Solutions for nonlinear systems of integro-differential equations that contain multiple integrals of (V F) and (F V) types with isolated singular kernels.

Raad N. Butris
Email: raad.butris@uod.ac
University of Duhok ,College of basic education, Department of mathematics, Kurdistan region, Iraq.

Hewa Selman Faris
Email: hewa.faris@uod.ac

University of Duhok ,College of basic education, Department of mathematics, Kurdistan region, Iraq.

ABSTRACT- The purpose of this work is to provide the solution for nonlinear systems of Integro-differential equations that contain multiple integrals. These equations include integral equations of (Volterra-Fredholm) and (Fredholm-Volterra) types with isolated singular kernels. We prove some theorems in the existence, uniqueness and stability by using both methods Picard approximation and Banach fixed point theorem in closed and bounded domains.

Keywords-Existence, uniqueness and Stability solutions, Integro-differential equations (Volterra-Fredholm) and (Fredholm-Volterra), Picard approximation method, Banach fixed point theorem.

I. Introduction

An integral equation is an equation in which the unknown function $u(t)$ to be determined appears under the integral sign. A typical form of an integral equation in $u(t)$ is of the form:-

$$u(t) = f(t) + \int_{\alpha(s)}^{\beta(s)} K(t,s)u(s)ds \qquad \dots (I)$$

where $K(t,s)$ is called the kernel of the integral equation and $\alpha(t)$ and $\beta(t)$ are the limits of integration. In (I), it is easily observed that the unknown function $u(t)$ appears under the integral sign as stated above, and out of the integral sign in most other cases as will be discussed later. It is important to point out that the kernel $K(t,s)$ and the function $f(t)$ in (I) are given in advance. Our goal is to determine $u(t)$ that will satisfy (I), and this may be achieved by using different techniques that will be discussed in the forthcoming chapters. The primary concern of this text will be focused on introducing these methods and techniques supported by illustrative and practical examples. Integral equations arise naturally in physics, chemistry, biology and engineering applications modeled by initial value problems for a finite interval $[a,b]$. More details about the sources and origins of integral equations can be found in [7,8,18,19,21,22].

Integro differential equations have virtually a wide influence in almost applied science as mechanical and physical systems, electrical and mathematical models and biological fields. Recently, authors have worked on both analytically and numerically methods for solving differential equations. Various analytical and numerical techniques suggested for solving diverse sorts of differential equations [1,2,3,4,5,10,16].The conditions of mentioned theorems are important in analysis theorems and results are often obtained by using fixed point theorems by successive approximations which are investigated [4,5,6, 12,15,20,22]. We consider the differential equation as base of our article and the solution methods based on theorems have become known as Picard approximation and Banach fixed point methods and a spectral method [9,11,12,13,15,17].

Butris [6], assumes both methods Picard approximation and Banach fixed point theorem studies the existence, uniqueness and stability solution of integro-differential equations which has the following form:-

$\frac{dx}{dt} = Ax + By + \int_{-\infty}^{t} K(t,s) f\big(s, x(s), y(s)\big) ds + \int_{a}^{b} G(t,s) g\big(s, x(s), y(s)\big) ds$

$\frac{dy}{dt} = Cx + Ey + \int_{-\infty}^{t} \varphi\big(t, s, x(s), y(s)\big) ds + \int_{a}^{b} \psi\big(t, s, x(s), y(s)\big) ds$

where $x \in D \subseteq R^n$ and $y \in D_1 \subseteq R^n$, D and D_1 are closed and bounded

Our works studied some theorems for a systems of non-linear integro differential equations of (Volterra-Fredholm) and (Fredholm-Volterra) which have the forms:-

$$\frac{dx}{dt} = \big(A_1 + B_1(t)\big)x + \big(A_2 + B_2(t)\big)y + f(t, x, y, u) \qquad \dots (V\ F)$$
$$\frac{dy}{dt} = \big(C_1 + D_1(t)\big)x + \big(C_2 + D_2(t)\big)y + g(t, x, y, v) \qquad \dots (F\ V)$$

where $0 < \tau \le s \le t \le T$ and $x \in G_0, y \in G_1, u \in G_u, v \in G_v$ and G_0, G_1 are closed and bounded domains subset of R^n also G_v, G_u are bounded domains subset of R^m.

Suppose that the vector functions $f(t, x, y, u)$ and $g(t, x, y, v)$ are defined and continuous on domains

$$\begin{aligned}(t, x, y, u) &\in R^n \times G_0 \times G_1 \times G_u = (-\infty, \infty) \times R^{2n} \times R^m \\ (t, x, y, v) &\in R^n \times G_0 \times G_1 \times G_v = (-\infty, \infty) \times R^{2n} \times R^m\end{aligned} \qquad \dots (1)$$

where

G_0: $\|x - x_0\| \le r_x$, G_1: $\|y - y_0\| \le r_y$, G_u: $\|u\| \le d_u$ and G_v: $\|v\| \le d_v$

and

$$\begin{aligned}u(t) &= \int_{-\infty}^{t} \int_{a}^{b} K_1(t,s) \psi_1\big(t, s, x(s), y(s), \rho(s)\big)\, dt\, ds \\ v(t) &= \int_{a}^{b} \int_{-\infty}^{t} K_2(t,s) \psi_2\big(t, s, x(s), y(s), \omega(s)\big)\, ds\, dt,\end{aligned} \qquad \dots (2)$$

$$\begin{aligned}\rho(s) &= \int_{h_1(s)}^{h_2(s)} \big(x(\tau) - y(\tau)\big)\, d\tau \\ \omega(s) &= \int_{h_3(s)}^{h_4(s)} \big(x(\tau) - y(\tau)\big)\, d\tau\end{aligned} \qquad \dots (3)$$

Assume that the vector functions, $f(t, x, y, u), g(t, x, y, v), \psi_1(t, s, x, y, w)$ and $\psi_2(t, s, x, y, v)$ satisfy the following inequalities

$$\|f(t, x, y, u)\| \le \vartheta_1, \|g(t, x, y, v)\| \le \vartheta_2 \qquad \dots (4)$$

$$\|f(t, x_1, y_1, u_1) - f(t, x_2, y_2, u_2)\| \le \Gamma_1 \|x_1 - x_2\|^{\alpha} + \Gamma_2 \|y_1 - y_2\|^{\beta} + \Gamma_3 \|u_1 - u_2\|^{\gamma} \qquad \dots (5)$$

$$\|g(t, x_1, y_1, v_1) - g(t, x_2, y_2, v_2)\| \le \Sigma_1 \|x_1 - x_2\|^{\alpha} + \Sigma_2 \|y_1 - y_2\|^{\beta} + \Sigma_3 \|v_1 - v_2\|^{\gamma} \qquad \dots (6)$$

$$\|\psi_1(t, s, x_1, y_1, \rho_1) - \psi_1(t, s, x_2, y_2, \rho_2)\| \le h_1 \|x_1 - x_2\|^{\alpha} + h_2 \|y_1 - y_2\|^{\beta} + h_3 \|\rho_1 - \rho_2\|^{\gamma} \qquad \dots (7)$$

$$\|\psi_2(t,s,x_1,y_1,\omega_1)-\psi_2(t,s,x_2,y_2,\omega_2)\| \leq l_1\|x_1-x_2\|^{\alpha}+l_2\|y_1-y_2\|^{\beta}+l_3\|\omega_1-\omega_2\|^{\gamma} \quad \dots (8)$$

for all $t\in[0,T]$, $x, x_1, x_2\in G_0$, $y, y_1, y_2 \in G_1$, $u, u_1, u_2 \in G_u$ and $v, v_1, v_2 \in G_v$ where ϑ_1, ϑ_2, Γ_1, Γ_2, Γ_3, $\Sigma_1, \Sigma_2, \Sigma_3$, h_1, h_2, h_3, l_1, l_2, l_3 and $0<\alpha,\beta,\gamma<1$.

The positive matrices $K_1(t,s)$ and $K_2(t,s)$ are the singular kernels for the equations (VF) and (FV) such that

$$\|K_1(t,s)\|\leq \delta_1 e^{-\gamma_1(t-s)}$$
$$\|K_2(t,s)\|\leq \delta_2 e^{-\gamma_2(t-s)} \quad \dots (9)$$

where δ_1, δ_2, γ_1, and γ_2 are positive constants .

Also $A_1=(A_{1ij})$, $A_2=(A_{2ij})$, $B_1=(B_{1ij})$, $B_2=(B_{2ij})$, $C_1=(C_{1ij})$, $C_2=(C_{2ij})$, $D_1=(D_{1ij})$ and $D_2=(D_{2ij})$ are non-negative matrices, $i,j=1,2,\dots,n$.

The non-empty sets were defined as follows

$$G_f=G_0-r_x=G_0-(Q_1+\mathrm{R}_1TH_1) \quad \dots (10)$$
$$G_g=G_1-r_y=G_1-(Q_2+\mathrm{R}_2TH_2) \quad \dots (11)$$
$$G_{\psi_1}=G_2-d_u \quad \dots (12)$$
$$G_{\psi_2}=G_3-d_v \quad \dots (13)$$

where

$$d_u=\left((h_1+h_3H_1{}^{\gamma})\big(Q_1+\mathrm{R}_1TH_1(t)\big)+(h_2+h_3H_1(t)^{\gamma})\big(Q_2+\mathrm{R}_2TH_2(t)\big)\,\big(\frac{\delta_1}{\gamma_1^2}(e^{-\gamma_1 a}-e^{-\gamma_1 b})e^{\gamma_1 T}\big)\right)$$

$$d_v=\left((l_1+l_3H_2(t)^{\gamma})\big(Q_1+\mathrm{R}_1TH_1(t)\big)\left(\frac{\delta_2(b-a)}{\gamma_2}\right)+(l_2+l_3H_2(t)^{\gamma})\big(Q_2+\mathrm{R}_2TH_2(t)\big)\left(\frac{\delta_2(b-a)}{\gamma_2}\right)\right)$$

and

$$\|x_0\|\|e^{A_1t}-I\|=Q_1, \|y_0\|\|e^{C_2t}-I\|=Q_2$$
$$\left\|e^{A_1(t-s)}\right\|\leq \mathrm{R}_1, \left\|e^{C_2(t-s)}\right\|\leq \mathrm{R}_2$$
$$H_1=\|h_2(t)-h_1(t)\|, H_2=\|h_4(t)-h_3(t)\|$$
$$H_1^*=\|B_1(t)\|\|x_0\|+\|A_2+B_2(t)\|\|y_0\|+\vartheta_1$$
$$H_2^*=\|C_1+D_1(t)\|\|x_0\|+\|D_2(t)\|\|y_0\|+\vartheta_2$$

Suppose that the sequences of continuous vectors functions $\{x_m(t)\}_{m=1}^{\infty}$ and $\{y_m(t)\}_{m=1}^{\infty}$ are defined by the following:-

$$x_{m+1}(t)=x_0e^{A_1t}+\int_0^t e^{A_1(t-s)}\left(B_1(s)x_m(s)+\big(A_2+B_2(s)\big)y_m(s)+f\big(s,x_m(s),y_m(s),u_m(t)\big)\right)ds$$

with

$$x(0)=x_0,\ \mathrm{m}=0,1,2,\dots \quad \dots (14)$$

where

$$u_m(t)=\int_{-\infty}^{t}\int_a^b K_1(t,s)\psi_1\big(t,s,x_m(s),y_m(s),\rho_m(s)\big)\,ds\,ds,\ \ \rho_m(s)=\int_{h_1(s)}^{h_2(s)}\big(x_m(\tau)-y_m(\tau)\big)\,d\tau,\ \mathrm{m}=1,2,\dots$$

and

$$y_{m+1}(t)=y_0e^{C_2t}$$

$$y_{m+1}(t) = y_0 e^{C_2 t}$$

$$+\int_0^t e^{C_2(t-s)}\Big(\big(C_1 + D_1(s)\big)x_m(s) + D_2(s)y_m(s) + g\big(s, x_m(s), y_m(s), v_m(t)\big)\Big)\,ds$$

with

$y(0) = y_0$, m=0,1,2,... ...(15)

where

$v_m(t) = \int_a^b \int_{-\infty}^t K_2(t,s)\psi_2\big(t,s,x_m(s),y_m(s),\omega_m(t)\big)\,ds\,ds$, $\omega_m(t) = \int_{h_3(s)}^{h_4(s)}\big(x_m(\tau) - y_m(\tau)\big)\,d\tau$, m= 1,2,...

Suppose that the greatest Eigen-value of the matrix $\varphi_\gamma(T) = \begin{pmatrix}\varphi_1(T) & \varphi_2(T)\\ \varphi_3(T) & \varphi_4(T)\end{pmatrix}$ less than one, that is

$$\max_\gamma\Big(\varphi_\gamma(T)\Big) = \frac{(\varphi_1(T)+\varphi_4(T)) + \sqrt{(\varphi_1(T)+\varphi_4(T))^2 - 4(\varphi_1(T)\varphi_4(T) - \varphi_2(T)\varphi_3(T))}}{2} < 1 \quad \text{... (16)}$$

where

$$\varphi_1(T) = \mathrm{R}_1 T\|B_1(t)\| + \mathrm{R}_1 T\Gamma_1 + \mathrm{R}_1 T\Gamma_3\left((h_1 + h_3 H_1{}^\gamma)\Big(\frac{\delta_1}{\gamma_1^2}(e^{-\gamma_1 a} - e^{-\gamma_1 b})e^{\gamma_1 T}\,\Big)\right)^\gamma$$

$$\varphi_2(T) = \mathrm{R}_1 T\|A_2 + B_2(t)\| + \mathrm{R}_1 T\Gamma_2 + \mathrm{R}_1 T\Gamma_3\left((h_2 + h_3 H_1(t)^\gamma)\Big(\frac{\delta_1}{\gamma_1^2}(e^{-\gamma_1 a} - e^{-\gamma_1 b})e^{\gamma_1 T}\,\Big)\right)^\gamma$$

$$\varphi_3(T) = \mathrm{R}_2 T\|C_1 + D_1(t)\| + \mathrm{R}_2 T\Sigma_1 + \mathrm{R}_2 T\Sigma_3\left((l_1 + l_3 H_2(t)^\gamma)\left(\frac{\delta_2(b-a)}{\gamma_2}\right)\right)^\gamma \quad \text{... (17)}$$

$$\varphi_4(T) = \mathrm{R}_2 T\|D_2(t)\| + \mathrm{R}_2 T\Sigma_2 + \mathrm{R}_2 T\Sigma_3\left((l_2 + l_3 H_2(t)^\gamma)\left(\frac{\delta_2(b-a)}{\gamma_2}\right)\right)^\gamma$$

Definition 1. [16] Let $\{\mathrm{f}_n\}$ be a sequence of real valued functions on a set S. We say that $\{\mathrm{f}_n\}$ converges uniformly on S to the function f, and we write $\mathrm{f}_n \to \mathrm{f}$ uniformly, if
for each $\varepsilon > 0$, there exists N in $\mathbb{N}$ such that, $\left|\mathrm{f}_{N+p}(x) - \mathrm{f}(x)\right| < \varepsilon \;\forall\, x \in S$ and all $p \in \mathbb{N}$.

Definition 2. [16] Let $\mathrm{f}: \mathrm{S} \to \mathbb{R}$. We say that f satisfies a Lipschitz condition, if there is a constant $C > 0$ such that , $|\mathrm{f}(x_1) - \mathrm{f}(x_2)| \le C|x_1 - x_2| \;\forall\, x_1, x_2 \in S$,where C is a Lipschitz constant .

Definition 3. [20] Let $\mathrm{f}: \mathrm{S} \to \mathbb{R}$. We say that f is a contraction mapping, if it is Lipschitz with a Lipschitz constant $0 < C < 1$.

Definition 4. [9] A function f satisfies a Hölder codition (Hölder inequality) of order β, $0 < \beta < 1$, on $[a,b] \in \mathbb{R}$,if there is a constant $K > 0$, so that for all $x_1, x_2 \in [a,b]$
$|\mathrm{f}(x_1) - \mathrm{f}(x_2)| \le K|x_1 - x_2|^\beta$.

Definition 5. [12] Suppose $x(t)$ be a continuous solution of differential equation and $\hat{x}(t)$ any other solution. If for each $\varepsilon > 0$, there exists $\delta > 0$ such that
$\|x(t_0) - \hat{x}(t_0)\| < \delta$ for some t_0 that satisfies $\|x(t) - \hat{x}(t)\| < \varepsilon \;\forall\, t \ge t_0$

Lemma 1. [18] Let $a_i \in \mathbb{R}$ and $q \in (0, \infty)$, then we obtained

(1) If $a_i \geq 0$ and $q \geq 1$, then for $1 \leq i \leq m$, $\sum_{i=1}^{m} a_i^q \leq (\sum_{i=1}^{m} a_i)^q \leq m^{q-1} \sum_{i=1}^{m} a_i^q$.
The reverse holds if $0 < q \leq 1$. Hence, for $1 \leq i \leq m$, $(\sum_{i=1}^{m} a_i)^q \leq \sum_{i=1}^{m} a_i^q$.

(2) If $a_i, b_i \in \mathbb{R}$ and $0 < q \leq 1$, then for $1 \leq i \leq m$, $\|a_i - b_i\|^q \leq \|a_i - b_i\|$.

Theorem 1. [20] Let S be a complete subset of $\mathbb{R}$. A contraction mapping $I: S \to S$ has a unique fixed point.

Theorem 2. [20] Suppose S be a Banach space and T is a contraction mapping in S. Then T has only one unique fixed point in S.

II. Existence solutions of integro-differential equations of (VF) and (FV) types.

The existence solutions for a systems (VF) and (FV) have proved by the following theorems.
Lemma2. Under the conditions and hypothesis for above, then the following inequalities:-

$$\text{i)} \|\rho_m(t) - \rho_{m-1}(t)\|^\gamma \leq \|x_m(t) - x_{m-1}(t)\|^\gamma H_1^\gamma + \|y_m(t) - y_{m-1}(t)\|^\gamma H_1^\gamma \quad ...(18)$$

and

$$\text{ii)} \|\omega_m(t) - \omega_{m-1}(t)\|^\gamma \leq \|x_m(t) - x_{m-1}(t)\|^\gamma H_2^\gamma + \|y_m(t) - y_{m-1}(t)\|^\gamma H_2^\gamma \quad ...(19)$$

are hold for all t$\in [0, T]$ and $m = 1,2,3, \dots$.

Proof (i). From Hölder condition and (14), we get

$$\|\rho_m(t) - \rho_{m-1}(t)\|^\gamma \leq \|x_m(t) - x_{m-1}(t)\|^\gamma H_1^\gamma + \|y_m(t) - y_{m-1}(t)\|^\gamma H_1^\gamma$$

$$\leq \|x_m(t) - x_{m-1}(t)\|^\gamma H_1^\gamma + \|y_m(t) - y_{m-1}(t)\|^\gamma H_1^\gamma.$$

for all t$\in [0, T]$ and $m = 1,2,3, \dots$.

Proof (ii). By the same way, from (15), we have

$$\|\omega_m(t) - \omega_{m-1}(t)\|^\gamma \leq \|x_m(s) - x_{m-1}(s)\|^\gamma H_2^\gamma + \|y_m(s) - y_{m-1}(s)\|^\gamma H_2^\gamma$$

$$\leq \|x_m(t) - x_{m-1}(t)\|^\gamma H_2^\gamma + \|y_m(t) - y_{m-1}(t)\|^\gamma H_2^\gamma$$

for all t$\in [0, T]$ and $m = 1,2,3, \dots$.

Theorem 3. Suppose that $u(t)$, $v(t)$ *and* $\psi_1(t, s, x, y, w)$, $\psi_2(t, s, x, y, v)$ be a continuous vector functions in the domain (1) and satisfies the inequalities(7) ,(8) , (9) and the relations (11),(12).Then the following inequalities :-

$$\text{i)} \|u_m(t) - u_{m-1}\| \leq \left(h_1 + h_3 H_1^\gamma\right)\left(\frac{\delta_1}{\gamma^2{}_1}(e^{-\gamma_1 a} - e^{-\gamma_1 b})e^{\gamma_1 T}\right)\|x_m(t) - x_{m-1}(t)\| + \left(h_2 + h_3 H_1^\gamma\right)\left(\frac{\delta_1}{\gamma^2{}_1}(e^{-\gamma_1 a} - e^{-\gamma_1 b})e^{\gamma_1 T}\right)\|y_m(t) - y_{m-1}(t)\| \quad \dots(20)$$

$$\text{ii)} \|v_m(t) - v_{m-1}\| \left(l_1 + l_3 H_2^\gamma\right)\left(\frac{\delta_2(b-a)}{\gamma_2}\right)\|x_m(t) - x_{m-1}(t)\| + \left(l_2 + l_3 H_2^\gamma\right)\left(\frac{\delta_2(b-a)}{\gamma_2}\right)\|y_m(t) - y_{m-1}(t)\| \quad ...(21)$$

are satisfied for all t$\in [0, T]$ *and* $m = 1,2,3, \dots$.

Proof (i) . From lemma2(i) and the functions $u_m(t)$ in (14) , we get

$$\|u_m(t)-u_{m-1}\| \le \left(\frac{\delta_1}{{\gamma^2}_1}(e^{-\gamma_1 a}-e^{-\gamma_1 b})e^{\gamma_1 T}\right)\left(h_1\|x_m(t)-x_{m-1}(t)\|^{\alpha}+h_2\|y_m(t)-y_{m-1}(t)\|^{\beta}+h_3\|\rho_m(t)-\rho_{m-1}(t)\|^{\gamma}\right)$$

$$\le \left(\frac{\delta_1}{{\gamma^2}_1}(e^{-\gamma_1 a}-e^{-\gamma_1 b})e^{\gamma_1 T}\right)\left(h_1\|x_m(t)-x_{m-1}(t)\|^{\alpha}+h_2\|y_m(t)-y_{m-1}(t)\|^{\beta}+h_3H_1^{\gamma}\|x_m(t)-x_{m-1}(t)\|^{\gamma}+h_3H_1^{\gamma}\|y_m(t)-y_{m-1}(t)\|^{\gamma}\right)$$

So that

$$\|u_m(t)-u_{m-1}\| \le \left(h_1+h_3H_1^{\gamma}\right)\left(\frac{\delta_1}{{\gamma^2}_1}(e^{-\gamma_1 a}-e^{-\gamma_1 b})e^{\gamma_1 T}\right)\|x_m(t)-x_{m-1}(t)\| + \left(h_2+h_3H_1^{\gamma}\right)\left(\frac{\delta_1}{{\gamma^2}_1}(e^{-\gamma_1 a}-e^{-\gamma_1 b})e^{\gamma_1 T}\right)\|y_m(t)-y_{m-1}(t)\|.$$

Proof(ii). Similarly, from lemma2(ii) and the functions $v_m(t)$ in (15) , we get

$$\|v_m(t)-v_{m-1}(t)\| \le \left(\frac{\delta_2(b-a)}{\gamma_2}\right)\left(l_1\|x_m(t)-x_{m-1}(t)\|^{\alpha}+l_2\|y_m(t)-y_{m-1}(t)\|^{\beta}+l_3\|\omega_m(t)-\omega_{m-1}(t)\|^{\gamma}\right)$$

$$\le \left(\frac{\delta_2(b-a)}{\gamma_2}\right)\left(l_1\|x_m(t)-x_{m-1}(t)\|^{\alpha}+l_2\|y_m(t)-y_{m-1}(t)\|^{\beta} + l_3\left(H_2^{\gamma}\|x_m(t)-x_{m-1}(t)\|^{\gamma}+H_2^{\gamma}\|y_m(t)-y_{m-1}(t)\|^{\gamma}\right)\right)$$

$$\le \left(\frac{\delta_2(b-a)}{\gamma_2}\right)\left(l_1\|x_m(t)-x_{m-1}(t)\|+l_2\|y_m(t)-y_{m-1}(t)\|+l_3H_2^{\gamma}\|x_m(t)-x_{m-1}(t)\| + l_3H_2^{\gamma}\|y_m(t)-y_{m-1}(t)\|\right)$$

Thus

$$\|v_m(t)-v_{m-1}(t)\| \le \left(l_1+l_3H_2^{\gamma}\right)\left(\frac{\delta_2(b-a)}{\gamma_2}\right)\|x_m(t)-x_{m-1}(t)\|+\left(l_2+l_3H_2^{\gamma}\right)\left(\frac{\delta_2(b-a)}{\gamma_2}\right)\|y_m(t)-y_{m-1}(t)\|.$$

Theorem 4. Suppose that $f(t,x,y,u)$ and $g(t,x,y,v)$ on the domain(1) are respectively continuous vector functions and satisfy the inequalities $(4)-(9)$ and relations $(10)-(13)$ and the condition(16). Then there exists a sequences of vector functions $\{x_m(t)\}_{m=0}^{\infty}$ and $\{y_m(t)\}_{m=0}^{\infty}$, uniformly convergent to the limit vector functions $x(t)$ *and* $y(t)$ which satisfying the following integral equations:-

$$x(t)=x_0e^{A_1t}+\int_0^t e^{A_1(t-s)}\left(B_1(s)x(s)+\left(A_2+B_2(s)\right)y(s)+f\left(s,x(s),y(s),u(t)\right)\right)ds \qquad \dots(22)$$

$$y(t)=y_0e^{C_2t}+\int_0^t e^{C_2(t-s)}\left(\left(C_1+D_1(s)\right)x(s)+D_2(s)y(s)+g\left(s,x(s),y(s),v(t)\right)\right)ds \qquad \dots(23)$$

Provided that

$$\begin{pmatrix} \|x(t)-x_0\| \\ \|y(t)-y_0\| \end{pmatrix} \le \begin{pmatrix} Q_1 + \mathrm{R}_1 T H_1^* \\ Q_2 + \mathrm{R}_2 T H_2^* \end{pmatrix} \quad \ldots(24)$$

$$\begin{pmatrix} \|x_m(t)-x(t)\| \\ \|y_m(t)-y(t)\| \end{pmatrix} \le \varphi_\gamma^m \left(E-\varphi_\gamma\right)^{-1} \Omega_1(T)\,. \quad \ldots(25)$$

Proof. From the sequence of functions (14) for $m = 1,2,3,\ldots$ and by mathematical induction, we have

$$\|x_m(t)-x_0\| \le \|x_0 e^{A_1 t} - x_0\| + \int_0^t \|e^{A_1(t-s)}\|\|B_1(s)x_0 + (A_2+B_2(t))y_0 + f(t,x_0,y_0,u_0(t))\|\,ds$$

$$\le Q_1 + \mathrm{R}_1 T(\|B_1(t)\|\|x_0\| + \|A_2+B_2(t)\|\|y_0\| + \vartheta_1)$$

$\le Q_1 + \mathrm{R}_1 T H_1^*$. That is $x_m(t) \in G_0\ \forall\, x_0 \in G_f$ and $y_0 \in G_g$.

Also from the sequence of functions (15) for $m = 1,2,3,\ldots$ and by mathematical induction, we get

$$\|y_m(t)-y_0\|$$

$$\le \|y_0\|\|e^{C_2 t} - I\| + \int_0^t \|e^{C_2(t-s)}\|\left(\|C_1+D_1(t)\|\|x_0\| + \|D_2(t)\|\|y_0\| + \|g(t,x_0,y_0,v_0(t))\|\right) ds$$

$$\le Q_2 + \mathrm{R}_2 T(\|C_1+D_1(t)\|\|x_0\| + \|D_2(t)\|\|y_0\| + \vartheta_2)$$

$\le Q_2 + \mathrm{R}_2 T H_2^*$ that is $y_m(t) \in G_1\ \forall\, x_0 \in G_f$ and $y_0 \in G_g$.

Next, we prove that the sequence of functions (14) and (15) convergent uniformly on the domains (10) and (11) respectively. Then for $m = 0,1,2,3,\ldots$ we obtain the following

$$\|x_{m+1}(t)-x_m(t)\| \le \mathrm{R}_1 t\big(\|B_1(t)\|\|x_m(t)-x_{m-1}(t)\| + \|A_2+B_2(t)\|\|y_m(t)-y_{m-1}(t)\| + \Gamma_1\|x_m(t)-x_{m-1}(t)\|^\alpha + \Gamma_2\|y_m(t)-y_{m-1}(t)\|^\beta + \Gamma_3\|u_m(t)-u_{m-1}(t)\|^\gamma\big)$$

$$\le \mathrm{R}_1 t\Bigg(\|B_1(t)\|\|x_m(t)-x_{m-1}(t)\| + \|A_2+B_2(t)\|\|y_m(t)-y_{m-1}(t)\| + \Gamma_1\|x_m(t)-x_{m-1}(t)\|^\alpha + \Gamma_2\|y_m(t)-y_{m-1}(t)\|^\beta + \Gamma_3\left(\left(h_1+h_3H_1^\gamma\right)\left(\frac{\delta_1}{\gamma^2{}_1}(e^{-\gamma_1 a}-e^{-\gamma_1 b})e^{\gamma_1 t}\right)\right)^\gamma \|x_m(t)-x_{m-1}(t)\|^\gamma + \Gamma_3\left(\left(h_2+h_3H_1^\gamma\right)\left(\frac{\delta_1}{\gamma^2{}_1}(e^{-\gamma_1 a}-e^{-\gamma_1 b})e^{\gamma_1 t}\right)\right)^\gamma \|y_m(t)-y_{m-1}(t)\|^\gamma\Bigg)$$

$$\le \left(\mathrm{R}_1 t\|B_1(t)\| + \mathrm{R}_1 t\Gamma_1 + \mathrm{R}_1 t\Gamma_3\left(\left(h_1+h_3H_1^\gamma\right)\left(\frac{\delta_1}{\gamma^2{}_1}(e^{-\gamma_1 a}-e^{-\gamma_1 b})e^{\gamma_1 t}\right)\right)^\gamma\right)\|x_m(t)-x_{m-1}(t)\| + \left(\mathrm{R}_1 t\|A_2+B_2(t)\| + \mathrm{R}_1 t\Gamma_2 + \mathrm{R}_1 t\Gamma_3\left(\left(h_2+h_3H_1^\gamma\right)\left(\frac{\delta_1}{\gamma^2{}_1}(e^{-\gamma_1 a}-e^{-\gamma_1 b})e^{\gamma_1 t}\right)\right)^\gamma\right)\|y_m(t)-y_{m-1}(t)\| \quad \ldots(26)$$

Also

$$\|y_{m+1}(t)-y_m(t)\| \leq R_2 t\big(\|C_1+D_1(t)\|\|x_m(t)-x_{m-1}(t)\|+\|D_2(t)\|\|y_m(t)-y_{m-1}(t)\| + \big\|g\big(s,x_m(s),y_m(s),v_m(s)\big)-g\big(s,x_m(s),y_m(s),v_m(s)\big)\big\|\big)$$

$$\leq R_2 t(\|C_1+D_1(t)\|\|x_m(t)-x_{m-1}(t)\|+\|D_2(t)\|\|y_m(t)-y_{m-1}(t)\|+\Sigma_1\|x_m(t)-x_{m-1}(t)\|^{\alpha} + \Sigma_2\|y_m(t)-y_{m-1}(t)\|^{\alpha}+\Sigma_3\|v_m(t)-v_{m-1}(t)\|^{\alpha})$$

$$\leq R_2 t\|C_1+D_1(t)\|\|x_m(t)-x_{m-1}(t)\|+R_2 t\|D_2(t)\|\|y_m(t)-y_{m-1}(t)\|+R_2 t\Sigma_1\|x_m(t)-x_{m-1}(t)\|+ R_2 t\Sigma_2\|y_m(t)-y_{m-1}(t)\|+R_2 t\Sigma_3\left(\left(l_1+l_3H_2^{\gamma}\right)\left(\frac{\delta_2(b-a)}{\gamma_2}\right)^{\gamma}\right)\|x_m(t)-x_{m-1}(t)\|+R_2 t\Sigma_3\left(\left(l_2+ l_3H_2^{\gamma}\right)\left(\frac{\delta_2(b-a)}{\gamma_2}\right)^{\gamma}\right)\|y_m(t)-y_{m-1}(t)\|$$

$$\leq \left(R_2 t\|C_1+D_1(t)\|+R_2 t\Sigma_1+R_2 t\Sigma_3\left(\left(l_1+l_3H_2^{\gamma}\right)\left(\frac{\delta_2(b-a)}{\gamma_2}\right)^{\gamma}\right)\right)\|x_m(t)-x_{m-1}(t)\| + \left(R_2 t\|D_2(t)\|+R_2 t\Sigma_2+R_2 t\Sigma_3\left(\left(l_2+l_3H_2^{\gamma}\right)\left(\frac{\delta_2(b-a)}{\gamma_2}\right)^{\gamma}\right)\right)\|y_m(t) - y_{m-1}(t)\| \qquad \dots(27)$$

Rewrite the inequalities (27) and (28) in a vector form:-

$$\Omega_{m+1}(t)\leq \varphi_{\gamma}(t)\Omega_m(t) \qquad \dots(28)$$

where

$$\Omega_{m+1}(t)=\begin{pmatrix}\|x_{m+1}(t)-x_m(t)\|\\ \|y_{m+1}(t)-y_m(t)\|\end{pmatrix},\ \Omega_m(t)=\begin{pmatrix}\|x_m(t)-x_{m-1}(t)\|\\ \|y_m(t)-y_{m-1}(t)\|\end{pmatrix},$$

$$\varphi_{\gamma}(t)=\begin{pmatrix}\varphi_1(t) & \varphi_2(t)\\ \varphi_3(t) & \varphi_4(t)\end{pmatrix}$$

and $\varphi_1(t)= R_1 t\|B_1(t)\|+R_1 t\Gamma_1+R_1 t\Gamma_3\left(\left(h_1+h_3H_1^{\gamma}\right)\left(\frac{\delta_1}{\gamma^2{}_1}(e^{-\gamma_1 a}-e^{-\gamma_1 b})e^{\gamma_1 t}\right)\right)^{\gamma}$,

$$\varphi_2(t)=R_1 t\|A_2+B_2(t)\|+R_1 t\Gamma_2+R_1 t\Gamma_3\left(\left(h_2+h_3H_1^{\gamma}\right)\left(\frac{\delta_1}{\gamma^2{}_1}(e^{-\gamma_1 a}-e^{-\gamma_1 b})e^{\gamma_1 t}\right)\right)^{\gamma},$$

$$\varphi_3(t)=R_2 t\|C_1+D_1(t)\|+R_2 t\Sigma_1+R_2 t\Sigma_3\left(\left(l_1+l_3H_2^{\gamma}\right)\left(\frac{\delta_2(b-a)}{\gamma_2}\right)^{\gamma}\right)$$

and

$$\varphi_4(t)=R_2 t\|D_2(t)\|+R_2 t\Sigma_2+R_2 t\Sigma_3\left(\left(l_2+l_3H_2^{\gamma}\right)\left(\frac{\delta_2(b-a)}{\gamma_2}\right)^{\gamma}\right)$$

Then by iterations of inequality (27) with taking the max. of two sides of it , we obtain that

$$\Omega_{m+1}(T)\leq \varphi_{\gamma}(T)\Omega_m(T)\leq\cdots\leq\varphi_{\gamma}^m(T)\Omega_1(T) \qquad \dots(29)$$

where

$\Omega_1(T) = \begin{pmatrix} Q_1 + R_1 T H_1^* \\ Q_2 + R_2 T H_2^* \end{pmatrix}$ and $\varphi_\gamma(T) = \begin{pmatrix} \varphi_1(T) & \varphi_2(T) \\ \varphi_3(T) & \varphi_4(T) \end{pmatrix}$.

From(27) and the condition (16) we find that

$\sum_{i=0}^{l-1}(\varphi_\gamma(T))^{m+i} \le \varphi_\gamma^m(T)\sum_{i=0}^{\infty}(\varphi_\gamma(T))^i \le \varphi_\gamma^m(T)(E - \varphi_\gamma(T))^{-1}$ and hence $\lim_{n\to\infty} \varphi_\gamma^m(t) = 0$.

Then the sequences of functions $\begin{pmatrix} x_m(t) \\ y_m(t) \end{pmatrix}$ convergent uniformly on the domains (10) and (11).

Let

$\lim_{m\to\infty} x_m(t) = x(t)$ and $\lim_{m\to\infty} y_m(t) = y(t)$.

Then from (26) , we can prove that the inequalities (24) and(25) are true for all $m \ge 0$.

III. Uniqueness solutions of integro-differential equations of (VF) and (FV) types.

The uniqueness solutions for a systems (VF) and (FV) are given by the following theorem.

Theorem 5. With the hypotheses and all conditions of the theorem 4, the vector solution of (VF) and (FV) is a unique on the domains (10) and (11).

Proof. Investigating the difference of both solutions $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ and $\begin{pmatrix} r(t) \\ w(t) \end{pmatrix}$ leads us to verify the uniqueness solution of (VF) and (FV) .

$$\|x(t) - r(t)\| \le \varphi_1(T)\,\|x(t) - r(t)\| + \varphi_2(T)\|y(t) - w(t)\| \qquad ...(30)$$

Same iterations are done for finding $\|y_I(t) - y_{II}(t)\|$. Thus

$$\|y(t) - w(t)\| \le \varphi_3(T)\,\|x(t) - r(t)\| + \varphi_4(T)\|y(t) - w(t)\| \qquad ...(31)$$

Rewrite (30) and (31) in a vector form , we have

$$\begin{pmatrix} \|x(t) - r(t)\| \\ \|y(t) - w(t)\| \end{pmatrix} \le \begin{pmatrix} \varphi_1(T) & \varphi_2(T) \\ \varphi_3(T) & \varphi_4(T) \end{pmatrix} \begin{pmatrix} \|x(t) - r(t)\| \\ \|y(t) - w(t)\| \end{pmatrix}$$

By the condition (16), we conclude that $x(t) = r(t)$ and $y(t) = w(t)$. This implies (VF) and (FV) has a unique solution.

V. Stability solutions of integro-differential equations of (VF) and (FV) types.

The stability solution of (VF) and (FV) will be introduced by the following theorem.

Theorem 6. Let $f(t, x, y, u)$ and $g(t, x, y, v)$ be continuous vector functions that satisfy the inequalities and conditions in theorem 4. Let $\begin{pmatrix} \hat{x}(t) \\ \hat{y}(t) \end{pmatrix}$ be another solution of (VF) and (FV). Then the solutions $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ is stable for all t$\ge$ 0.

Proof. We define $\begin{pmatrix} \hat{x}(t) \\ \hat{y}(t) \end{pmatrix}$ a solution of (VF) and (FV). Then we obtain the following

$$\|x(t) - \hat{x}(t)\|$$

$$\le \|e^{A_1 t}\|\|x_0 - \hat{x}_0\| + R_1 T\big(\|B_1(t)\|\|x(t) - \hat{x}(t)\| + \|A_2 + B_2(t)\|\|y(t) - \hat{y}(t)\| + \Gamma_1\|x(t) - \hat{x}(t)\|^\alpha + \Gamma_2\|y(t) - \hat{y}(t)\|^\beta + \Gamma_3\|u(t) - \hat{u}(t)\|^\gamma\big)$$

$$\leq \|e^{A_1 t}\|\|x_0 - \hat{x}_0\| + R_1 T\Bigg(\|B_1(t)\|\|x(t) - \hat{x}(t)\| + \|A_2 + B_2(t)\|\|y(t) - \hat{y}(t)\| + \Gamma_1\|x(t) - \hat{x}(t)\|^{\alpha} + \Gamma_2\|y(t) - \hat{y}(t)\|^{\beta} + \Gamma_3\left(\left(h_1 + h_3 H_1^{\gamma}\right)\left(\frac{\delta_1}{{\gamma^2}_1}(e^{-\gamma_1 a} - e^{-\gamma_1 b})e^{\gamma_1 T}\right)\right)^{\gamma}\|x(t) - \hat{x}(t)\| + \Gamma_3\left(\left(h_2 + h_3 H_1^{\gamma}\right)\left(\frac{\delta_1}{{\gamma^2}_1}(e^{-\gamma_1 a} - e^{-\gamma_1 b})e^{\gamma_1 T}\right)\right)^{\gamma}\|y(t) - \hat{y}(t)\|\Bigg)$$

$$\leq \|e^{A_1 t}\|\|x_0 - \hat{x}_0\| + \left(R_1 T\|B_1(t)\| + R_1 T\Gamma_1 + R_1 T\Gamma_3\left(\left(h_1 + h_3 H_1^{\gamma}\right)\left(\frac{\delta_1}{{\gamma^2}_1}(e^{-\gamma_1 a} - e^{-\gamma_1 b})e^{\gamma_1 T}\right)\right)^{\gamma}\right)\|x(t) - \hat{x}(t)\| + \left(R_1 T\|A_2 + B_2(t)\| + R_1 T\Gamma_2 + R_1 T\Gamma_3\left(\left(h_2 + h_3 H_1^{\gamma}\right)\left(\frac{\delta_1}{{\gamma^2}_1}(e^{-\gamma_1 a} - e^{-\gamma_1 b})e^{\gamma_1 T}\right)\right)^{\gamma}\right)\|y(t) - \hat{y}(t)\| \quad \ldots (31)$$

By the same duplications we have

$$\|y(t) - \hat{y}(t)\| \leq \|e^{C_2 t}\|\|y_0 - \hat{y}_0\| + \left(R_2 T\|C_1 + D_1(t)\| + R_2 T\Sigma_1 + R_2 T\Sigma_3\left(\left(l_1 + l_3 H_2^{\gamma}\right)\left(\frac{\delta_2(b-a)}{\gamma_2}\right)\right)^{\gamma}\right)\|x(t) - \hat{x}(t)\| + \left(R_2 T\|D_2(t)\| + R_2 T\Sigma_2 + R_2 T\Sigma_3\left(\left(l_2 + l_3 H_2^{\gamma}\right)\left(\frac{\delta_2(b-a)}{\gamma_2}\right)\right)^{\gamma}\right)\|y(t) - \hat{y}(t)\|$$

Collect a bove in a vector form as follows:-

$$\begin{pmatrix}\|x(t) - \hat{x}(t)\| \\ \|y(t) - \hat{y}(t)\|\end{pmatrix} \leq \begin{pmatrix}\|e^{A_1 t}\|\|x_0 - \hat{x}_0\| \\ \|e^{C_2 t}\|\|y_0 - \hat{y}_0\|\end{pmatrix} + \begin{pmatrix}\varphi_1(T) & \varphi_2(T) \\ \varphi_3(T) & \varphi_4(T)\end{pmatrix}\begin{pmatrix}\|x(t) - \hat{x}(t)\| \\ \|y(t) - \hat{y}(t)\|\end{pmatrix}$$

.

Thus

$$\begin{pmatrix}\|x(t) - \hat{x}(t)\| \\ \|y(t) - \hat{y}(t)\|\end{pmatrix} \leq \begin{pmatrix}\|e^{A_1 t}\|\|x_0 - \hat{x}_0\| \\ \|e^{C_2 t}\|\|y_0 - \hat{y}_0\|\end{pmatrix} + \varphi_{\gamma}^{m}(T)\begin{pmatrix}\|x(t) - \hat{x}(t)\| \\ \|y(t) - \hat{y}(t)\|\end{pmatrix}.$$

From the definition of stability we have $\begin{pmatrix}\|x_0 - \hat{x}_0\| \\ \|y_0 - \hat{y}_0\|\end{pmatrix} \leq \begin{pmatrix}\delta_1 \\ \delta_2\end{pmatrix}$, then

$$\begin{pmatrix}\|x(t) - \hat{x}(t)\| \\ \|y(t) - \hat{y}(t)\|\end{pmatrix} \leq \begin{pmatrix}\|e^{A_1 t}\| & 0 \\ 0 & \|e^{C_2 t}\|\end{pmatrix}\begin{pmatrix}\delta_1 \\ \delta_2\end{pmatrix} + \varphi_{\gamma}^{m}(T)\begin{pmatrix}\|x(t) - \hat{x}(t)\| \\ \|y(t) - \hat{y}(t)\|\end{pmatrix}$$

Then $\begin{pmatrix}\|x(t) - \hat{x}(t)\| \\ \|y(t) - \hat{y}(t)\|\end{pmatrix} \leq \begin{pmatrix}\varepsilon_1 \\ \varepsilon_2\end{pmatrix}$, where $\begin{pmatrix}\delta_1 \\ \delta_2\end{pmatrix} = \begin{pmatrix}\varepsilon_1 / \|e^{A_1 t}\| \\ \varepsilon_2 / \|e^{C_2 t}\|\end{pmatrix}$.

This tends to that $\begin{pmatrix}x(t) \\ y(t)\end{pmatrix}$ is a stable solution of (VF) and (FV) for all t$\geq$ 0.

VI.Existence and uniqueness solution of (VF) and (FV).

In this section, we prove the existence and uniqueness theorem of (VF) and (FV) by using Banach fixed point theorem.

.Theorem 7 (Banach fixed point theorem). Suppose that $f(t,x,y,u)$ and $g(t,x,y,v)$ are defined by $((V\ F),(F\ V))$ and continuous in domains (10) and (11) that satisfy the inequalities and conditions in theorem 3. Then the system $((V\ F),(F\ V))$ has a unique solution that satisfies the Banach fixed point theorem.

Proof. Suppose that $(S,\|.\|)$ be a Banach space on $C[0,T]$ and T^* be a mapping on S by the following vector functions

$$T^*x(t) = x_0 e^{A_1 t} + \int_0^t e^{A_1(t-s)} \left(B_1(s)x(s) + \left(A_2 + B_2(s)\right)y(s) + f(s, x(s), y(s), u(s))\right) ds$$
$$T^*y(t) = y_0 e^{C_2 t} + \int_0^t e^{C_2(t-s)} \left(\left(C_1 + D_1(s)\right)x(s) + D_2(s)y(s) + g\left(s, x(s), y(s), v(s)\right)\right) ds \quad \text{... (32)}$$

Easily to prove $T^*: C[0,T] \to C[0,T]$. Now, we shall to prove that T^* is a contraction mapping on $[0,T]$.

Let $x(t), z(t), y(t)$ and $w(t)$ are vector functions on $[0,T]$, then

$$\|T^*x(t) - T^*z(t)\| = \max_{t\in[0,T]} \{|T^*x(t) - T^*z(t)|\}$$

and

$$\|T^*y(t) - T^*v(t)\| = \max_{t\in[0,T]} \{|T^*y(t) - T^*v(t)|\}$$

Thus

$$\|T^*x\,(t) - T^*z(t)\| \leq \varphi_1(T)\|x(t) - z(t)\| + \varphi_2(T)\|y(t) - w(t)\| \quad \text{... (33)}$$

and

$$\|T^*y(t) - T^*w(t)\| \leq\leq \varphi_3(T)\|x_1(t) - x_2(t)\| + \varphi_4(T)\|y_1(t) - y_2(t)\| \quad \text{... (34)}$$

Rewrite (31) and (32) in a vector form, we get

$$\begin{pmatrix} \|T^*x\,(t) - T^*z(t)\| \\ \|T^*y\,(t) - T^*w(t)\| \end{pmatrix} \leq \begin{pmatrix} \varphi_1(T) & \varphi_2(T) \\ \varphi_3(T) & \varphi_4(T) \end{pmatrix} \begin{pmatrix} \|x\,(t) - z(t)\| \\ \|y\,(t) - w(t)\| \end{pmatrix}$$

From condition (16) , then T^* is a contraction mapping.
From Banach fixed point theorem then there exists fixed points $x(t)$ and $y(t)$ such that $T^*x(t) = x(t)$ and $T^*y(t) = y(t)$. Therefore (22) and (23) are a unique solutions of $(VF) and\ (FV)$.

Remak . We can state and prove a theorems 4,5,6,7 by using a Lipchitz condition for nonlinear systems of Integro-differential equations that contain multiple integrals (VF) and (FV), if α, β and γ, are equal to one.

VII. Conclusion.

In this work, we study the solution of nonlinear systems of Integro-differential equations that contain multiple integrals of (Volterra-Fredholm) and (Fredholm-Volterra) types with isolated singular kernels. We prove some theorems in the existence, uniqueness and stability by using both methods Picard approximation and Banach fixed point theorem in closed and bounded domains.

References

[1] Ahmad, B., Nieto, J. J.,A class of differential equations of fractional order with multi-point boundary conditions. Georgian Math. J. 2014, 21, 243–248. https://doi.org/10.1515/gmj.,(2014).

[2] Ahmad, B., Nieto, J. J., Existence results for a coupled system of nonlinear fractional differential equations with three-point boundary conditions Comput.Math. Appl. 58, 1838-1843 (2009) https://doi.org/10.1016/j.camwa.2009.07.091

[3] Benchohra, M., Cabada, A., Seba, D., An existence result for nonlinear fractional differential equations on https://doi.org/10.1155/2009/628916 Banach spaces.Bound.Value Probl.(2009).

[4] Butris R. N., Rafeeq A. Sh., Existence and uniqueness solution of a Boundary value problems for integro-differential equation with parameter, Italian journal of pure and applied mathematics, 37(2017), 431-440.

[5] Butris R. N., Haji H. M., Existence, uniqueness and stability solutions for nonlinear systems of differential equations with higher derivatives, China, Journal of Xi'an University of Architecture &. Technology, Volume XII, Issue IV, (2020), , Page No: 3069-3088.

[6] Butris R. N., Haji H. M., , Existence, uniqueness and stability solution of Volterra- Friedholm of integro-differential equations, International Journal of Mechanical Engineering and Technology, Volume 10, Issue 07, July(2019), pp. 68-81.

[7] Coddington, E. A., An introduction to ordinary differential equations, Prentice Hall, Inc., Englewood Cliffs, N. J., (1961).

[8] Coppell, W. A., Stability and asymptotic behavior of differential equation, D. C. Heath, Boston, (1965).

[9] Fima C. K., Introduction to Stochastic Calculus with Applications, Second Edition, Imperial College Press 57 Shelton Street, London, WC2H9HE, 2005, 163-167.

[10] Goma, E. A., Successive Approximation Method of two points boundary problems, Ukraine, Math. , J., Tom (33), Kiev, 800-835, (1976).

[11] Hadid, S. B., Local and global existence theoremson differential equations of non-integer order.*Journal of Fractional Calculus*, vol. 7, pp. 101–105, 1995.

[12] Houshang H. S., Basic Real Analysis, Second Edition, ©Springer Science + Business Media, New York, 2014949237, U. S. A., 2013, 163-167.

[13] Majid A., " Modified Neumann series for solving Fredholm integral equation", Applied Mathematical Sciences, Vol. 3, No. 13, pp 623 –628,(2009).

[14] Maleknejad K. and Mahmoudi Y., " Taylor polynomial solution of high order nonlinear Volterra-Fredholm integro-differential equations ", Applied Mathematics Sciences, Vol. 24, pp 566-572, (1957).

[15] Pachpatte, B., G., Multidimensional Integral Equations and inequalities, Volume 9, 57 Shri Niketen Colony, Aurangabad, India, (2011).

[16] Plaat, O., Ordinary differential equations, Holden Day, Inc. San Francisco, Cambridge, London, Amsterdam, (1971).

[17] Pachpatie B.," On a Certain Volterra-Fredholm type integral equation Jouranl of inqualilties in pure and applied mathematics, Vol. 9, pp 1-12, ", (2008)..

[18] Parviz S., Mehdi S. and Mohammed H.," Solution of Fredholm Integrodifferential equations system by Adomian decomposition method ", Journal of Moder Mathematics and Statistics, pp 98-101, (2008).

[19] Pachpatie B., " On certain Volterra integral and integro-differential equations", Ser. Mathematics Inform., Vol. 23, pp 1-12 (2008).

[20] Rama, M. M., Ordinary differential equations theory and applications, United Kingdom, (1981).

[21] Savun, I., Stability of systems of differential equation and biological applications, M. Sc. Thesis, Eastern Mediterranean University, Gazimağusa, North Cyprus, (2010).

[22] Tao T., " On spectral methods for Volterra integral equation and the convergence analysis ", Journal of Computational Mathematics,Vol. 26 No. 6, pp 825-837, (2008).