Solving of Transcendental Equation $\sqrt{2z-4}=\sqrt{x+\sqrt{C}y}\pm\sqrt{x-\sqrt{C}y}$ by means of the method of Continued Fraction for the Choices of $C=m^2\pm 4$

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Abstract- The Transcendental equation with three unknowns is given by $\sqrt{2z-4}=\sqrt{x+\sqrt{C}y}\pm\sqrt{x-\sqrt{C}y}$ is considered and analyzed for finding different set of integer solutions by means of continued fraction method, under numerous patterns with some numerical examples.

Key words - Transcendental equation, Integer solutions, Continued fraction, Pell equation, Recurrence Relation.

I. INTRODUCTION

Diophantine equations have versatile field of research. Many Diophantine equations are in algebraic form. In this paper, we try to find the integer solutions for the given transcendental equation by using continued fraction method. Here we consider the Pell equation of $X^2-CY^2=1$ for different values of C, where ($C\neq 1$) is a positive non square integer. Also, we enumerate some of the most salient qualities of simple continued fraction. I referred the journal [6] in detailed but slightly flawed book [3] as the main source of an inspiration for numerous experiments I have made in this problem.

II. PRELIMINARIES

Definition 2.1 Continued Fraction Expansion

A simple continued fraction expansion of order n is an expression of the form

$$a_0+\cfrac{1}{a_1+\cfrac{1}{a_2+\cfrac{1}{a_3+\cdots}}} \quad \cdots+\frac{1}{a_n}$$

and it can be abbreviated as $[a_0;a_1,a_2,\dots,a_n]$ and a_0 may be positive or negative or zero. $\sqrt{C}=[a_0;a_1,a_2,\dots,a_l]$ denote the Continued fraction expansion of period length l. Set $A_{-2}=0, A_{-1}=1, A_k=a_kA_{k-1}+A_{k-2}$ and $B_{-2}=1, B_{-1}=2, B_k=a_kB_{k-1}+B_{k-2}$ for $k>0$. $P_k=\frac{A_k}{B_k}$ is the k^{th} convergent of $\sqrt{C}$.

Now we give the fundamental solution of the equation $x^2-Cy^2=\pm 1$ by means of the period length of the continued fraction expansion of $\sqrt{C}$.

Lemma 2.1 Let l be the period length of continued fraction expansion of $\sqrt{C}$. If l is even, then the fundamental solution to the equation $x^2-Cy^2=1$ is given by

$$x_1+y_1\sqrt{C}=p_{l-1}+q_{l-1}\sqrt{C}$$

and the equation $x^2-Cy^2=-1$ has no integer solution. If l is odd, then the fundamental solution to the equation $x^2-Cy^2=1$ is given by

$$x_1+y_1\sqrt{C}=p_{2l-1}+q_{2l-1}\sqrt{C}$$

and the fundamental solution to the equation $x^2-Cy^2=-1$ is given by

$$x_1+y_1\sqrt{C}=p_{l-1}+q_{l-1}\sqrt{C}$$

Now, we give continued fraction expansion of $\sqrt{C}$ for $C=m^2\pm 4$. The proof of the following two theorems is easy and they can be found many text books on number theory as an exercise (see, for example [5])

Theorem 2.1 Let $m>1$. Then

$$\sqrt{m^2+4}=\begin{cases}\left[m;\ \overline{\frac{m}{2},2m}\right], & if\ m\ is\ even\\ \left[m;\ \overline{\frac{m-1}{2},1,1,\frac{m-1}{2},2m}\right], & if\ m\ is\ odd\end{cases}$$

Theorem 2.2 Let $m>3$. Then

$$\sqrt{m^2-4}=\begin{cases}\left[m-1;\ \overline{1,\frac{m-4}{2},1,2(m-1)}\right], & if\ m\ is\ even\ and\ m\neq 4\\ \left[m-1;\ \overline{1,\frac{m-3}{2},2,\frac{m-3}{2},1,2(m-1)}\right], & if\ m\ is\ odd\\ [3;\ \overline{2,6}], & if\ m=4\end{cases}$$

Now we give the following two corollaries from [8].

Corollary 2.1 Let $m>1$ and $C=m^2+4$. Then the fundamental solution to the equation $x^2-Cy^2=1$ is

$$x_1+y_1\sqrt{C}=\begin{cases}\frac{m^2+2}{2}+\frac{m}{2}\sqrt{C}, & if\ m\ is\ even\\ \frac{m^6+6m^4+9m^2+2}{2}+\frac{m^5+4m^3+3m}{2}\sqrt{C}, & if\ m\ is\ odd\end{cases}$$

Corollary 2.2 Let $m>3$ and $C=m^2-4$. Then the fundamental solution to the equation $x^2-Cy^2=1$ is

$$x_1 + y_1\sqrt{C} = \begin{cases} \frac{m^2-2}{2} + \frac{m}{2}\sqrt{C}, & if\ m\ is\ even \\ \frac{m^3-3m}{2} + \frac{m^2-1}{2}\sqrt{C}, & if\ m\ is\ odd \end{cases}$$

III. DESCRIPTION OF METHOD

The Transcendental equation $\sqrt{2z-4} = \sqrt{x+\sqrt{C}y} \pm \sqrt{x-\sqrt{C}y}$

Consider the Transcendental equation $\sqrt{2z-4} = \sqrt{x+\sqrt{C}y} \pm \sqrt{x-\sqrt{C}y}$ (1)

Squaring on both sides, we get

$z-2 = x \pm \sqrt{x^2 - Cy^2}$ (2)

Take $x^2 - Cy^2 = \alpha^2$, so that $z = x + 2 \pm \alpha$

Take $\alpha = 1$ and therefore

$x^2 = Cy^2 + 1\ \ and\ z = x + 2 \pm 1$ (3)

That is $z = x + 3$ or $Z = x + 1$

The choices of C, we get all the solutions for (3).

The choices of C (Non square integer) are $m^2 + 4$ and $m^2 - 4$.

IV. MAIN THEOREMS

Theorem 4. Let $\mathrm{m} \geq 1$ be any integer, and let $\mathrm{C} = \mathrm{m}^2 + 4$.

$$(x_1, y_1) = \begin{cases} (\frac{m^2+2}{2}, \frac{m}{2}), & if\ m\ is\ even \\ (\frac{m^6+6m^4+9m^2+2}{2}, \frac{m^5+4m^3+3m}{2}), & if\ m\ is\ odd \end{cases}$$

be the fundamental solution of the first equation of (3). Set $\{(x_n, y_n)\}$, where

$$\frac{x_n}{y_n} = \begin{cases} \left[m;\ \overline{\frac{m}{2}, 2m}\right], & if\ m\ is\ even \\ \left[m;\ \overline{\frac{m-1}{2}, 1, 1, \frac{m-1}{2}, 2m}\right], & if\ m\ is\ odd \end{cases} \quad (4)$$

for $\mathrm{n} \geq 2$. Then (x_n, y_n) is a solution of $x^2 - (m^2+4)y^2 = 1$ and so (x_n, y_n, z_n) is the solution of the equation (3), where $z_n = x_n + 2 \pm 1$. The consecutive solutions (x_n, y_n, z_n) and $(x_{n+1}, y_{n+1}, z_{n+1})$ satisfy

$$x_{n+1} = \begin{cases} \frac{(m^2+2)x_n + (m^2+4m)y_n}{2}, & if\ m\ is\ even \\ \frac{\left((m^3+3m)^2+2\right)x_n + m(m^2+1)(m^2+3)(m^2+4)y_n}{2}, & if\ m\ is\ odd \end{cases}$$

$$y_{n+1} = \begin{cases} \frac{mx_n + (m^2+2)y_n}{2},, & if\ m\ is\ even \\ \frac{m(m^2+1)(m^2+3)x_n + ((m^3+3m)^2+2)y_n}{2}, & if\ m\ is\ odd \end{cases}$$

$$z_{n+1} = \begin{cases} \frac{(m^2+2)x_n + (m^2+4m)y_n}{2} + 2 \pm 1, & if\ m\ is\ even \\ \frac{((m^3+3m)^2+2)x_n + m(m^2+1)(m^2+3)(m^2+4)y_n}{2} + 2 \pm 1, & if\ m\ is\ odd \end{cases}$$

with $n \geq 1$.

Proof.

Case (i) Let $\mathrm{m} \geq 1$ be an even integer. By the Corollary 2.1 $x_1 + y_1\sqrt{C} = \frac{m^2+2}{2} + \frac{m}{2}\sqrt{C}$ be the fundamental solution of the first equation of (3). Indeed $\left(\frac{m^2+2}{2}\right)^2 - (m^2+4)\left(\frac{m}{2}\right)^2 = 1$. Now we assume that (x_n, y_n) is a solution of

$x^2-(m^2+4)y^2=1$. Applying (4), we get

$$\frac{x_{n+1}}{y_{n+1}}=m+\cfrac{1}{\left(\frac{m}{2}\right)+\cfrac{1}{2m+\cfrac{1}{\left(\frac{m}{2}\right)+\cfrac{1}{2m+\cdots}}}}\;\ldots+\frac{1}{2m}$$

$$=m+\cfrac{1}{\left(\frac{m}{2}\right)+\cfrac{1}{m+m+\cfrac{1}{\left(\frac{m}{2}\right)+\cfrac{1}{2m+\cdots}}}}\;\ldots+\frac{1}{2m}$$

$$=m+\cfrac{1}{\left(\frac{m}{2}\right)+\cfrac{1}{m+\frac{x_n}{y_n}}}$$

$$=m+\frac{x_n+my_n}{\frac{m}{2}x_n+\left(\frac{m^2+2}{2}\right)y_n}$$

$$=\frac{\frac{(m^2+2)x_n+(m^2+4m)y_n}{2}}{\frac{mx_n+(m^2+2)y_n}{2}}$$

Thus,

$$\frac{x_{n+1}}{y_{n+1}}=\frac{\frac{(m^2+2)x_n+(m^2+4m)y_n}{2}}{\frac{mx_n+(m^2+2)y_n}{2}}$$

Therefore,

$$x_{n+1}=\frac{(m^2+2)x_n+(m^2+4m)y_n}{2}\ and\ y_{n+1}=\frac{mx_n+(m^2+2)y_n}{2}$$

Now,

$$x_{n+1}^2-(m^2+4)y_{n+1}^2=\frac{(m^2+2)^2x_n^2+(m^2+4m)^2y_n^2+2(m^2+2)(m^2+4m)x_ny_n}{4}$$
$$-(m^2+4)\frac{m^2x_n^2+(m^2+2)^2y_n^2+2m(m^2+2)x_ny_n}{4}$$
$$=\frac{(m^2+2)^2(x_n^2-(m^2+4)y_n^2)-m^2(m^2+4)(x_n^2-(m^2+4)y_n^2)}{4}$$
$$=\frac{(m^2+2)^2(1)-m^2(m^2+4)(1)}{4}=1$$

Therefore (x_{n+1},y_{n+1}) is a solution of $x^2-(m^2+4)y^2=1$.
From the second equation of (3), $z_{n+1}=x_{n+1}+2\pm 1$.
That is,

$$z_{n+1}=\frac{(m^2+2)x_n+(m^2+4m)y_n}{2}+2\pm 1$$

Thus, $(x_{n+1},y_{n+1},z_{n+1})=\left(\frac{(m^2+2)x_n+(m^2+4m)y_n}{2},\frac{mx_n+(m^2+2)y_n}{2},\frac{(m^2+2)x_n+(m^2+4m)y_n}{2}+2\pm 1\right)$ is the solution set of (1) in which $x^2-Cy^2=x^2-(m^2+4)y^2=1$.

Case (ii) Let m ≥ 1 be an odd integer. By the Corollary 2.1, $x_1 + y_1\sqrt{C} = \frac{m^6+6m^4+9m^2+2}{2} + \frac{m^5+4m^3+3m}{2}\sqrt{C}$ be the fundamental solution of the first equation of (3). Indeed $\left(\frac{m^6+6m^4+9m^2+2}{2}\right)^2 - (m^2+4)\left(\frac{m^5+4m^3+3m}{2}\right)^2 = 1$. Now we assume that (x_n, y_n) is a solution of $x^2 - (m^2+4)y^2 = 1$. Applying (4), we get

$$\frac{x_{n+1}}{y_{n+1}} = m + \cfrac{1}{\left(\frac{m-1}{2}\right) + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{\left(\frac{m-1}{2}\right) + \cfrac{1}{2m + \cfrac{1}{\left(\frac{m-1}{2}\right) + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{\left(\frac{m-1}{2}\right) + \cfrac{1}{2m + \cfrac{1}{\cdots}}}}}}}}}}$$

$$= m + \cfrac{1}{\left(\frac{m-1}{2}\right) + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{\left(\frac{m-1}{2}\right) + \cfrac{1}{2m + \cfrac{1}{\left(\frac{m-1}{2}\right) + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{\left(\frac{m-1}{2}\right) + \cfrac{1}{m + \frac{x_n}{y_n}}}}}}}}}}$$

On simplifying, we get,

$$\frac{x_{n+1}}{y_{n+1}} = \frac{\frac{(m^2(m^2+3)^2+2)x_n + \left(m(m^2+4)(m^2+3)(m^2+1)\right)y_n}{2}}{\frac{m(m^2+3)(m^2+1)x_n + (m^2(m^2+3)^2+2)y_n}{2}}$$

Thus,

$$x_{n+1} = \frac{(m^2(m^2+3)^2+2)x_n + \left(m(m^2+4)(m^2+3)(m^2+1)\right)y_n}{2}$$

$$y_{n+1} = \frac{m(m^2+3)(m^2+1)x_n + (m^2(m^2+3)^2+2)y_n}{2}$$

$$x_{n+1}^2 - (m^2+4)y_{n+1}^2 = \frac{(m^2(m^2+3)^2+2)^2x_n^2 + m^2(m^2+4)^2(m^2+3)^2(m^2+1)^2y_n^2 + 2(m^2(m^2+3)^2+2)\left(m(m^2+4)(m^2+3)(m^2+1)\right)x_ny_n}{4} - (m^2+4)\frac{m^2(m^2+3)^2(m^2+1)^2x_n^2 + (m^2(m^2+3)^2+2)^2y_n^2 + 2m(m^2+3)(m^2+1)(m^2(m^2+3)^2+2)x_ny_n}{4}$$

$$= \frac{(m^2(m^2+3)^2+2)^2(x_n^2-(m^2+4)y_n^2) - m^2(m^2+4)(m^2+3)^2(m^2+1)^2(x_n^2-(m^2+4)y_n^2)}{4}$$

$$= \frac{(m^2(m^2+3)^2+2)^2(1) - m^2(m^2+4)(m^2+3)^2(m^2+1)^2(1)}{4} = 1$$

Therefore (x_{n+1}, y_{n+1}) is a solution of $x^2 - (m^2+4)y^2 = 1$.

From the second equation of (3), $z_{n+1} = x_{n+1} + 2 \pm 1$.

That is, $z_{n+1} = \frac{\left(m^2(m^2+3)^2+2\right)x_n + \left(m(m^2+4)(m^2+3)(m^2+1)\right)y_n}{2} + 2 \pm 1$

Thus, $(x_{n+1}, y_{n+1}, z_{n+1}) =$

$$\left(\begin{array}{c}\frac{\left(m^2(m^2+3)^2+2\right)x_n+\left(m(m^2+4)(m^2+3)(m^2+1)\right)y_n}{2}, \frac{\left(m^2(m^2+3)^2+2\right)x_n+\left(m(m^2+4)(m^2+3)(m^2+1)\right)y_n}{2}, \\ \frac{\left(m^2(m^2+3)^2+2\right)x_n+\left(m(m^2+4)(m^2+3)(m^2+1)\right)y_n}{2} + 2 \pm 1\end{array}\right)$$ is the solution set of (1)

in which $x^2 - Cy^2 = x^2 - (m^2+4)y^2 = 1$

By the cases (i) and (ii), we get

$$x_{n+1} = \begin{cases} \frac{(m^2+2)x_n + (m^2+4m)y_n}{2}, & if\ m\ is\ even \\ \frac{((m^3+3m)^2+2)x_n + m(m^2+1)(m^2+3)(m^2+4)y_n}{2}, & if\ m\ is\ odd \end{cases}$$

$$y_{n+1} = \begin{cases} \frac{mx_n + (m^2+2)y_n}{2}, & if\ m\ is\ even \\ \frac{m(m^2+1)(m^2+3)x_n + ((m^3+3m)^2+2)y_n}{2}, & if\ m\ is\ odd \end{cases}$$

$$z_{n+1} = \begin{cases} \frac{(m^2+2)x_n + (m^2+4m)y_n}{2} + 2 \pm 1, & if\ m\ is\ even \\ \frac{((m^3+3m)^2+2)x_n + m(m^2+1)(m^2+3)(m^2+4)y_n}{2} + 2 \pm 1, & if\ m\ is\ odd \end{cases}$$

for $n \geq 1$.

Theorem 4.2. Let $\mathrm{m} > 4$ be any integer, and let $\mathrm{C} = \mathrm{m}^2 - 4$. $(x_1, y_1) = \begin{cases} \left(\frac{m^2-2}{2}, \frac{m}{2}\right), & if\ m\ is\ even \\ \left(\frac{m^2-3m}{2}, \frac{m^2-1}{2}\right), & if\ m\ is\ odd \end{cases}$ be the fundamental solution of the first equation of (3). Set $\{(x_n, y_n)\}$, where

$$\frac{x_n}{y_n} = \begin{cases} \left[m-1;\ \overline{1, \frac{m-4}{2}, 1, 2(m-1)}\right], & if\ m\ is\ even \\ \left[m-1;\ \overline{1, \frac{m-3}{2}, 2, \frac{m-3}{2}, 1, 2(m-1)}\right], & if\ m\ is\ odd \end{cases} \quad (5)$$

for $\mathrm{n} \geq 2$. Then (x_n, y_n) is a solution of $x^2 - (m^2-4)y^2 = 1$ and so (x_n, y_n, z_n) is a solution of (3), where $z_n = x_n + 2 \pm 1$. The consecutive solutions (x_n, y_n, z_n) and $(x_{n+1}, y_{n+1}, z_{n+1})$ satisfy

$$x_{n+1} = \begin{cases} \frac{(m^2-2)x_n + m(m^2-4)y_n}{2}, & if\ m\ is\ even \\ \frac{m(m^2-3)x_n + (m^2-1)(m^2-4)y_n}{2}, & if\ m\ is\ odd \end{cases}$$

$$y_{n+1} = \begin{cases} \frac{mx_n + (m^2-2)y_n}{2}, & if\ m\ is\ even \\ \frac{(m^2-1)x_n + m(m^2-3)y_n}{2}, & if\ m\ is\ odd \end{cases}$$

$$z_{n+1} = \begin{cases} \frac{(m^2-2)x_n + m(m^2-4)y_n}{2} + 2 \pm 1, & if\ m\ is\ even \\ \frac{m(m^2-3)x_n + (m^2-1)(m^2-4)y_n}{2} + 2 \pm 1, & if\ m\ is\ odd \end{cases}$$

with $n \geq 1$.

Proof

Case (i) Let $\mathrm{m} > 4$ be an even integer. By the Corollary 2.2 $x_1 + y_1\sqrt{C} = \frac{m^2-2}{2} + \frac{m}{2}\sqrt{C}$ be the fundamental solution of the first equation of (3). Indeed $\left(\frac{m^2-2}{2}\right)^2 - (m^2-4)\left(\frac{m}{2}\right)^2 = 1$. Now we assume that (x_n, y_n) is a solution of $x^2 - (m^2-4)y^2 = 1$. Applying (5), we get

$$\frac{x_{n+1}}{y_{n+1}} = (m-1) + \cfrac{1}{1 + \cfrac{1}{\left(\frac{m-4}{2}\right) + \cfrac{1}{1 + \cfrac{1}{2(m-1) + \cfrac{1}{1 + \cfrac{1}{\left(\frac{m-4}{2}\right) + \cdots \atop \dots\dots + \frac{1}{2(m-1)}}}}}}}$$

$$= (m-1) + \cfrac{1}{1 + \cfrac{1}{\left(\frac{m-4}{2}\right) + \cfrac{1}{1 + \cfrac{1}{(m-1) + (m-1) + \cfrac{1}{1 + \cfrac{1}{\left(\frac{m-4}{2}\right) + \cdots \atop \dots\dots + \frac{1}{2(m-1)}}}}}}}$$

$$= (m-1) + \cfrac{1}{1 + \cfrac{1}{\left(\frac{m-4}{2}\right) + \cfrac{1}{1 + \cfrac{1}{(m-1) + \frac{x_n}{y_n}}}}}$$

$$= (m-1) + \frac{\left(\frac{m-2}{2}\right)x_n + \left(\frac{m^2-2m-2}{2}\right)y_n}{\frac{m}{2}x_n + \left(\frac{m^2-2}{2}\right)y_n}$$

$$= m + \frac{x_n + my_n}{\frac{m}{2}x_n + \left(\frac{m^2+2}{2}\right)y_n}$$

$$= \frac{\frac{(m^2-2)x_n + (m^2-4m)y_n}{2}}{\frac{mx_n + (m^2-2)y_n}{2}}$$

Thus,

$$\frac{x_{n+1}}{y_{n+1}} = \frac{\frac{(m^2-2)x_n + (m^2-4m)y_n}{2}}{\frac{mx_n + (m^2-2)y_n}{2}}$$

Therefore,

$$x_{n+1} = \frac{(m^2-2)x_n + (m^2-4m)y_n}{2} \text{ and } y_{n+1} = \frac{mx_n + (m^2-2)y_n}{2}$$

Now,

$$x_{n+1}^2 - (m^2-4)y_{n+1}^2 = \frac{(m^2-2)^2x_n^2 + (m^2-4m)^2y_n^2 + 2(m^2-2)(m^2-4m)x_ny_n}{4} - (m^2-4)\frac{m^2x_n^2 + (m^2-2)^2y_n^2 + 2m(m^2-2)x_ny_n}{4}$$

$$= \frac{(m^2-2)^2(x_n^2-(m^2-4)y_n^2)-m^2(m^2-4)(x_n^2-(m^2-4)y_n^2)}{4}$$
$$= \frac{(m^2-2)^2(1)-m^2(m^2-4)(1)}{4} = 1$$

Therefore (x_{n+1}, y_{n+1}) is a solution of $x^2-(m^2-4)y^2 = 1$.
From the second equation of (3), $z_{n+1} = x_{n+1} + 2 \pm 1$.
That is,

$$z_{n+1} = \frac{(m^2-2)x_n + (m^2-4m)y_n}{2} + 2 \pm 1$$

Thus, $(x_{n+1}, y_{n+1}, z_{n+1}) = \left(\frac{(m^2-2)x_n+(m^2-4m)y_n}{2}, \frac{mx_n+(m^2-2)y_n}{2}, \frac{(m^2-2)x_n+(m^2-4m)y_n}{2} + 2 \pm 1\right)$ is the solution set of (1) in which $x^2 - Cy^2 = x^2-(m^2-4)y^2 = 1$.

Case (ii) Let m > 4 be an odd integer. By the Corollary 2.2, $x_1 + y_1\sqrt{C} = \frac{m^3-3m}{2} + \frac{m^2-1}{2}\sqrt{C}$ be the fundamental solution of the first equation of (3). Indeed $\left(\frac{m^3-3m}{2}\right)^2 - (m^2-4)\left(\frac{m^2-1}{2}\right)^2 = 1$. Now we assume that (x_n, y_n) is a solution of $x^2-(m^2-4)y^2 = 1$. Applying (5), we get

$$\frac{x_{n+1}}{y_{n+1}} = (m-1) + \cfrac{1}{1+\cfrac{1}{\left(\frac{m-3}{2}\right)+\cfrac{1}{2+\cfrac{1}{\left(\frac{m-3}{2}\right)+\cfrac{1}{1+\cfrac{1}{2(m-1)+\cfrac{1}{1+\cfrac{1}{\left(\frac{m-3}{2}\right)+\cdots}}}}}}}} \quad \ldots\ldots + \frac{1}{2(m-1)}$$

$$= (m-1) + \cfrac{1}{1+\cfrac{1}{\left(\frac{m-3}{2}\right)+\cfrac{1}{2+\cfrac{1}{\left(\frac{m-3}{2}\right)+\cfrac{1}{1+\cfrac{1}{(m-1)+\frac{x_n}{y_n}}}}}}}$$

$$= (m-1) + \frac{\left(\frac{m^2-2m-1}{2}\right)x_n + \left(\frac{m^3-2m^2-3m+4}{2}\right)y_n}{\frac{(m^2-1)}{2}x_n + \left(\frac{m^3-3m}{2}\right)y_n}$$
$$= \frac{\frac{m(m^2-3)x_n + (m^2-1)(m^2-4)y_n}{2}}{\frac{(m^2-1)x_n + m(m^2-3)y_n}{2}}$$

Thus,

$$\frac{x_{n+1}}{y_{n+1}} = \frac{\frac{m(m^2-3)x_n + (m^2-1)(m^2-4)y_n}{2}}{\frac{(m^2-1)x_n + m(m^2-3)y_n}{2}}$$

Therefore,

$$x_{n+1} = \frac{m(m^2-3)x_n + (m^2-1)(m^2-4)y_n}{2} \text{ and } y_{n+1} = \frac{(m^2-1)x_n + m(m^2-3)y_n}{2}$$

Now,

$$x_{n+1}^2-(m^2-4)y_{n+1}^2=\frac{m^2(m^2-3)^2x_n^2+(m^2-4)(m^2-1)^2y_n^2+2m(m^2-3)(m^2-1)(m^2-4)x_ny_n}{4}$$
$$-(m^2-4)\frac{(m^2-1)^2x_n^2+m^2(m^2-3)^2y_n^2+2m(m^2-1)(m^2-3)x_ny_n}{4}$$
$$=\frac{m^2(m^2-3)^2(x_n^2-(m^2-4)y_n^2)-(m^2-4)(m^2-1)^2(x_n^2-(m^2-4)y_n^2)}{4}$$
$$=\frac{m^2(m^2-3)^2(1)-(m^2-1)^2(m^2-4)(1)}{4}=1$$

Therefore (x_{n+1},y_{n+1}) is a solution of $x^2-(m^2-4)y^2=1.$
From the second equation of (3), $z_{n+1}=x_{n+1}+2\pm1.$
That is,

$$z_{n+1}=\frac{m(m^2-3)x_n+(m^2-1)(m^2-4)y_n}{2}+2\pm1$$

Thus,
$(x_{n+1},y_{n+1},z_{n+1})=\left(\frac{m(m^2-3)x_n+(m^2-1)(m^2-4)y_n}{2},\frac{(m^2-1)x_n+m(m^2-3)y_n}{2},\frac{m(m^2-3)x_n+(m^2-1)(m^2-4)y_n}{2}+2\pm1\right)$ is the solution set of (1) in which $x^2-Cy^2=x^2-(m^2-4)y^2=1.$
By the cases (i) and (ii), we get

$$x_{n+1}=\begin{cases}\dfrac{(m^2-2)x_n+m(m^2-4)y_n}{2}, & if\ m\ is\ even\\ \dfrac{m(m^2-3)x_n+(m^2-1)(m^2-4)y_n}{2}, & if\ m\ is\ odd\end{cases}$$
$$y_{n+1}=\begin{cases}\dfrac{mx_n+(m^2-2)y_n}{2}, & if\ m\ is\ even\\ \dfrac{(m^2-1)x_n+m(m^2-3)y_n}{2}, & if\ m\ is\ odd\end{cases}$$
$$z_{n+1}=\begin{cases}\dfrac{(m^2-2)x_n+m(m^2-4)y_n}{2}+2\pm1, & if\ m\ is\ even\\ \dfrac{m(m^2-3)x_n+(m^2-1)(m^2-4)y_n}{2}+2\pm1, & if\ m\ is\ odd\end{cases}$$

for $n\geq1$.

V. CONCLUSION

In this paper, we gave all the possible non-negative integer solutions for the equation (3) by using continued fraction method. And it is interesting to see that the researcher can also proceed for further results in this problem.

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