HZ Invariants of New Types of Graphs

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Abstract

In this paper, we study the HZ-invariant for four new operation of graphs which are related to some special well-known graphs which is introduced by Sarkar et al. [9] in 2017.
Keywords: graph invariant, join, subdivision, total graph.
MSC: 05C12, 05C76

1 Introduction

All the graphs considered in this paper are connected and simple. For vertex $u \in V(G)$, the degree of the vertex u in G, denoted by $d_G(u)$, is the number of edges incident to u in G. A topological invariant of a graph is a parameter related to the graph; it does not depend on labeling or pictorial representation of the graph. In theoretical chemistry, molecular structure descriptors are used for modeling physicochemical, pharmacologic, toxicologic, biological and other properties of chemical compounds [3]. Several types of such indices exist, especially those based on vertex and edge distances. Two of these invariants are known under various names, the most commonly used ones are the first and second Zagreb invariants. They are defined as $M_1(G) = \sum_{u\in V(G)} d_G^2(u) = \sum_{uv\in E(G)} (d_G(u) + d_G(v))$ and $M_2(G) = \sum_{uv\in E(G)} d_G(u)d_G(v)$, respectively. The F-invariant of a graph G is defined as $F = F(G) = \sum_{u\in V(G)} d_G^3(u)$.

Shirdel et al.[6] introduced a variant of the first Zagreb invariant called HZ-invariant which is defined as $HZ(G) = \sum_{uv\in E(G)} (d(u) + d(v))^2$. In [6], the HZ-invariants of the Cartesian product, composition, join and disjunction of graphs are obtained. Various upper and lower bounds for HZ-invariant are computed in [7]. The HZ-invariants of some classes of chemical graphs are obtained in [4, 5]. Pattabiraman and Vijayaragavan have obtained the HZ-invariant of some

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special classes of graphs[8]. In this paper, we present the results for HZ-invariants of vertex join graphs.

2 Preliminaries

For a connected graph G, there are four related graphs as follows:

(i) The subdivision graph $S(G)$ is the graph obtained from G by replacing each edge of G by a path of length two.

(ii) $R(G)$ is obtained from G by adding a new vertex corresponding to each edge of G, then joining each new vertex to the end vertices of the corresponding edge.

(iii) $Q(G)$ is obtained from G by inserting a new vertex into each edge of G, then joining with edges those pairs of new vertices on adjacent edges of G.

(iv) The total graph $T(G)$ has as its vertices the edges and vertices of G. Adjacency in $T(G)$ is defined as adjacency or incidence for the corresponding elements of G.

The join of two graphs G_1 and G_2, denoted by $G_1 \vee G_2$, is the union $G_1 \cup G_2$ together with all the edges joining $V(G_1)$ and $V(G_2)$. Eliasi and Taeri [1] introduced the four operations of the graphs based on the Cartesian product of two graphs. Sarala et al. [2] introduced the new operations of the graphs based on the composition of these graphs. In this sequence, Sarkar et al. [9] another type of operations of graphs based on join of graphs.

3 Vertex new join graphs

Let G_1 and G_2 be two vertex disjoint graphs. The vertex S-join graph is obtained from $S(G_1)$ and G_2 by joining each vertex of $V(G_1)$ to every vertex of G_2 and is denoted by $G_1 \vee_S G_2$. Similarly, the *vertex R-join graph*, denoted by $G \vee_S G_2$, is obtained from $R(G_1)$ and G_2 by joining each vertex of $V(G_1)$ to every vertex of G_2.

The proof of the following lemma is follows from the structure of the graphs $G_1 \vee_S G_2$ and $G_1 \vee_R G_2$.

Lemma 3.1. *Let G_1 and G_2 be two connected graphs. Then*

(i) *The degree of a vertex v in $G_1 \vee_S G_2$ is* $d_{G_1 \vee_S G_2}(v) = \begin{cases} d_{G_1}(v) + |V(G_2)|, & if\, v \in V(G_1) \\ d_{G_2}(v) + |V(G_1)|, & if\, v \in V(G_2) \\ 2, & if\, v \in I(G_1). \end{cases}$

(ii) *The degree of a vertex v in $G_1 \vee_R G_2$ is* $d_{G_1 \vee_R G_2}(v) = \begin{cases} d_{G_1}(v) + |V(G_2)|, & if\, v \in V(G_1) \\ d_{G_2}(v) + |V(G_1)|, & if\, v \in V(G_2) \\ 2, & if\, v \in I(G_1). \end{cases}$ ■

Lemma 3.2. *[10] Let* f *be a convex function on the interval* I *and* $x_1, x_2, \ldots, x_n \in I$. *Then* $f\Big(\frac{x_1+x_2+\ldots+x_n}{n}\Big) \leq \frac{f(x_1)+f(x_2)+\ldots f(x_n)}{n}$, *with equality if and only if* $x_1 = x_2 = \ldots = x_n$. ■

Now we find the exact value of HZ-invariant of the graph $G_1 \vee_S G_2$.

Theorem 3.3. *Let* G_i *be a graph with* n_i *vertices and* m_i *edges,* $i = 1, 2$. *Then* $HZ(G_1 \vee_S G_2) = HZ(G_2) + F(G_1) + (3n_2 + 4)M_1(G_1) + 5n_1M_1(G_2) + 4(n_1 + n_2)(m_1 + m_2) + 2m_1(4m_2 + (n_2 + 2)^2) + 4n_1^2m_2 + (n_1 + n_2)^2n_1n_2$.

Proof. From the definition of HZ-invariant, we have

$$HZ(G_1 \vee_S G_2) = \sum_{uv\in E(G_1\vee_S G_2)} \Big(d_{G_1\vee_S G_2}(u) + d_{G_1\vee_S G_2}(v)\Big)^2.$$

The edge set of $G_1 \vee_S G_2$ can be partitioned into three subsets, namely,

$$E_1 = \{uv \in E(G_1 \vee_S G_2) | uv \in E(S(G_1))\},$$
$$E_2 = \{uv \in E(G_1 \vee_S G_2) | uv \in E(G_2)\} \text{ and}$$
$$E_1 = \{uv \in E(G_1 \vee_S G_2) | u \in V(G_1),\ v \in V(G_2)\}.$$

The contribution of the edges in E_1 to the HZ-invariant of $G_1 \vee_S G_2$ is given by

$$\begin{aligned} HZ(G_1 \vee_S G_2) &= \sum_{uv\in E_1)} \Big(d_{G_1\vee_S G_2}(u) + d_{G_1\vee_S G_2}(v)\Big)^2 \\ &= \sum_{uv\in E(S(G_1))} \Big(d_{G_1}(v) + n_2 + 2\Big)^2, \text{by Lemma 3.1}(i) \\ &= \sum_{v\in V(G_1)} d_{G_1}(v)\Big(d_{G_1}(v)^2 + 2d_{G_1}(v)(n_2 + 2) + (n_2 + 2)^2\Big) \end{aligned}$$

From the definitions of first Zagreb invariant and F-invariant, we have

$$HZ(G_1 \vee_S G_2) = F(G_1) + 2M_1(G_1)(n_2 + 2) + 2m_1(n_2 + 2)^2.$$

Similarly, the contribution of the edges in E_2 to the HZ-invariant of $G_1 \vee_S G_2$ is given by

$$\begin{aligned} HZ(G_1 \vee_S G_2) &= \sum_{uv\in E_2} \Big(d_{G_1\vee_S G_2}(u) + d_{G_1\vee_S G_2}(v)\Big)^2 \\ &= \sum_{uv\in E(G_2)} \Big(d_{G_2}(u) + n_1 + d_{G_2}(v) + n_1\Big)^2, \text{by Lemma 3.1}(i) \\ &= \sum_{uv\in E(G_2)} \Big((d_{G_2}(u) + d_{G_2}(v))^2 + 4n_1^2 + 4n_1(d_{G_2}(u) + d_{G_2}(v))\Big) \end{aligned}$$

From the definition of first Zagreb invariant, we have

$$= HZ(G_2) + 4n_1M_1(G_2) + 4n_1^2m_2.$$

Also, for the edges in E_3, contribution to the HZ-invariant of $G_1 \vee_S G_2$ is given by

$$\begin{aligned} HZ(G_1 \vee_S G_2) &= \sum_{u\in V(G_1)} \sum_{v\in V(G_2)} \left(d_{G_1\vee_S G_2}(u) + d_{G_1\vee_S G_2}(v)\right)^2 \\ &= \sum_{u\in V(G_1)} \sum_{v\in V(G_2)} \left(d_{G_1}(u) + n_2 + d_{G_2}(v) + n_1\right)^2, \text{by Lemma 3.1}(i) \\ &= \sum_{u\in V(G_1)} \sum_{v\in V(G_2)} \Big(d_{G_1}(u)^2 + d_{G_2}(v)^2 + (n_1+n_2)^2 \\ &\quad +2d_{G_1}(u)d_{G_2}(v) + 2d_{G_1}(u)(n_1+n_2) + 2d_{G_2}(v)(n_1+n_2)\Big) \\ &= n_2M_1(G_1) + n_1M_1(G_2) + n_1n_2(n_1+n_2)^2 + 8m_1m_2 + 4(n_1+n_2)(m_1+m_2). \end{aligned}$$

The desired expression for the HZ-invariant of $G_1 \vee_S G_2$ is obtained by summing the above three expression. ■

Next, we calculate the exact value for HZ-invariant of the graph $G_1 \vee_R G_2$.

Theorem 3.4. *Let G_i be a graph with n_i vertices and m_i edges, $i = 1, 2$. Then* $HZ(G_1 \vee_R G_2) = 4HZ(G_1) + HZ(G_2) + (12n_2 + 16)M_1(G_1) + 5n_1M_1(G_2) + 4(n_1 + n_2)(2n_2m_1 + m_2n_1) + 16m_1m_2 + (n_1 + n_2)^2 n_1n_2 + n_2^2(4m_1 + n_1) + 16n_2m_1 + 4n_1^2 m_2$.

Proof. The edge set of $G_1 \vee_R G_2$ can be partitioned into three subsets

$$\begin{aligned} E_1 &= \{uv \in E(G_1 \vee_R G_2) | uv \in E(G_1)\}, \\ E_2 &= \{uv \in E(G_1 \vee_R G_2) | uv \in E(G_2)\}, \\ E_3 &= \{uv \in E(G_1 \vee_R G_2) | u \in I(G_1),\ v \in V(G_1)\} \text{ and} \\ E_4 &= \{uv \in E(G_1 \vee_R G_2) | u \in V(G_1),\ v \in V(G_2)\}. \end{aligned}$$

The contribution of the edges in E_1 to the HZ-invariant of $G_1 \vee_R G_2$ is given by

$$\begin{aligned} HZ(G_1 \vee_R G_2) &= \sum_{uv\in E_1} \left(d_{G_1\vee_R G_2}(u) + d_{G_1\vee_R G_2}(v)\right)^2 \\ &= \sum_{uv\in E(G_1)} \left(2d_{G_1}(u) + n_2 + 2d_{G_1}(v) + n_2\right)^2, \text{by Lemma 3.1}(ii) \\ &= \sum_{uv\in E(G_1)} \left(4(d_{G_1}(u) + d_{G_1}(v))^2 + 8n_2(d_{G_1}(u) + d_{G_1}(v)) + 4n_2^2\right) \\ &= 4HM(G_1) + 8n_2M_1(G_1) + 4m_1n_2^2. \end{aligned}$$

The contribution of the edges in E_2 to the HZ-invariant of $G_1 \vee_R G_2$ is given by

$$\begin{aligned} HZ(G_1 \vee_R G_2) &= \sum_{uv\in E_2} \left(d_{G_1\vee_R G_2}(u) + d_{G_1\vee_R G_2}(v)\right)^2 \\ &= \sum_{uv\in E(G_2)} \left(d_{G_2}(u) + n_1 + d_{G_2}(v) + n_1\right)^2, \text{by Lemma 3.1}(ii) \\ &= \sum_{uv\in E(G_2)} \left((d_{G_2}(u) + d_{G_2}(v))^2 + 4n_1^2 + 4n_1(d_{G_2}(u) + d_{G_2}(v))\right) \\ &= HZ(G_2) + 4n_1M_1(G_2) + 4n_1^2m_2. \end{aligned}$$

The contribution of the edges in E_3 to the HZ-invariant of $G_1 \vee_R G_2$ is given by

$$\begin{aligned} HZ(G_1 \vee_R G_2) &= \sum_{u\in I(G_1),\, v\in V(G_1)} \left(d_{G_1\vee_R G_2}(u) + d_{G_1\vee_R G_2}(v)\right)^2 \\ &= \sum_{u\in I(G_1),\, v\in V(G_1)} \left(d_{I(G_1)}(u) + 2d_{G_1}(v) + n_2\right)^2, \text{by Lemma 3.1}(ii) \\ &= \sum_{v\in V(G_1)} \left(2d_{G_1}(v) + 2d_{G_1}(v) + n_2\right)^2 \\ &= \sum_{u\in I(G_1),\, v\in V(G_1)} \left(16d_{G_1}(v)^2 + 8n_2 d_{G_1}(v) + n_2^2\right) \\ &= 16M_1(G_1) + n_1 n_2^2 + 16m_1 n_2. \end{aligned}$$

The contribution of the edges in E_4 to the HZ-invariant of $G_1 \vee_R G_2$ is given by

$$\begin{aligned} HZ(G_1 \vee_R G_2) &= \sum_{u\in V(G_1)} \sum_{v\in V(G_2)} \left(d_{G_1\vee_R G_2}(u) + d_{G_1\vee_R G_2}(v)\right)^2 \\ &= \sum_{u\in V(G_1)} \sum_{v\in V(G_2)} \left(2d_{G_1}(u) + n_2 + d_{G_2}(v) + n_1\right)^2, \text{by Lemma 3.1}(ii) \\ &= 4n_2 M_1(G_1) + n_1 M_1(G_2) + (n_1+n_2)^2 n_1 n_2 + 16m_1 m_2 \\ &\quad +4(n_1+n_2)^2(m_1 n_2 + n_1 m_2). \end{aligned}$$

By summing the above four expression, the desired expression for the HZ-invariant of $G_1 \vee_R G_2$ is obtained. ■

The vertex Q-join graph of G_1 and G_2 is obtained from $Q(G_1)$ and G_2 by joining each vertex of $V(G_1)$ to every vertex of G_2 and is denoted by $G_1 \vee_Q G_2$. Similarly, the vertex T-join graph, denoted by $G \vee_T G_2$, is obtained from $T(G_1)$ and G_2 by joining each vertex of $V(G_1)$ to every vertex of G_2. The line graph $L(G)$ of G is the graph whose vertices correspond to the edges of G with two vertices being adjacent if and only if the corresponding edges in G have a vertex in common.

The proof of the following lemma is follows from the structure of the graphs $G_1 \vee_Q G_2$ and $G_1 \vee_T G_2$.

Lemma 3.5. *Let G_1 and G_2 be two connected graphs. Then*

(i) *The degree of v in $G_1 \vee_Q G_2$ is* $d_{G_1\vee_Q G_2}(v) = \begin{cases} d_{G_1}(v) + |V(G_2)|, & \text{if } v \in V(G_1) \\ d_{G_2}(v) + |V(G_1)|, & \text{if } v \in V(G_2) \\ d_{G_1}(x) + d_{G_1}(y), & \text{if } e = xy \in I(G_1). \end{cases}$

(ii) *The degree of v in $G_1 \vee_T G_2$ is* $d_{G_1\vee_T G_2}(v) = \begin{cases} 2d_{G_1}(v) + |V(G_2)|, & \text{if } v \in V(G_1) \\ d_{G_2}(v) + |V(G_1)|, & \text{if } v \in V(G_2) \\ d_{G_1}(x) + d_{G_1}(y), & \text{if } e = xy \in I(G_1). \end{cases}$ ■

Here we compute the exact values of HZ-invariant of the graphs $G_1 \vee_Q G_2$ and $G_1 \vee_T G_2$.

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Theorem 3.6. *Let G_i be a graph with n_i vertices and m_i edges, $i = 1,2$. Then* $HZ(G_1 \vee_Q G_2) = HZ(L(G_1)) + 4HZ(G_1) + HZ(G_2) + 8F(G_1) + (9n_2 - 24)M_1(G_1) + 5n_1M_1(G_2) + 16M_2(G_1) + 16m_1 + 4(n_2^2 m_1 + n_1^2 m_2) + 8m_1m_2 + n_1n_2(n_1 + n_2)^2 + 4(n_1 + n_2)(m_1n_2 + m_2n_1)$.

Proof. From the definition of HZ-invariant, we have

$$HZ(G_1 \vee_Q G_2) = \sum_{uv \in E(G_1 \vee_Q G_2)} \Big(d_{G_1 \vee_Q G_2}(u) + d_{G_1 \vee_Q G_2}(v)\Big)^2.$$

The edge set of $G_1 \vee_Q G_2$ can be partitioned into three subsets

$$E_1 = \{uv \in E(G_1 \vee_Q G_2) | u, v \in I(G_1)\},$$
$$E_2 = \{uv \in E(G_1 \vee_Q G_2) | u \in V(G_1),\ v \in I(G_1)\},$$
$$E_3 = \{uv \in E(G_1 \vee_Q G_2) | uv \in E(G_2)\} \text{ and}$$
$$E_4 = \{uv \in E(G_1 \vee_Q G_2) | u \in V(G_1),\ v \in V(G_2)\}.$$

The contribution of the edges in E_1 to the HZ-invariant of $G_1 \vee_Q G_2$ is given by

$$\begin{aligned} HZ(G_1 \vee_Q G_2) &= \sum_{u,v \in I(G_1)} \Big(d_{G_1 \vee_Q G_2}(u) + d_{G_1 \vee_Q G_2}(v)\Big)^2 \\ &= \sum_{u,v \in I(G_1)} \Big(d_{I(G_1)}(u) + d_{I(G_1)}(v)\Big)^2, \text{by Lemma 3.5}(i) \\ &= \sum_{uv,vw \in E(G_1)} \Big(d_{G_1}(u) + d_{G_1}(v) + d_{G_1}(v) + d_{G_1}(w)\Big)^2 \\ &= \sum_{e=uv, f=vw \in E(L(G_1))} \Big(d_{L(G_1)}(e) + d_{L(G_1)}(f) + 4\Big)^2 \\ &= \sum_{e,f \in E(L(G_1))} \Big((d_{L(G_1)}(e) + d_{L(G_1)}(f))^2 + (d_{L(G_1)}(e) + d_{L(G_1)}(f)) + 16\Big) \\ &= HZ(L(G_1)) + 8M_1(L(G_1)) + 16\,|E(L(G_1))|\,. \end{aligned}$$

Using $M_1(L(G_1)) = F(G_1) - 4M_1(G_1) + 2M_2(G_1) + 4m_1$ and $|E(L(G_1))| = \frac{M_1(G_1)}{2} - m_1$, we obtain

$$HZ(G_1 \vee_Q G_2) = HZ(L(G_1)) + 8F(G_1) - 24M_1(G_1) + 16M_2(G_1) + 16m_1.$$

Similarly, the contribution of the edges in E_2 to the HZ-invariant of $G_1 \vee_Q G_2$ is given by

$$\begin{aligned} HZ(G_1 \vee_Q G_2) &= \sum_{u \in V(G_1)} \sum_{v \in I(G_1)} \Big(d_{G_1 \vee_Q G_2}(u) + d_{G_1 \vee_Q G_2}(v)\Big)^2 \\ &= \sum_{u \in V(G_1)} \sum_{v \in I(G_1)} \Big(d_{G_1}(u) + d_{I(G_1)}(v)\Big)^2 \\ &= \sum_{uv \in E(G_1)} \Big(d_{G_1}(u) + n_2 + d_{G_1}(v) + n_2 + d_{G_1}(u) + d_{G_1}(v)\Big)^2, \text{by Lemma 3.5}(i) \\ &= 4HM(G_1) + 4n_2^2 m_1 + 8n_2M_1(G_1). \end{aligned}$$

The contribution of the edges in E_3 to the HZ-invariant of $G_1 \vee_Q G_2$ is given by

$$\begin{aligned} HZ(G_1 \vee_Q G_2) &= \sum_{uv \in E(G_2)} \left(d_{G_1 \vee_Q G_2}(u) + d_{G_1 \vee_Q G_2}(v)\right)^2 \\ &= \sum_{uv \in E(G_2)} \left(d_{G_2}(u) + n_1 + d_{G_2}(v) + n_1\right)^2, \text{by Lemma 3.5}(i) \\ &= \sum_{uv \in E(G_2)} \left((d_{G_2}(u) + d_{G_2}(v))^2 + 4n_1^2 + 4n_1(d_{G_2}(u) + d_{G_2}(v))\right) \\ &= HZ(G_2) + 4n_1 M_1(G_2) + 4n_1^2 m_2. \end{aligned}$$

Also, the contribution of the edges in E_4 to the HZ-invariant of $G_1 \vee_Q G_2$ is given by

$$\begin{aligned} HZ(G_1 \vee_Q G_2) &= \sum_{u \in V(G_1)} \sum_{v \in V(G_2)} \left(d_{G_1 \vee_Q G_2}(u) + d_{G_1 \vee_Q G_2}(v)\right)^2 \\ &= \sum_{u \in V(G_1)} \sum_{v \in V(G_2)} \left(d_{G_1}(u) + n_2 + d_{G_2}(v) + n_1\right)^2, \text{by Lemma 3.5}(i) \\ &= \sum_{u \in V(G_1)} \sum_{v \in V(G_2)} \Big(d_{G_1}(u)^2 + d_{G_2}(v)^2 + (n_1 + n_2)^2 \\ &\quad +2d_{G_1}(u)d_{G_2}(v) + 2d_{G_1}(u)(n_1 + n_2) + 2d_{G_2}(v)(n_1 + n_2)\Big)^2 \\ &= n_2 M_1(G_1) + n_1 M_1(G_2) + n_1 n_2 (n_1 + n_2)^2 + 8m_1 m_2 + 4(n_1 + n_2)(m_1 + m_2). \end{aligned}$$

The desired expression for the HZ-invariant of $G_1 \vee_Q G_2$ is obtained by summing the above four expression. ■

Theorem 3.7. *Let G_i be a graph with n_i vertices and m_i edges, $i = 1, 2$. Then* $HZ(G_1 \vee_T G_2) = HZ(l(G_1)) + 8F(G_1) + 16M_2(G_1) + 13HM(G_1) + HZ(G_2) + 24(n_2 + 1)M_1(G_1) + 5n_1 M_1(G_2) + 16m_1 + 16m_1 m_2 + n_1 n_2 (n_1 + n_2)^2 + 4(n_1 + n_2)(2n_2 m_1 + n_1 m_2) + 4n_2^2(2m_1 + m_2)$.

Proof. The edge set of $G_1 \vee_T G_2$ can be partitioned into five subsets

$$E_1 = \{uv \in E(G_1 \vee_T G_2) | u, v \in I(G_1)\},$$
$$E_2 = \{uv \in E(G_1 \vee_T G_2) | u \in V(G_1),\ v \in I(G_1)\},$$
$$E_3 = \{uv \in E(G_1 \vee_T G_2) | uv \in E(G_2)\},$$
$$E_4 = \{uv \in E(G_1 \vee_T G_2) | u \in V(G_1),\ v \in V(G_2)\} \text{ and}$$
$$E_5 = \{uv \in E(G_1 \vee_T G_2) | uv \in E(G_1)\}$$

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The contribution of the edges in E_1 to the HZ-invariant of $G_1 \vee_T G_2$ is given by

$$\begin{aligned} HZ(G_1 \vee_T G_2) &= \sum_{u,v\in I(G_1)} \left(d_{G_1\vee_T G_2}(u) + d_{G_1\vee_T G_2}(v)\right)^2 \\ &= \sum_{u,v\in I(G_1)} \left(d_{I(G_1)}(u) + d_{I(G_1)}(v)\right)^2 \\ &= \sum_{uv,vw\in E(G_1)} \left(d_{G_1}(u) + d_{G_1}(v) + d_{G_1}(v) + d_{G_1}(w)\right)^2 \\ &= \sum_{e=uv, f=vw\in L(G_1)} \left(d_{L(G_1)}(e) + d_{L(G_1)}(f) + 4\right)^2 \\ &= \sum_{e,f\in L(G_1)} \left((d_{L(G_1)}(e) + d_{L(G_1)}(f))^2 + (d_{L(G_1)}(e) + d_{L(G_1)}(f)) + 16\right) \\ &= HZ(L(G_1)) + 8M_1(L(G_1)) + 16\,|E(L(G_1))| \\ &= HZ(L(G_1)) + 8F(G_1) - 24M_1(G_1) + 16M_2(G_1) + 16m_1. \end{aligned}$$

The contribution of the edges in E_2 to the HZ-invariant of $G_1 \vee_T G_2$ is given by

$$\begin{aligned} HZ(G_1 \vee_T G_2) &= \sum_{u\in V(G_1)} \sum_{v\in I(G_1)} \left(d_{G_1\vee_T G_2}(u) + d_{G_1\vee_T G_2}(v)\right)^2 \\ &= \sum_{u\in V(G_1)} \sum_{v\in I(G_1)} \left(d_{G_1}(u) + d_{I(G_1)}(v)\right)^2 \\ &= \sum_{uv\in E(G_1)} \left(2d_{G_1}(u) + n_2 + 2d_{G_1}(v) + n_2 + d_{G_1}(u) + d_{G_1}(v)\right)^2 \\ &= 9HM(G_1) + 4n_2^2 m_1 + 12n_2 M_1(G_1). \end{aligned}$$

The contribution of the edges in E_3 to the HZ-invariant of $G_1 \vee_T G_2$ is given by

$$\begin{aligned} HZ(G_1 \vee_T G_2) &= \sum_{uv\in E(G_2)} \left(d_{G_1\vee_T G_2}(u) + d_{G_1\vee_T G_2}(v)\right)^2 \\ &= \sum_{uv\in E(G_2)} \left(d_{G_2}(u) + n_1 + d_{G_2}(v) + n_1\right)^2 \\ &= \sum_{uv\in E(G_2)} \left((d_{G_2}(u) + d_{G_2}(v))^2 + 4n_1^2 + 4n_1(d_{G_2}(u) + d_{G_2}(v))\right) \\ &= HZ(G_2) + 4n_1 M_1(G_2) + 4n_1^2 m_2. \end{aligned}$$

The contribution of the edges in E_4 to the HZ-invariant of $G_1 \vee_T G_2$ is given by

$$\begin{aligned} HZ(G_1 \vee_T G_2) &= \sum_{u\in V(G_1)} \sum_{v\in V(G_2)} \left(d_{G_1\vee_T G_2}(u) + d_{G_1\vee_T G_2}(v)\right)^2 \\ &= \sum_{u\in V(G_1)} \sum_{v\in V(G_2)} \left(2d_{G_1}(u) + n_2 + d_{G_2}(v) + n_1\right)^2 \\ &= 4n_2 M_1(G_1) + n_1 M_1(G_2) + n_1 n_2 (n_1 + n_2)^2 \\ &\quad +16m_1 m_2 + 4(n_1 + n_2)(2m_1 n_2 + m_2 n_1). \end{aligned}$$

The contribution of the edges in E_5 to the HZ-invariant of $G_1 \vee_T G_2$ is given by

$$\begin{aligned} HZ(G_1 \vee_T G_2) &= \sum_{uv \in E(G_1)} \left(d_{G_1 \vee_T G_2}(u) + d_{G_1 \vee_T G_2}(v)\right)^2 \\ &= \sum_{uv \in E(G_1)} \left(2d_{G_1}(u) + n_2 + 2d_{G_1}(v) + n_2\right)^2 \\ &= \sum_{uv \in E(G_1)} \left(4(d_{G_1}(u) + d_{G_1}(v))^2 + 8n_2(d_{G_1}(u) + d_{G_1}(u)) + 4n_2^2\right) \\ &= 4HZ(G_1) + 8n_2 M_1(G_1) + 4m_1 n_2^2. \end{aligned}$$

The desired expression for the HZ-invariant of $G_1 \vee_T G_2$ is obtained by summing the above five expression. ■

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