PROPERTIES OF UNIT REGULAR INVERSE MONOIDS

V. K. SREEJA

Department of Mathematics, Amrita Vishwa Vidyapeetham , Amritapuri, India

sreeja@amritapuri.amrita.edu

Abstract

Let S be a unit regular semigroup with group of units $G = G(S)$ and semilattice of idempotents $E = E(S)$. Then for every $x \in S$ there is a $u \in G$ such that $x = xux$. Then both xu and ux are idempotents and we can write $x = u^{-1}ux$ or $x = xuu^{-1}$.Thus every element of a unit regular inverse monoid is a product of a group element and an idempotent. It is evident that every L-class and every R-class of an inverse semigroup contains exactly one idempotent where L and R are two of Greens relations. In this paper we study about some properties of unit regular inverse monoids.

Keywords

Inverse monoids, Unit regular monoids, Green's relation.

Mathematics Subject Classification

20M17, 20M18, 20M32

Preliminaries

Throughout this paper let $E = E(S)$ denote the semilattice of idempotents and $G = G(S)$ denote the group of units of the regular monoid S.

Proposition 1.1 ([2]). *Let S be a regular monoid. Then S is unit regular if and only if for* each $x \in S$ *there is an idempotent* $e \in E$ and $u \in G$ *such that* x =eu

Proposition 1.2 ([2]). *Let S be a regular monoid with group of units G. Then if* $e \in E$, *then* $ueu^{-1} \in E$ for all $u \in G$.

Proposition 1.3. ([1]). Let S *be an* inverse monoid with G = G(S) and E = E(S). Then the following conditions are equivalent on S. (i) S is unit regular (ii) $L_e = Ge$ for every $e \in E$. (iii) $R_e = eG$ for every $e \in E$, where L_e is the L-class containing e and R_e is the R-class containing e.

Proposition 1.4. ([2]). The homomorphic images of unit regular monoids are unit regular**.**

Proposition 1.5. ([4]). Let S be a regular monoid . Then the set $G_x = \{u \in G : xu\ L\ x\}$ for $x \in S$ is a group.

Proposition 1.6. ([4]). Let S be a regular monoid . Then the set $G'_x = \{u \in G : ux\ R\ x\}$ for $x \in S$ is a group.

Definition 1.7. A semigroup S is left simple if $Sa = S$ for all $a \in S$ and is called right simple if $aS = S$ for all $a \in S$.

Definition 1.8. A semigroup S with group of units G is strongly unit regular if for any $x \in S$ there is a $u \in G$ such that $x = xux$ and for any $e, f \in E(S)$ with $e\, D\, f$, there exists $v \in G$ such that $f = vev^{-1}$.

If S is a monoid with G as group of units then we can define the relations R_G and L_G on S as follows; $x\, R_G\, y \Leftrightarrow x = yu$ for some $u \in G$. $x\, L_G\, y \Leftrightarrow x = uy$ for some $u \in G$. A unit regular monoid S is said to be R-strongly unit regular if $R = R_G$ and is said to be L-strongly unit regular if $L = L_G$ on S, where R and L are Green's equivalences on S.

Unit regular inverse monoids

Proposition 2.1. *Let $\phi : S \to T$ be a homomorphism from a R-strongly unit regular semigroup S into a semigroup T. Then $\phi(S)$ is R-strongly unit regular.*

Proof: Since S is R-strongly unit regular, any $x \in S$ can be written as $x = eu$. Hence $\phi(x) = \phi(e)\phi(u)$. Since $\phi(S)$ is an unit regular monoid, by Proposition 1.4, and since $\phi(e)$, is an idempotent, $\phi(u)$ is a unit we get $\phi(S)$ is R-strongly unit regular.

Proposition 2.2. Suppose S is a inverse unit regular semigroup .Then the following conditions hold:

(i) $R_e R_f \subseteq R_{eufu^{-1}}$, where $e, f \in E(S)$

(ii) $L_e L_f \subseteq L_{eufu^{-1}}$.

Proof: (i) **Let** $x \in R_e$ and $y \in R_f$**.** Then $x = eu$ and $y = fv$**,** by Proposition 1.3. Hence $xy = eufv = eufu^{-1}uv$**.** So $xy \in R_{eufu^{-1}}$ **,** by Proposition 1.3.
(ii) Proof of (ii) follows dually.

Remark. Suppose S has E-centralising group of units**,** then $ufu^{-1} = f$ and hence $R_e R_f \subseteq R_{ef}$ and $L_e L_f \subseteq L_{ef}$.

Theorem 2.3. Let S be a inverse unit regular semigroup with E-centralising group of units. Then the following conditions hold :

(i) Each ***R***-class is a group.

(ii) Each ***L***_class is a group .

(iii) Each **D**-class is a completely regular semigroup.

Proof: (i) Each *R*-class contains a unique idempotent *e*. Let *x* belongs to an *R*-class. Then $x = eu$ by Proposition 1.3. . Also $x^{-1} = u^{-1}e$.. Evidently *e* acts as the identity element of *R*.

Proof of (ii) follows dually.

(iii) Each **D**-class in a semigroup is a union of ***R***-classes and a union of ***L***-classes. Since each ***R***-class and each ***L***-class is a group the result follows.

Lemma 2.4. Let *S* be an inverse unit regular monoid with group of units equal to *G* and let $G_f = \{u \in G : fu \, L \, f\}$. Then for every $u \in G_f$, $fu = uf$.

Proof : We have $uf = ufu^{-1}u$. But ufu^{-1} is an idempotent and since $u \in G_f$ implies $u^{-1} \in G_f$ we get $ufu^{-1} L \; fu^{-1} L \; f$. Hence $ufu^{-1}. = f$, since S is an inverse monoid. So $uf = fu$.

Lemma 2.5 Let *S* be an inverse unit regular monoid with group of units equal to *G* and let $G_f = \{u \in G : fu \, L \, f\}$, where *f* is an idempotent of *S*. Then fG_f is a subgroup of S with *f* acting as the identity element.

Proof : Let $fu_1, fu_2 \in fG_f$ **.** Then $fu_1 fu_2 = fu_1u_2$ **,** by Lemma 2.4. Also $(fu_1)^{-1} = u_1^{-1} f$. Also $(fu_1)\,(fu_1)^{-1} = (fu_1)^{-1}\,(fu_1) = f$. So $f\,G_f$ is a subgroup of S with *f* acting as the identity element

Theorem 2.6. Let *S* be a unit regular inverse monoid . Then $T_f = fG_f \cup G_f$ is a unit regular inverse submonoid of *S* with G_f as group of units of T_f and the set of idempotents of T_f as $\{f\}$**.**

Proof : The proof follows from Lemma 2.4 and Lemma 2.5.

The proofs of the following lemmas and theorem follows dually.

Lemma 2.7. Let *S* be an inverse unit regular monoid with group of units equal to *G* and let $G'_f = \{u \in G : uf \, R \, f\}$. Then for every $u \in G'_f$, $fu = uf$.

Lemma 2.8 Let *S* be an inverse unit regular monoid with group of units equal to *G* and let $G_f = \{u \in G : uf' \, R \, f\}$, where *f* is an idempotent of *S*. Then $G'_f f$ is a subgroup of S with *f* acting as the identity element.

Theorem 2.9. Let *S* be a unit regular inverse monoid . Then $T'_f = G'_f f \cup G'_f$ is a unit regular inverse submonoid of *S* with G'_f as group of units of T'_f and the set of idempotents of T'_f as $\{f\}$**.**

Proposition 2.10. Let S be an unit regular inverse monoid. Then $G_e = \{u \in G : veu = e \text{ for some v} \in \text{G}\}$ and $G'_e = \{v \in G : veu = e \text{ for some u} \in \text{G}\}$.

Proof : $G_e = \{u \in G : eu\ L\ e\}$. Let $\{u \in G : veu = e \text{ for some v} \in \text{G}\}$ be G_1. Then we can prove that $G_e = G_1$. For that let $u \in G_e$.Then $eu\ L\ e$. Hence by Proposition 1.3. we get that $eu = ve$ for some $v \in G.$ Hence $v^{-1}eu = e.$ So $u \in G_1$. Thus $G_e \subseteq G_1$. Conversely assume that $u \in G_1$. Then $v\,eu = e$ for some $v \in G.$ Hence $eu = v^{-1}e$. But $v^{-1}e\ L\ e.$ So $eu\ L\ e.$ Hence $u \in G_e$. So $G_1 \subseteq G_e$. Hence $G_e = G_1$. So $G_e = \{u \in G : veu = e \text{ for some v} \in \text{G}\}$. Similarly by Proposition 1.3 we have $G'_e = \{v \in G : veu = e \text{ for some } u \in G\}$.

Proposition 2.11. Let *S* be a unit regular monoid which is right simple with *G* as the group of units of *S* and *E* = *E*(*S*) as the set of idempotents of *S*. Then *S* is a group .

Proof : Since *S* is right simple **,** $R_e = S$ **,** for any $e \in E(S)$. Hence $1 = eu$, for some $u \in G$ **,**since S is a unit regular monoid, $R_1 = G$ **.** Since $1 \in E(S)$ **,** we get $G = S.$

Lemma 2.12. Let *S* be a unit regular inverse monoid with *E* as the set of idempotents of *S* and *G* as the group of units of *S*. Then $x\ D\ y$ if and only if there exist $u,v \in G$ such that $x = vyu.$

Proof : Since $x\ D\ y$, there exists $z \in S$ such that $x\ R\ z\ L\ y$. Hence there exist $u, v \in G$ such that $x = zu$, and $z = vy$, by Proposition 1.3. Hence $x = vyu.$

The next Theorem is just a refinement of Theorem 2.3.

Theorem 2.13. Let *S* be a unit regular inverse monoid which is also strongly unit regular with *E* as the set of idempotents of *S* and *G* as the *E*-centralising group of units of *S*. Then each *D*-class is a group.

Proof : Each *D*-class contains only one idempotent, since if $e\ D\ f$, where $e, f \in E$, then $f = u^{-1}eu$, where $u \in G.$ Since *G* is *E*-centralizing we get $eu = ue.$ Hence $f = e.$ Let $x, y \in D_e$. Then $x = v_1 e u_1$, by Lemma 2.12. Since *G* is *E*-centralising $x = eu, y = ev$, where $u, v \in G.$ Hence $xy = euv \in D_e$. Also $xe = ex = x$ and if $x = eu,$ where $u \in G, e \in E$ then $x^{-1} = eu^{-1}$. Hence each *D*-class D_e is a group with *e* acting as the identity element.

References

[1] Chen S.Y. and S.C. Hsieh, *Factorizable inverse semigroups*, Semigroup forum Vol 8(1974), 283 -297

[2]. Hickey J.B and M.V. Lawson, *Unit regular monoids*, University of Glasgow, Department of Mathematics.

[3] Nambooripad K.S.S., *The natural partial order on a regular semigroup*, Proc. Edinburgh Math. Soc (1980), 249-260.

[4] V.K. Sreeja and A.R.Rajan , *Construction of Certain unit regular orthodox submonoids*, Southeast Asian Bulletin of Mathematics, 38 (4), 907–916 (2014).

[5] V.K. Sreeja and A.R.Rajan , *Some Properties of Regular Monoids*, Southeast Asian Bulletin of Mathematics, 39 (6), 891–902 (2015).