A REVIEW ON DIRICHLET PROBLEM AND ITS SOLUTION IN THE ASPECTS OF POTENTIAL THEORY

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ABSTRACT - This paper is a review of various methods used to solve the Dirichlet problem in the aspects of potential theory. We discuss how the solution is been arrived by considering the Dirichlet problem in the view of Classical (R^n), Axiomatic and Discrete potential theory.

Key words - Dirichlet problem, infinite network, harmonic functions, superharmonic and subharmonic functions, balayage.

1. INTRODUCTION

In the study of potential theory (Classical and axiomatic) Dirichlet problem is one of the important problem. The Dirichlet problem in the classical case i.e, in R^n [1] is to find the harmonic function h inside the domain D and h = ψ on the boundary of domain D, where ψ is a real valued function defined on the boundary. Solution to the problem is obtained in both bounded and unbounded domains [2] and [3].

The Perron Weiner method introduced in 1924 by Oskar Perron is developed and used in axiomatic potential theory to solve the generalized Dirichlet problem. For f being a real valued function on a set L of filters F associated to saturated hyperharmonic functions on a connected locally compact hausdroff space Ω [4] and [5]. The lower envelope $\underline{H}_f$ and the upper envelope $\overline{H}_f$ are proved to be equal at a point and hence leading to the resolutivity of f. Hence H_f is called the generalized solution.

In discrete case, Let B be a subset of an infinite network G [6]. Suppose f is a function defined on the boundary of subset B denoted by $\partial B \ni u \leq f \leq v$, where v and u are real valued functions on B such that $u \leq v$. Then the problem is to prove that there exist a function h on B such that $u \leq h \leq v$ on B, $h = f$ on ∂B and $\Delta h(x) = 0$ (harmonic) at each $x \in \dot{B}$ (interior of B) [7].

2. CLASSICAL POTENTIAL THEORY

In [8], Dirichlet problem is worked on an arbitrary open subset of R^n ($n \geq 3$) with non compact boundaries. Let Ω be a subset of R^n the problem is carried out by assuming that there exist atleast one connected component Ω, which extends upto infinity. The solution is framed using the PWB (Perron's method, Wiener and Brelot refinements) method.

The main result is the investigation of the regularity of infinity for Ω which concludes whether there exists a unique solution (or) infinitely many solutions for the Dirichlet Problem, according as the Wiener series converges (or) diverges. The uniqueness of a bounded solution of Dirichlet problem is investigated by "thinness" of the complement of Ω at infinity.

According to the Kelvin transformation counterpart it is concluded that regularity or irregularity of infinity has no effect in the uniqueness of the solution and as per measure theoretical counterpart the existence of unique (or) infinitely many solution is based on the existence of equilibrium measure for the complementary set Ω^C.

It is concluded that if the wiener series diverges then the regularity of infinity implies that there exist a unique, bounded solution. Conversely if the infinity is irregular as wiener series converges then there is infinitely many solutions. The classification discussed relates the difference between bounded and unbounded open sets with compact boundaries.

3. DISCRETE POTENTIAL THEORY

Dirichlet problem is considered in an infinite network in **[9]**. The techniques used here are based on classical and axiomatic potential theory on the complex plane and Riemann surfaces. Solution to the problem is approximated by using lower and upper functions called subharmonic and superharmonic functions.

Taking Markov chains and Infinite electrical grids as examples of infinite graph the functions on it are studied. By infinite network they mean a triple (X,Y,t) where X and Y are countable collections of vertices and edges, f is a non-negative number associated with any pair of vertices x and y such that $t(x, y) > 0$ iff there exist an edge joining x and y. Poisson modification of superharmonic functions is constructed and used in defining greatest harmonic minorant further arriving at the solution of Dirichlet problem.

The relation between the charges and potential on the conductors is significant for two conductors in isolation and is expressed as condenser principle and is obtained as a corollary of the classical Dirichlet solution. The distribution of mass on E (subset of infinite network) is done so that there is no mass on E and in such a way that there is no change in the potential outside E. Poincare's method is used in redistribution of mass and this process is called Balayage which is presented as a Dirichlet solution.

In **[10]**, Generalised Dirichlet problem has been solved with respect to an arbitrary subset of vertices of an infinite network N. This paper is a study of functions from the classical potential theoretic point of view.

Generalized Dirichlet problem is solved under certain restricted conditions. "For F being an arbitrary set in X and E be a subset of X. Let $E \subset \dot{F}$ (collection of all interior points of F) and $f \geq 0$ be a function defined on F / E. Suppose there exists a superharmonic function $u \geq 0$ on F such that $u \geq f$ on F / E. Then there exist a function $h \geq 0$ on F such that $0 \leq h \leq u$ on F, $\triangle h = 0$ on E and $h = f$ on F / E; moreover, if h_1 is another function on F such that $\triangle h_1 = 0$ on E and $h_1 = f$ on F/E, then $h_1 \geq h$ on F (i.e, h is the smallest of such functions)."

As a consequence of the generalized version of Dirichlet problem and its solution considered in an infinite network, some important results of potential theoretic concepts are obtained: Classical Dirichlet Problem, Potential-dominated Dirichlet Problem, Domination Principle, Greens potential, Poisson Kernal, Balayage, Green Kernal on a set, Dirichlet Poisson solution, Generalized Capacitary functions.

In general, Minimum principle is used for proving the uniqueness of the solution of Dirichlet problem considered in an infinite network in [9] and [10].

The solution of Dirichlet problem and construction of superharmonic functions in the context of discrete potential theory is been discussed in **[11]**. A Cartier Tree T is a countably infinite connected graph which is locally finite and without loops. Dirichlet problem is investigated by considering a cartier tree T. A function $g_e(x)$ in T for a vertex e is defined and proved to be superharmonic. A finite set of connected vertices in T is considered and the first part of the problem $h(x) = f(x)$ is proved by the defined family of subharmonic functions and superharmonic functions, Further the harmonicity is proved by considering a vertex in the interior of tree and solving with subharmonic functions. Uniqueness of the solution is explained through maximum principle with the remark that the Dirichlet solution is obtained for unbounded sets also, but uniqueness cannot be obtained for the same.

Green's first formula for a tree is proved referring the result of [6]. Some consequences of Green's formula are obtained which also includes local representation of harmonic functions and flux at infinity .

In **[12]**, Dirichlet principle in the context of a Cartier tree is framed keeping Dirichlet principle in the context of an infinite network as a model [13]. The problem is investigated considering a Cartier tree. A finite set of vertices is considered in a Cartier tree and Dirichlet semi norm is constructed using green's formula. Dirichlet solution is obtained as a projection on the closed subspace of harmonic functions.

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