Relation Between the SDD Invariant and Splice Graphs

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Abstract

The SDD invariant is one of the 148 discrete Adriatic indices contributed many years ago. In this paper, we present the relations between the SDD invariant and splice graph of two given graphs.
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1 Introduction

Molecular descriptors have found applications in modeling several physicochemical properties in $QSAR$ and $QSPR$ studies [2, 8]. A particularly many type of molecular descriptors are defined as functions of the structure of the underlying molecular graph, such as the Wiener invariant [21], the Zagreb invariants [4] and Balaban invariant [1]. Damir Vukicević et al. [19] proved that many of these descriptors are defined the sum of individual bond contributions. Among the 148 discrete Adriatic invariants studied in [19], whose predictive properties were evaluated against the benchmark datasets of the Internation Academy of Mathematical Chemistry [15], 20 invariants were selected as significant predictors of physicochemical properties. One of these useful discrete adriatic indices is the symmetric division deg (SDD) invariant which is defined as $SDD(\Gamma) = \sum_{xy\in E(\Gamma)} \left(\frac{\lambda_\Gamma(x)}{\lambda_\Gamma(y)} + \frac{\lambda_\Gamma(y)}{\lambda_\Gamma(x))}\right)$, where $\lambda_\Gamma(x)$ and $\lambda_\Gamma(y)$ are the degrees of vertices x and y, respectively. Among all the existing molecular descriptors, SDD invariant has the best correlating ability for predicting the total surface area of polychlorobiphenys [19].

Vasilyev [20] provided the different types of lower and upper bounds of symmetric division deg invariant in some classes of graphs and determined the corresponding extremal graphs. Palacios

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[7] found a new upper bound for the symmetric division deg invariant of a graph Γ with n vertices, in terms of the inverse degree invariant,that is attained by all regular,all complete multipartite graphs, $K_{b_1,b_2,...,b_l}$, and all $(s-1,t)$-regular graphs of order s, where $1 = t < s-1$. Several papers have been appeared in literature addressing the mathematical aspects of this descriptor; for example see [5, 6, 10, 11]. In this paper, we present the relations between the SDD invariant and splice graph of two given graphs.

2 Main Results

The maximum and minimum degrees of Γ, respectively denoted by $\Delta(\Gamma)$ and $\delta(\Gamma)$. The degree of a vertex x is denoted by $\lambda_\Gamma(x)$.

Let Γ_1 and Γ_2 be two simple connected graphs with disjoint vertex sets $V(\Gamma_1)$ and $V(\Gamma_2)$ and edge sets $E(\Gamma_1)$ and $E(\Gamma_2)$, respectively. Let $x_1 \in V(\Gamma_1)$ and $y_1 \in V(\Gamma_2)$. Then the splice graph $\Gamma_1 * \Gamma_2$ of Γ_1 and Γ_2 by vertices x_1 and y_1, respectively, is defined by identifying the vertices x_1 and y_1 in the union of Γ_1 and Γ_2. One can clearly observe that, for splice graphs $\Gamma_1 * \Gamma_2$, the total number of vertices is $|V(\Gamma_1)|+|V(\Gamma_2)|-1$ while the total number of edges is $|E(\Gamma_1)|+|E(\Gamma_2)|$, see Figure 1. $\lambda(S(x))$ is the dehree of selected vertex and $Mr(\Gamma_1 * \Gamma_2)$ is the merged vertex in $\Gamma_1 * \Gamma_2$.

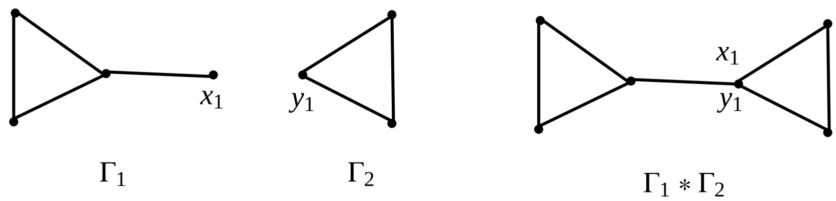

Figure 1. The graphs Γ_1, Γ_2 and its splice graph

Let Γ_1 and Γ_2 be two simple connected graphs with disjoint graphs and let $x_1 \in V(\Gamma_1)$ and $y_1 \in V(\Gamma_2)$. The vertex subdivision splice graph of Γ_1 and Γ_2 is denoted by $\Gamma_1 *_V \Gamma_2$ and obtained from $S(\Gamma_1)$ and one copy of Γ_2 which is identifying the vertices $x_1 \in V(\Gamma_1)$ and $y_1 \in V(\Gamma_2)$ in the union of $S(\Gamma_1)$ and Γ_2, see Figure 2.

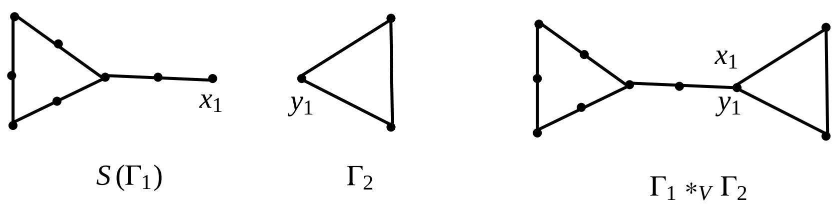

Figure 2. The graphs $S(\Gamma_1)$, Γ_2 and its splice graph

Theorem 2.1. *Let Γ_1 and Γ_2 be two given graphs. Then $\alpha \le SDD(\Gamma_1 *_V \Gamma_2) \le \beta$, where*

$\alpha = \left(\frac{(2m_1-\delta(\Gamma_1))(\delta(\Gamma_1)^2+4)}{2\delta(\Gamma_1)}\right) + 2\Big(m_2 - \delta(\Gamma_2)\Big) + \left(\frac{(\delta(\Gamma_1)+\delta(\Gamma_2))^2(\delta(\Gamma_1)+2)+4\delta(\Gamma_1)+2\delta(\Gamma_2)^2}{2(\delta(\Gamma_1)+\delta(\Gamma_2))}\right)$ *and*
$\beta = \left(\frac{(\Delta(\Gamma_1)^2+4)(2m_1-\Delta(\Gamma_1))}{2\Delta(\Gamma_1)}\right) + 2\Big(m_2 - \Delta(\Gamma_2)\Big) + \left(\frac{(\Delta(\Gamma_1)+\Delta(\Gamma_2))^2(\Delta(\Gamma_1)+2)+4\Delta(\Gamma_1)+2\Delta(\Gamma_2)^2}{2(\Delta(\Gamma_1)+\Delta(\Gamma_2))}\right)$.

Proof. By the definition of SDD index

$$\begin{aligned} SDD(\Gamma_1 *_V \Gamma_2) &= \sum_{xy\in E(\Gamma_1 *_V \Gamma_2),\ x\in V(\Gamma_1), y\in C(\Gamma_1)} \frac{\lambda_{\Gamma_1}(x)^2 + 2^2}{(\lambda_{\Gamma_1}(x))(2)} + \sum_{xy\in E(\Gamma_1 *_V \Gamma_2),\ x,y\in V(\Gamma_2)} \frac{\lambda_{\Gamma_2}(x)^2 + \lambda_{\Gamma_2}(y)^2}{\lambda_{\Gamma_2}(x)\lambda_{\Gamma_2}(y)} \\ &+ \sum_{xy\in E(\Gamma_1 *_V \Gamma_2),\ x\in Mr(\Gamma_1 *_V \Gamma_2), y\in C(\Gamma_1)} \frac{(\lambda_{\Gamma_1}(x) + \lambda_{\Gamma_2}(y))^2 + 2^2}{(\lambda_{\Gamma_1}(x) + \lambda_{\Gamma_2}(y))(2)} \\ &+ \sum_{xy\in E(\Gamma_1 *_V \Gamma_2),\ x\in Mr(\Gamma_1 *_V \Gamma_2), y\in V(\Gamma_2)} \frac{(\lambda_{\Gamma_1}(x) + \lambda_{\Gamma_2}(y))^2 + \lambda_{\Gamma_2}(w)^2}{(\lambda_{\Gamma_1}(x) + \lambda_{\Gamma_2}(y))\lambda_{\Gamma_2}(w)} \\ &= \sum_{xy\in E(\Gamma_1 *_V \Gamma_2),\ x\in V(\Gamma_1), y\in C(\Gamma_1)} \frac{\lambda_{\Gamma_1}(x)^2 + 4}{2\lambda_{\Gamma_1}(x)} + \sum_{xy\in E(\Gamma_1 *_V \Gamma_2),\ x,y\in V(\Gamma_1)} \frac{\lambda_{\Gamma_2}(x)^2 + \lambda_{\Gamma_2}(y)^2}{\lambda_{\Gamma_2}(x)\lambda_{\Gamma_2}(y)} \\ &+ \sum_{xy\in E(\Gamma_1 *_V \Gamma_2),\ x\in Mr(\Gamma_1 *_V \Gamma_2), y\in C(\Gamma_1)} \frac{(\lambda_{\Gamma_1}(x) + \lambda_{\Gamma_2}(y))^2 + 4}{2(\lambda_{\Gamma_1}(x) + \lambda_{\Gamma_2}(y))} \\ &+ \sum_{xy\in E(\Gamma_1 *_V \Gamma_2),\ x\in Mr(\Gamma_1 *_V \Gamma_2), y\in V(\Gamma_2)} \frac{(\lambda_{\Gamma_1}(x) + \lambda_{\Gamma_2}(y))^2 + \lambda_{\Gamma_2}(w)^2}{(\lambda_{\Gamma_1}(x) + \lambda_{\Gamma_2}(y))\lambda_{\Gamma_2}(w)} \qquad (1) \end{aligned}$$

According to edge partitions of the graph $\Gamma_1 *_V \Gamma_2$, we have

$$\begin{aligned} SDD(\Gamma_1 *_V \Gamma_2) &= (2m_1 - \lambda_{\Gamma_1}(S(x)))\Big(\frac{\lambda_{\Gamma_1}(x)^2 + 4}{2\lambda_{\Gamma_1}(x)}\Big) + (m_2 - \lambda_{\Gamma_2}(S(x)))\Big(\frac{\lambda_{\Gamma_2}(x)^2 + \lambda_{\Gamma_2}(y)^2}{\lambda_{\Gamma_2}(x)\lambda_{\Gamma_2}(y)}\Big) \\ &+ \lambda_{\Gamma_1}(S(x))\Big(\frac{(\lambda_{\Gamma_1}(x) + \lambda_{\Gamma_2}(y))^2 + 4}{2(\lambda_{\Gamma_1}(x) + \lambda_{\Gamma_2}(y))}\Big) + \lambda_{\Gamma_2}(S(x)))\Big(\frac{(\lambda_{\Gamma_1}(x) + \lambda_{\Gamma_2}(y))^2 + \lambda_{\Gamma_2}(w)^2}{(\lambda_{\Gamma_1}(x) + \lambda_{\Gamma_2}(y))\lambda_{\Gamma_2}(w)}\Big) \\ &\leq (2m_1 - \Delta(\Gamma_1))\Big(\frac{\Delta(\Gamma_1)^2 + 4}{2\Delta(\Gamma_1)}\Big) + (m_2 - \Delta(\Gamma_2))\Big(\frac{2\Delta(\Gamma_2)^2}{\Delta(\Gamma_2)^2}\Big) \\ &+ \Delta(\Gamma_1)\Big(\frac{(\Delta(\Gamma_1) + \Delta(\Gamma_2))^2 + 4}{2(\Delta(\Gamma_1) + \Delta(\Gamma_2))}\Big) + \Delta(\Gamma_2)\Big(\frac{(\Delta(\Gamma_1) + \Delta(\Gamma_2))^2 + \Delta(\Gamma_2)^2}{(\Delta(\Gamma_1) + \Delta(\Gamma_2))\Delta(\Gamma_2)}\Big). \qquad (2) \end{aligned}$$

Hence

$$\begin{aligned} SDD(\Gamma_1 *_V \Gamma_2) &\leq (2m_1 - \Delta(\Gamma_1))\Big(\frac{\Delta(\Gamma_1)^2 + 4}{2\Delta(\Gamma_1)}\Big) + 2(m_2 - \Delta(\Gamma_2)) \\ &+ \Delta(\Gamma_1)\Big(\frac{(\Delta(\Gamma_1) + \Delta(\Gamma_2))^2 + 4}{2(\Delta(\Gamma_1) + \Delta(\Gamma_2))}\Big) + \frac{(\Delta(\Gamma_1) + \Delta(\Gamma_2))^2 + \Delta(\Gamma_2)^2}{(\Delta(\Gamma_1) + \Delta(\Gamma_2))} \\ &= (2m_1 - \Delta(\Gamma_1))\Big(\frac{\Delta(\Gamma_1)^2 + 4}{2\Delta(\Gamma_1)}\Big) + 2(m_2 - \Delta(\Gamma_2)) \\ &+ \frac{1}{2(\Delta(\Gamma_1) + \Delta(\Gamma_2))}\Big((\Delta(\Gamma_1) + \Delta(\Gamma_2))^2(\Delta(\Gamma_1) + 2) + 4\Delta(\Gamma_1) + 2\Delta(\Gamma_2)^2\Big). \end{aligned}$$

One can analogously compute the following lower bound.

$$\begin{aligned} SDD(\Gamma_1 *_V \Gamma_2) &\leq (2m_1 - \delta(\Gamma_1))\Big(\frac{\delta(\Gamma_1)^2 + 4}{2\delta(\Gamma_1)}\Big) + 2(m_2 - \delta(\Gamma_2)) \\ &+ \frac{1}{2(\delta(\Gamma_1) + \delta(\Gamma_2))}\Big((\delta(\Gamma_1) + \delta(\Gamma_2))^2(\delta(\Gamma_1) + 2) + 4\delta(\Gamma_1) + 2\delta(\Gamma_2)^2\Big). \end{aligned}$$

Let $r_2 \in C(\Gamma_1)$ be the inserted vertex of $S(\Gamma_1)$ and $y_1 \in V(\Gamma_2)$. The edge subdivision splice graph of Γ_1 and Γ_2 is denoted by $\Gamma_1 *_E \Gamma_2$ that is obtained from $S(\Gamma_1)$ and one copy of Γ_2 identifying the vertices r_2 and y_1 in the union of $S(\Gamma_1)$ and Γ_2, see Figure 3.

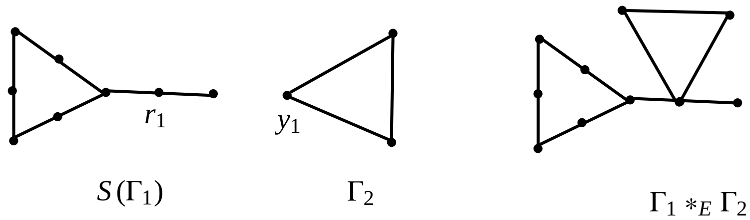

Figure 3. The graphs $S(\Gamma_1)$, Γ_2 and its edge subdivision splice graph

Theorem 2.2. *Let Γ_1 and Γ_2 be two given graphs. Then $\alpha_1 \leq SDD(\Gamma_1 *_E \Gamma_2) \leq \beta_1$, where* $\alpha_1 = \left(\frac{(m_1-1)(\Delta(\Gamma_1)^2+4)}{\Delta(\Gamma_1)}\right) + 2(m_2 - \Delta(\Gamma_2)) + \left(\frac{2((\Delta(\Gamma_1)+2)^2+\Delta(\Gamma_2)^2)+\Delta(\Gamma_1)((\Delta(\Gamma_2)+2)^2+\Delta(\Gamma_2)^2)}{\Delta(\Gamma_1)(\Delta(\Gamma_2)+2)}\right)$ *and*

$$\beta_1 = \left(\frac{(m_1-1)(\delta(\Gamma_1)^2+4)}{\delta(\Gamma_1)}\right) + 2(m_2 - \delta(\Gamma_2)) + \left(\frac{2((\delta(\Gamma_1)+2)^2+\delta(\Gamma_2)^2)+\delta(\Gamma_1)((\delta(\Gamma_2)+2)^2+\delta(\Gamma_2)^2)}{\delta(\Gamma_1)(\delta(\Gamma_2)+2)}\right).$$

Proof. Consider

$$\begin{aligned} SDD(\Gamma_1 *_E \Gamma_2) &= \sum_{xy\in E(\Gamma_1 *_E \Gamma_2),\ x\in V(\Gamma_1), y\in C(\Gamma_1)} \frac{\lambda_{\Gamma_1}(x)^2 + 2^2}{(\lambda_{\Gamma_1}(x))(2)} + \sum_{xy\in E(\Gamma_1 *_E \Gamma_2),\ x,y\in V(\Gamma_2)} \frac{\lambda_{\Gamma_2}(x)^2 + \lambda_{\Gamma_2}(y)^2}{\lambda_{\Gamma_2}(x)\lambda_{\Gamma_2}(y)} \\ &+ \sum_{xy\in E(\Gamma_1 *_E \Gamma_2),\ x\in Mr(\Gamma_1 *_E \Gamma_2), y\in V(\Gamma_1)} \frac{(\lambda_{\Gamma_2}(y)+2)^2 + \lambda_{\Gamma_1}(x)^2}{(\lambda_{\Gamma_2}(y)+2)\lambda_{\Gamma_1}(x)} \\ &+ \sum_{xy\in E(\Gamma_1 *_E \Gamma_2),\ x\in Mr(\Gamma_1 *_E \Gamma_2), y\in V(\Gamma_2)} \frac{(\lambda_{\Gamma_2}(x)+2)^2 + \lambda_{\Gamma_2}(y)^2}{(\lambda_{\Gamma_2}(x)+2)\lambda_{\Gamma_2}(y)}. \end{aligned}$$

According to edge partitions of the graph $\Gamma_1 *_E \Gamma_2$, we have

$$\begin{aligned} SDD(\Gamma_1 *_E \Gamma_2) &= (2m_1 - 2)\Big(\frac{\lambda_{\Gamma_1}(x)^2 + 2^2}{(\lambda_{\Gamma_1}(x))(2)}\Big) + (m_2 - \lambda_{\Gamma_2}(S(x)))\Big(\frac{\lambda_{\Gamma_2}(x)^2 + \lambda_{\Gamma_2}(y)^2}{\lambda_{\Gamma_2}(x)\lambda_{\Gamma_2}(y)}\Big) \\ &+2\Big(\frac{(\lambda_{\Gamma_2}(y)+2)^2 + \lambda_{\Gamma_1}(x)^2}{(\lambda_{\Gamma_2}(y)+2)\lambda_{\Gamma_1}(x)}\Big) + \lambda_{\Gamma_2}(S(x))\Big(\frac{(\lambda_{\Gamma_2}(x)+2)^2 + \lambda_{\Gamma_2}(y)^2}{(\lambda_{\Gamma_2}(x)+2)\lambda_{\Gamma_2}(y)}\Big) \\ &\leq (2m_1 - 2)\Big(\frac{\Delta(\Gamma_1)^2 + 4}{2\Delta(\Gamma_1)}\Big) + (m_2 - \Delta(\Gamma_2))\Big(\frac{2\Delta(\Gamma_2)^2}{\Delta(\Gamma_2)^2}\Big) \\ &+2\Big(\frac{(\Delta(\Gamma_2)+2)^2 + \Delta(\Gamma_1)^2}{(\Delta(\Gamma_2)+2)\Delta(\Gamma_1)}\Big) + \Delta(\Gamma_2)\Big(\frac{(\Delta(\Gamma_2)+2)^2 + \Delta(\Gamma_2)^2}{(\Delta(\Gamma_2)+2)\Delta(\Gamma_2)}\Big). \end{aligned} \tag{3}$$

Hence

$$\begin{aligned} SDD(\Gamma_1 *_E \Gamma_2) &\leq (m_1 - 1)\Big(\frac{\Delta(\Gamma_1)^2 + 4}{\Delta(\Gamma_1)}\Big) + 2(m_2 - \Delta(\Gamma_2)) \\ &+2\Big(\frac{(\Delta(\Gamma_2)+2)^2 + \Delta(\Gamma_1)^2}{(\Delta(\Gamma_2)+2)\Delta(\Gamma_1)}\Big) + \frac{(\Delta(\Gamma_2)+2)^2 + \Delta(\Gamma_2)^2}{(\Delta(\Gamma_2)+2)\Delta(\Gamma_2)} \\ &= (m_1 - 1)\Big(\frac{\Delta(\Gamma_1)^2 + 4}{\Delta(\Gamma_1)}\Big) + 2(m_2 - \Delta(\Gamma_2)) \\ &+\frac{1}{\Delta(\Gamma_1)(\Delta(\Gamma_2)+2)}\Big(2((\Delta(\Gamma_1)+2)^2 + \Delta(\Gamma_2)^2) + \Delta(\Gamma_1)((\Delta(\Gamma_2)+2)^2 + \Delta(\Gamma_2)^2)\Big). \end{aligned}$$

One can analogously compute the following lower bound.

$$SDD(\Gamma_1 *_E \Gamma_2) \geq (m_1-1)\Big(\frac{\delta(\Gamma_1)^2+4}{\delta(\Gamma_1)}\Big)+2(m_2-\delta(\Gamma_2)) + \frac{1}{\delta(\Gamma_1)(\delta(\Gamma_2)+2)}\Big(2((\delta(\Gamma_1)+2)^2+\delta(\Gamma_2)^2)+\delta(\Gamma_1)((\delta(\Gamma_2)+2)^2+\delta(\Gamma_2)^2)\Big).$$

Let $b_1 \in V(\Gamma_1)$ and $y_1 \in V(\Gamma_2)$. The vertex neighborhood subdivision splice of Γ_1 and Γ_2 is denoted by $\Gamma_1 *_{VN} \Gamma_2$ and obtained from $S(\Gamma_1)$ and $\lambda(b_1)$ copies of Γ_2 and identifying the neighborhood vertices of b_1. For $y_1 \in V(\Gamma_2)$, the union of the corresponding neighborhood separated vertices $b_1 \in V(\Gamma_1)$ of $S(\Gamma_1)$, see Figure 4.

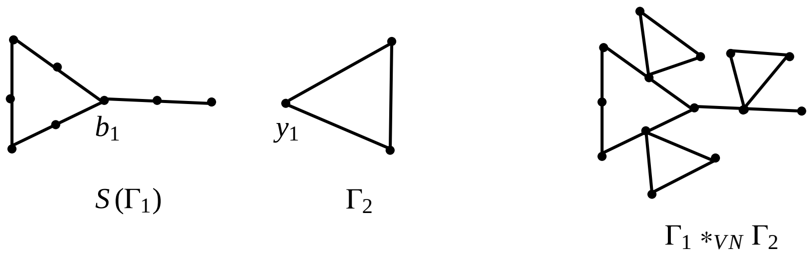

Figure 4. The graphs $S(\Gamma_1)$, Γ_2 and itsvertex neighborhood subdivision splice

Theorem 2.3. *Let* Γ_1 *and* Γ_2 *be two given graphs. Then* $\alpha_2 \leq SDD(\Gamma_1 *_{VN} \Gamma_2) \leq \beta_2$, *where* $\alpha_2 = \Big(\frac{(m_1-\Delta(\Gamma_1))(\Delta(\Gamma_1)^2+4)}{\Delta(\Gamma_1)}\Big)+2\Delta(\Gamma_1)(m_2-\Delta(\Gamma_2))+\frac{2\Big(\Delta(\Gamma_1)^2+(2+\Delta(\Gamma_2))^2\Big)+\Delta(\Gamma_1)\Big((2+\Delta(\Gamma_2))^2+(\Delta(\Gamma_2))^2\Big)}{(2+\Delta(\Gamma_2))}$ *and* $\beta_2 = \Big(\frac{(m_1-\delta(\Gamma_1))(\delta(\Gamma_1)^2+4)}{\delta(\Gamma_1)}\Big)+2\delta(\Gamma_1)(m_2-\delta(\Gamma_2))+\frac{2\Big(\delta(\Gamma_1)^2+(2+\delta(\Gamma_2))^2\Big)+\delta(\Gamma_1)\Big((2+\delta(\Gamma_2))^2+(\delta(\Gamma_2))^2\Big)}{(2+\delta(\Gamma_2))}$.

Proof. By the definition of SDD index

$$\begin{aligned} SDD(\Gamma_1 *_{VN} \Gamma_2) &= \sum_{xy\in E(\Gamma_1 *_{VN}\Gamma_2),\ x\in V(\Gamma_1), y\in C(\Gamma_1)} \frac{\lambda_{\Gamma_1}(x)^2+2^2}{(\lambda_{\Gamma_1}(x))(2)} + \sum_{xy\in E(\Gamma_1 *_{VN}\Gamma_2),\ x,y\in V(\Gamma_2)} \frac{\lambda_{\Gamma_2}(x)^2+\lambda_{\Gamma_2}(y)^2}{\lambda_{\Gamma_2}(x)\lambda_{\Gamma_2}(y)} \\ &+ \sum_{xy\in E(\Gamma_1 *_{VN}\Gamma_2),\ x\in Mr(\Gamma_1 *_{VN}\Gamma_2), y\in V(\Gamma_1)} \frac{\lambda_{\Gamma_1}(x)^2+(2+\lambda_{\Gamma_2}(y))^2}{(\lambda_{\Gamma_1}(x)(2+\lambda_{\Gamma_2}(y))} \\ &+ \sum_{xy\in E(\Gamma_1 *_{VN}\Gamma_2),\ x\in Mr(\Gamma_1 *_{VN}\Gamma_2), y\in V(\Gamma_2)} \frac{(2+\lambda_{\Gamma_2}(x))^2+(\lambda_{\Gamma_2}(y))^2}{(2+\lambda_{\Gamma_2}(x))\lambda_{\Gamma_2}(y)}. \end{aligned}$$

According to edge partitions of the graph $\Gamma_1 *_{VN} \Gamma_2$, we have

$$\begin{aligned} SDD(\Gamma_1 *_{VN} \Gamma_2) &= 2(m_1-\lambda_{\Gamma_1}(S(x)))\Big(\frac{\lambda_{\Gamma_1}(x)^2+4}{2\lambda_{\Gamma_1}(x)}\Big)+\lambda_{\Gamma_1}(S(x))(m_2-\lambda_{\Gamma_2}(S(x)))\Big(\frac{\lambda_{\Gamma_2}(x)^2+\lambda_{\Gamma_2}(y)^2}{\lambda_{\Gamma_2}(x)\lambda_{\Gamma_2}(y)}\Big) \\ &+2\lambda_{\Gamma_1}(S(x))\Big(\frac{\lambda_{\Gamma_1}(x)^2+(2+\lambda_{\Gamma_2}(y))^2}{(\lambda_{\Gamma_1}(x)(2+\lambda_{\Gamma_2}(y))}\Big)+\lambda_{\Gamma_1}(S(x))\lambda_{\Gamma_2}(S(x))\Big(\frac{(2+\lambda_{\Gamma_2}(x))^2+(\lambda_{\Gamma_2}(y))^2}{(2+\lambda_{\Gamma_2}(x))\lambda_{\Gamma_2}(y)}\Big) \\ &\leq 2(m_1-\Delta(\Gamma_1))\Big(\frac{\Delta(\Gamma_1)^2+4}{2\Delta(\Gamma_1)}\Big)+\Delta(\Gamma_1)(m_2-\Delta(\Gamma_2))\Big(\frac{\Delta(\Gamma_2)^2+\Delta(\Gamma_2)^2}{\Delta(\Gamma_2)\Delta(\Gamma_2)}\Big) \\ &+2\Delta(\Gamma_1)\Big(\frac{\Delta(\Gamma_1)^2+(2+\Delta(\Gamma_2))^2}{\Delta(\Gamma_1)(2+\Delta(\Gamma_2))}\Big)+\Delta(\Gamma_1)\Delta(\Gamma_2)\Big(\frac{(2+\Delta(\Gamma_2))^2+(\Delta(\Gamma_2))^2}{(2+\Delta(\Gamma_2))\Delta(\Gamma_2)}\Big). \end{aligned}$$

Hence

$$\begin{aligned} SDD(\Gamma_1 *_{NN} \Gamma_2) &\leq (m_1 - \Delta(\Gamma_1))\Big(\frac{\Delta(\Gamma_1)^2 + 4}{\Delta(\Gamma_1)}\Big) + 2\Delta(\Gamma_1)(m_2 - \Delta(\Gamma_2)) \\ &\quad + 2\Big(\frac{\Delta(\Gamma_1)^2 + (2 + \Delta(\Gamma_2))^2}{(2 + \Delta(\Gamma_2))}\Big) + \Delta(\Gamma_1)\Big(\frac{(2 + \Delta(\Gamma_2))^2 + (\Delta(\Gamma_2))^2}{(2 + \Delta(\Gamma_2))}\Big) \\ &= \Big(\frac{(m_1 - \Delta(\Gamma_1))(\Delta(\Gamma_1)^2 + 4)}{\Delta(\Gamma_1)}\Big) + 2\Delta(\Gamma_1)(m_2 - \Delta(\Gamma_2)) \\ &\quad + \frac{2\Big(\Delta(\Gamma_1)^2 + (2 + \Delta(\Gamma_2))^2\Big) + \Delta(\Gamma_1)\Big((2 + \Delta(\Gamma_2))^2 + (\Delta(\Gamma_2))^2\Big)}{(2 + \Delta(\Gamma_2))}. \end{aligned}$$

One can analogously compute the following lower bound.

$$\begin{aligned} SDD(\Gamma_1 *_{VN} \Gamma_2) &\geq \Big(\frac{(m_1 - \delta(\Gamma_1))(\delta(\Gamma_1)^2 + 4)}{\delta(\Gamma_1)}\Big) + 2\delta(\Gamma_1)(m_2 - \delta(\Gamma_2)) \\ &\quad + \frac{2\Big(\delta(\Gamma_1)^2 + (2 + \delta(\Gamma_2))^2\Big) + \delta(\Gamma_1)\Big((2 + \delta(\Gamma_2))^2 + (\delta(\Gamma_2))^2\Big)}{(2 + \delta(\Gamma_2))}. \end{aligned}$$

Let $p_1 \in C(\Gamma_1)$ be the inserted vertex of $S(\Gamma_1)$ and $y_1 \in V(\Gamma_2)$. Then the edge neighborhood subdivision splice of Γ_1 and Γ_2 is denoted by $\Gamma_1 *_{EN} \Gamma_2$ that is obtained from $S(\Gamma_1)$ and two copies of Γ_2 identifying the vertices p_1. For $y_1 \in V(\Gamma_2)$, the union of the corresponding neighbourhood separated vertices p_1 of $S(\Gamma_1)$, see Figure 5.

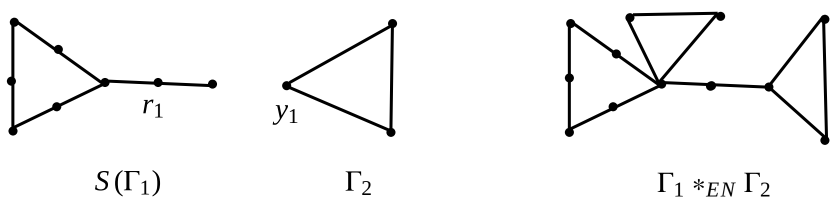

Figure 5. The graphs $S(\Gamma_1)$, Γ_2 and its edge neighborhood subdivision splice graph

Theorem 2.4. *Let* Γ_1 *and* Γ_2 *be two given graphs. Then* $\alpha_3 \leq SDD(\Gamma_1 *_{EN} \Gamma_2) \leq \beta_3$, *where* $\alpha_3 = \Big(\frac{(m_1 - \Delta(\Gamma_1))(\Delta(\Gamma_1)^2+4)}{\Delta(\Gamma_1)}\Big) + 4(m_2 - \Delta(\Gamma_2)) + \frac{\Delta(\Gamma_1)\Big((\Delta(\Gamma_1)+\Delta(\Gamma_2))^2+4\Big)+2\Big((\Delta(\Gamma_1)+\Delta(\Gamma_2))^2+\Delta(\Gamma_2)^2\Big)}{(\Delta(\Gamma_1)+\Delta(\Gamma_2))}$ *and* $\beta_3 = \Big(\frac{(m_1 - \delta(\Gamma_1))(\delta(\Gamma_1)^2+4)}{\delta(\Gamma_1)}\Big) + 4(m_2 - \delta(\Gamma_2)) + \frac{\delta(\Gamma_1)\Big((\delta(\Gamma_1)+\delta(\Gamma_2))^2+4\Big)+2\Big((\delta(\Gamma_1)+\delta(\Gamma_2))^2+\delta(\Gamma_2)^2\Big)}{(\delta(\Gamma_1)+\delta(\Gamma_2))}$.

Proof. Consider

$$\begin{aligned} SDD(\Gamma_1 *_{EN} \Gamma_2) &= \sum_{xy \in E(\Gamma_1 *_{EN} \Gamma_2),\ x \in V(\Gamma_1), y \in C(\Gamma_1)} \frac{\lambda_{\Gamma_1}(x)^2 + 2^2}{(\lambda_{\Gamma_1}(x))(2)} + \sum_{xy \in E(\Gamma_1 *_{EN} \Gamma_2),\ x,y \in V(\Gamma_2)} \frac{\lambda_{\Gamma_2}(x)^2 + \lambda_{\Gamma_2}(y)^2}{\lambda_{\Gamma_2}(x)\lambda_{\Gamma_2}(y)} \\ &\quad + \sum_{xy \in E(\Gamma_1 *_{EN} \Gamma_2),\ x \in Mr(\Gamma_1 *_{EN} \Gamma_2), y \in C(\Gamma_1)} \frac{(\lambda_{\Gamma_1}(x) + \lambda_{\Gamma_2}(y))^2 + 2^2}{(\lambda_{\Gamma_1}(x) + \lambda_{\Gamma_2}(y))(2)} \\ &\quad + \sum_{xy \in E(\Gamma_1 *_{EN} \Gamma_2),\ x \in M(\Gamma_1 *_{EN} \Gamma_2), y \in V(\Gamma_2)} \frac{(\lambda_{\Gamma_1}(x) + \lambda_{\Gamma_2}(w))^2 + \lambda_{\Gamma_2}(y)^2}{(\lambda_{\Gamma_1}(x) + \lambda_{\Gamma_2}(w))\lambda_{\Gamma_2}(y)}. \end{aligned}$$

According to edge partitions of the graph $\Gamma_1 *_{EN} \Gamma_2$, we have

$$\begin{aligned} SDD(\Gamma_1 *_{EN} \Gamma_2) &= 2(m_1 - \lambda_{\Gamma_1}(S(e))\Big(\frac{\lambda_{\Gamma_1}(x)^2+4}{2\lambda_{\Gamma_1}(x)}\Big) + 2(m_2 - \lambda_{\Gamma_2}(S(x))\Big(\frac{\lambda_{\Gamma_2}(x)^2+\lambda_{\Gamma_2}(y)^2}{\lambda_{\Gamma_2}(x)\lambda_{\Gamma_2}(y)}\Big) \\ &\quad +2\lambda_{\Gamma_1}(S(e)\Big(\frac{(\lambda_{\Gamma_1}(x)+\lambda_{\Gamma_2}(y))^2+4}{2(\lambda_{\Gamma_1}(x)+\lambda_{\Gamma_2}(y))}\Big) + 2\lambda_{\Gamma_2}(S(x)))\Big(\frac{(\lambda_{\Gamma_1}(x)+\lambda_{\Gamma_2}(y))^2+\lambda_{\Gamma_2}(w)^2}{(\lambda_{\Gamma_1}(x)+\lambda_{\Gamma_2}(y))\lambda_{\Gamma_2}(w)}\Big) \\ &\leq 2(m_1 - \Delta(\Gamma_1))\Big(\frac{\Delta(\Gamma_1)^2+4}{2\Delta(\Gamma_1)}\Big) + 2(m_2 - \Delta(\Gamma_2))\Big(\frac{\Delta(\Gamma_2)^2+\Delta(\Gamma_2)^2}{\Delta(\Gamma_2)\Delta(\Gamma_2)}\Big) \\ &\quad +2\Delta(\Gamma_1)\Big(\frac{(\Delta(\Gamma_1)+\Delta(\Gamma_2))^2+4}{2(\Delta(\Gamma_1)+\Delta(\Gamma_2))}\Big) + 2\Delta(\Gamma_2)\Big(\frac{(\Delta(\Gamma_1)+\Delta(\Gamma_2))^2+\Delta(\Gamma_2)^2}{(\Delta(\Gamma_1)+\Delta(\Gamma_2))\Delta(\Gamma_2)}\Big). \end{aligned}$$

Hence

$$\begin{aligned} SDD(\Gamma_1 *_{EN} \Gamma_2) &\leq (m_1 - \Delta(\Gamma_1))\Big(\frac{\Delta(\Gamma_1)^2+4}{\Delta(\Gamma_1)}\Big) + 4(m_2 - \Delta(\Gamma_2)) \\ &\quad +\Delta(\Gamma_1)\Big(\frac{(\Delta(\Gamma_1)+\Delta(\Gamma_2))^2+4}{(\Delta(\Gamma_1)+\Delta(\Gamma_2))}\Big) + 2\Big(\frac{(\Delta(\Gamma_1)+\Delta(\Gamma_2))^2+\Delta(\Gamma_2)^2}{(\Delta(\Gamma_1)+\Delta(\Gamma_2))}\Big) \\ &= (m_1 - \Delta(\Gamma_1))\Big(\frac{\Delta(\Gamma_1)^2+4}{\Delta(\Gamma_1)}\Big) + 4(m_2 - \Delta(\Gamma_2)) \\ &\quad + \frac{\Delta(\Gamma_1)\Big((\Delta(\Gamma_1)+\Delta(\Gamma_2))^2+4\Big) + 2\Big((\Delta(\Gamma_1)+\Delta(\Gamma_2))^2+\Delta(\Gamma_2)^2\Big)}{(\Delta(\Gamma_1)+\Delta(\Gamma_2))}. \end{aligned}$$

One can analogously compute the following lower bound.

$$\begin{aligned} SDD(\Gamma_1 *_{EN} \Gamma_2) &\geq (m_1 - \delta(\Gamma_1))\Big(\frac{\delta(\Gamma_1)^2+4}{\delta(\Gamma_1)}\Big) + 4(m_2 - \delta(\Gamma_2)) \\ &\quad + \frac{\delta(\Gamma_1)\Big((\delta(\Gamma_1)+\delta(\Gamma_2))^2+4\Big) + 2\Big((\delta(\Gamma_1)+\delta(\Gamma_2))^2+\delta(\Gamma_2)^2\Big)}{(\delta(\Gamma_1)+\delta(\Gamma_2))}. \end{aligned}$$

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