2})\right)}\right]\right]=0 \qquad (6.2)$$

$$\frac{\partial \log L}{\partial p} = \left[\sum_{i=1}^{n}\left[\frac{2(1+z_i)-(2+\theta z_i^{\,2})}{\left(2p(1+z_i)+(1-p)(2+\theta z_i^{\,2})\right)}\right]\right] = 0 \qquad (6.3)$$

MLEs of θ & p cannot be obtained by solving above complex equations as these equations are not in closed form. So we solve above equations by using iteration method through R software.

7. Applications of Sanna Distribution:-

We fitted Sanna distribution and its related distributions to two real life data sets to check the superiority of our model over its related models.

Data Set 1: Here we consider a data set initially analysed by Chhikara& Folks (1977). The maintenance data set given in table 9 represents active repair times (in hours) for an airborne communication transceiver. Sinha (1986) used this data set to compute the bayes estimate considering Lindley's method under non informative prior. Betro & Rotondi (1991) have also analysed this data set.

Table 1: Active repair times (in hours) of 46 transceiver.

0.2	0.3	0.5	0.5	0.5	0.5	0.6	0.6
0.7	0.7	0.7	0.8	0.8	1.0	1.0	1.0
1.0	1.1	1.3	1.5	1.5	1.5	1.5	2.0
2.0	2.2	2.5	2.7	3.0	3.0	3.3	3.3
4.0	4.0	4.5	4.7	5.0	5.4	5.4	7.0
7.5	8.8	9.0	10.3	22.0	24.5		

Data Set 2: The data set given in table 2 corresponds to remission times (in months) of a random sample of 128 bladder cancer patients given in Lee and Wang (2003). The data set is given as follows

Table2: Remission times (in months) of 128 patients of Bladder cancer.

0.08	2.09	3.48	4.87	6.94	8.66	13.11	23.63	21.73	0.20
2.22	3.52	4.98	6.99	9.02	13.29	0.40	2.26	3.57	5.06
7.09	9.22	13.80	25.74	0.50	2.46	3.64	5.09	7.26	9.47
14.24	25.82	0.51	2.54	3.70	5.17	7.28	9.74	14.76	26.31
0.81	2.62	3.82	5.32	7.32	10.06	14.77	32.15	2.64	3.88
5.32	7.39	10.34	14.83	34.26	0.90	2.69	4.18	5.34	7.59
10.66	15.96	36.66	1.05	2.69	4.23	5.41	7.62	10.75	16.62
43.01	1.19	2.75	4.26	5.41	7.63	17.12	46.12	1.26	2.83
4.33	5.49	7.66	11.25	17.14	79.05	1.35	2.87	5.62	7.87
11.64	17.36	1.40	3.02	4.34	5.71	7.93	11.79	18.10	1.46
4.40	5.85	8.26	11.98	19.13	1.76	3.25	4.50	6.25	8.37
12.02	2.02	3.31	4.51	6.54	8.53	12.03	20.28	2.02	3.36
6.76	12.07	2.07	3.36	6.93	8.65	12.63	22.69		

These data sets are used here only for illustrative purposes. The required numerical evaluations are carried out using R software version R i386 3.3.2. We have fitted Sanna distribution, Rayleigh distribution & Quasi Akash distribution to these two real life data sets. The summary statistic of these two data sets is given in table 3. The MLEs of the parameters with standard errors in parentheses, model functions are displayed in table 4 for these two data sets. The corresponding likelihood ratio statistic, log-likelihood values, AIC, AICC, HQIC, BIC, Kolmogorov statistic & Shannon's entropy are given in table 5 & 6 for data sets 1 & 2 respectively.

Table 3: Summary statistic of data sets 1 & 2.

Data Set	No. of observations	Min.	First quartile	median	mean	Third quartile	Max.
Data Set 1	46	0.200	0.800	1.750	3.607	4.375	24.500
Data Set 2	128	0.080	3.348	6.395	9.366	11.838	79.050

Table 4: ML Estimates, Standard Error of Estimates in parenthesis, model function of related models and proposed model for data sets 1 & 2.

Data set	**Distribution**	**ML Estimates with Standard errors**	**Model Function**
Data set 1	**Sanna Distribution (SD)**	$\hat{\theta} = 0.48859334$ (0.06278918) $\hat{p} = 0.85214734$ (0.25642684)	$\frac{\theta^2 e^{-\theta z}}{2(\theta+1)}\left[2p(1+z)+(1-p)(2+\theta z^2)\right]$
	Quasi Akash Distribution (QAD)	$\hat{\theta} = 0.59076079$ (0.05832089)	$\frac{\theta^2}{2(\theta+1)}\left(2+\theta z^2\right)e^{-\theta z}$
	Rayleigh Distribution (RD)	$\hat{\theta} = 4.2965485$ (0.3167457)	$\frac{z e^{-z^2/\left(2\theta^2\right)}}{\theta^2}$

Data set 2	**Sanna Distribution (SD)**	$\hat{\theta} = 0.2006745$ (0.0171143) $\hat{p} = 0.9425194$ (0.1477759)	$\frac{\theta^2 e^{-\theta z}}{2(\theta+1)}\left[2p(1+z)+(1-p)(2+\theta z^2)\right]$
	Quasi Akash Distribution (QAD)	$\hat{\theta} = 0.26888667$ (0.01484086)	$\frac{\theta^2}{2(\theta+1)}\left(2+\theta z^2\right)e^{-\theta z}$
	Rayleigh Distribution (RD)	$\hat{\theta} = 9.9316968$ (0.4389231)	$\frac{ze^{-z^2/(2\theta^2)}}{\theta^2}$

Table 5: Model comparison, Likelihood ratio of proposed model and its related models for data set 1.

Distribution	$-\log L$	**AIC**	**BIC**	**AICC**	**HQIC**	**Shanon entropy *H(X)***	**Kolmogorov statistic (D)**	**Likelihood Ratio**
Sanna Distribution (SD)	109.775	223.5500	227.20	223.82	224.92	2.38	0.2299	6.208
Quasi Akash Distribution (QAD	112.879	227.7584	229.58	227.84	228.44	2.45	0.2436	
Rayleigh Distribution (RD)	149.832	301.6653	303.493	301.75	302.350	3.25	0.44235	

Table 6: Model comparison, Likelihood ratio of proposed model and related models for data set 2.

Distribution	$-\log L$	AIC	BIC	AICC	HQIC	Shanon entropy $H(X)$	Kolmogorov Statistic (D)	Likelihood Ratio
Sanna Distribution (SD)	419.44	842.88	848.59	842.98	845.20	3.27	0.11576	
Quasi Akash Distribution (QAD)	429.36	860.72	863.57	860.75	861.87	3.35	0.16043	19.84
Rayleigh Distribution (RD)	491.265	984.531	987.383	984.563	985.69	3.83	0.3521	

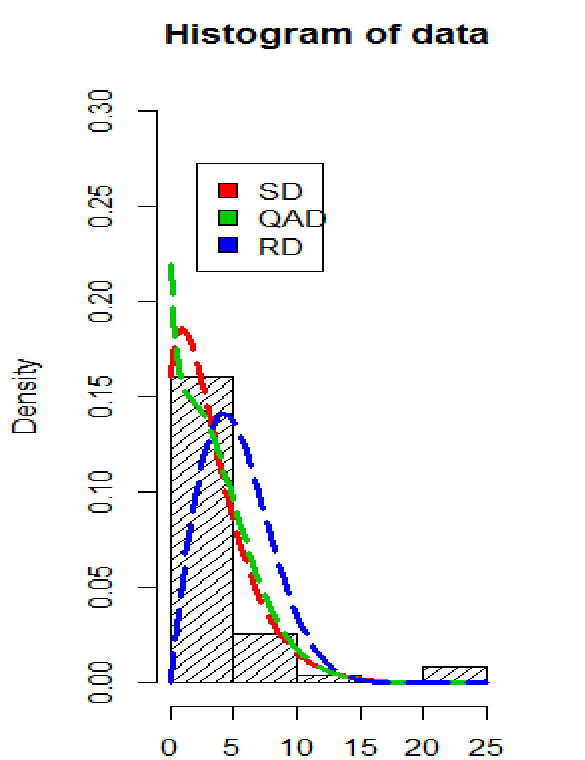

Figure 4 Graph of data set 1 fitted by proposed model and other related mo

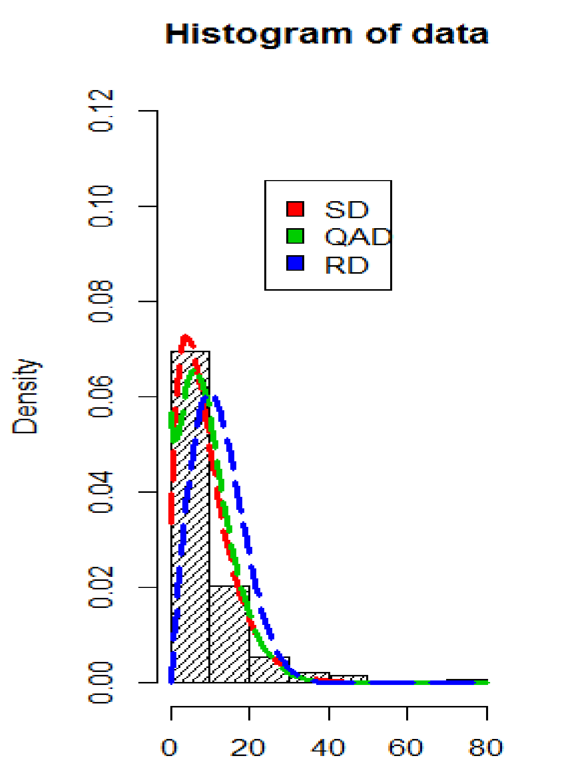

Figure 5 Graph of data set 2 fitted by proposed model and other related mod

For testing the goodness of fit of proposed model over Rayleigh distribution and Quasi-Akash distribution to the two data sets we computed Kolmogorov statistic. The model which possesses lesser value of Kolmogorov statistic better better to data sets. From tables 5 & 6 it can be seen that Sanna distribution possesses the least value of Kolmogorov statistic as compared to Quasi Akash distribution and Rayleigh distribution, for data sets 1 & 2 respectively. Hence Sanna model fits better to both the data sets.

To test whether mixing parameter p plays a significant role or not and for checking superiority of Sanna distribution over Quasi Akash distribution and Rayleigh distribution for data sets 1 & 2 we computed likelihood ratio (LR) statistic for data sets 1 & 2 in tables 5 & 6 respectively. For testing $H_0 : p = 0$ versus $H_1 : p \neq 0$ the LR statistic for testing H_0 is $\omega_1 = 2\{L(\hat{\Theta}) - L(\hat{\Theta}_0)\} = 6.208$ for data set 1 & $\omega_2 = 2\{L(\hat{\Theta}) - L(\hat{\Theta}_0)\} = 19.84$ for data set 2 where $\hat{\Theta}$ and $\hat{\Theta}_0$ are MLEs under H_1 and H_0. LR statistic $\omega \sim (\chi_{(1)}{}^2 (\alpha = 0.05) = 3.841)$ as $n \to \infty$, where 1= degrees of freedom is the difference in dimensionality. From table 5, 6 $\omega_1 = 6.208 > 3.841$ & $\omega_2 = 19.84 > 3.841$ at 5% level of significance for all the two data sets, so we reject H_0 and conclude that mixing parameter p plays statistically a significant role.

In order to compare the Sanna Distribution with Quasi Akash Distribution & Rayleigh Distribution, We compute the criteria like AIC (Akaike information criterion), AICC (corrected Akaike information criterion) , BIC (Bayesian information criterion) & HQIC which represent the loss of information resulting from fitting probability models to data. The better distribution corresponds to lesser AIC, AICC, BIC & HQIC values. Also we computed the Shannon's entropy *(H(X))* which represents the average uncertainty. The better model possesses lesser Shannon's entropy value.

AIC = 2k-2log$L$ AICC = AIC+ $\dfrac{2k(k+1)}{n-k-1}$

BIC = k log$n$-2log$L$ HQIC=2k log(log(n))+2 log L

$$H(X) = -\frac{\log L}{n}$$

where k is the number of parameters in the statistical model, n is the sample size and -2log$L$ is the maximized value of the log-likelihood function under the considered model. From Tables 5 & 6, it has been observed that the Sanna distribution possesses the lesser AIC, AICC, BIC, HQIC and *H(X)* values as compared to Quasi Akash distribution & Rayleigh distribution for data sets 1 & 2 respectively. Hence we can conclude that the Sanna distribution leads to a better fit than the Quasi Akash distribution & Rayleigh distribution for data sets 1 & 2 respectively.

8. Special Cases

Case I: If we put $p = 0$, then Sanna distribution (2.6) reduces to Quasi Akash distribution with shape parameter= 2 and having probability density function as:

$$f(z) = \frac{\theta^2}{2(\theta+1)}\left(2+\theta z^2\right)e^{-\theta z} \qquad z > 0, \theta > 0$$

Case II: For $p = 1$, Sanna distribution (2.6) reduces to one parameter Lindley distribution with probability density function given as

$$f_1(z) = \frac{\theta^2}{(\theta+1)}(1+z)e^{-\theta z} \qquad z > 0, \theta > 0$$

9. Conclusion

We developed Sanna distribution as a genuine mixture of one parameter Lindley distribution with parameter θ and Quasi Akash distribution with scale parameter θ and shape parameter 2. We obtained the important statistical properties like moments, reliability, moment generating function, order statistics etc., of our formulated model. We also obtained the estimates of unknown parameters of our proposed model by method of maximum likelihood estimation. For testing the supremacy of our model over its competitive models and for testing the significance of mixing parameter we fitted our model to two real life data sets and concluded that our model fits better to these data sets as proposed model possesses lesser AIC, BIC, AICC, HQIC, D, Shannon entropy values for both the data sets. So our model finds applicability in real life.

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