Numerical Analysis Based Differential Equations and Error Redact

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Abstract- In this paper we discussed the methods used for solving the integrodifferential equations and hence, we illustrated the numerical comparison of these methods. Usually, the best exploitation of the differential equations is for describing the functionality of the varied system. Many applications are employing the concept of these methods i.e. all the sciences, engineering, economic, health sciences, business administration, etc. For last decades many approaches have been introduced for discussing the nature of these methods and hence new technologies have been invented in this regard. Usually, the application uses this method is having a complex nature and volume i.e. huge systems.

It is clear that solving such complicated systems is very difficult so that computers simulation are exploited also the development of new technologies of the numerical method has become very demanded.

Prior to computers programming revolution, many technologies were providing the facilitation to solve the differential equations such as the approximation method. Unlike the time after the programming revolution, many decades ago very large manpower was demanded for complex systems numerical solution.

Keywords- differential equation, Runga-Kutta Method, Ordinary Differential Equations, Partial Differential Equation, Tylor's method, boundary value problem.

I. INTRODUCTION

Such functions contain an unknown variables equations derivative are termed to differential equations DE

$$x' = \sin(z), (x')^4 - y^2 + 2xy - x^2 = 0$$

$$x' + x^3 + y = 0$$

All these are first-order differential equations.

Another form of differential equations are containing the derivatives in the higher degree (order) these derivatives may belong to unknown functions [4]. The easiest way to solve them is to initiate the integration directly.

Samples:

$$x' = \sin(z) \rightarrow x = -\cos(z) + C$$

$$x'' = 6z + e^z$$

$$x' = 3z^2 + e^z$$

$$x = y^3 + e^z + c_1 Z + c_2$$

In these examples of differential equations, we can notice that the first sample includes a single variable while the second sample is having two different variables (parameters).

II. Differential Equation Categorization

Ordinary Differential Equations-

Such type termed to the equations involving one function with the derivation of it. These differential equations have a major impact of physics and engineering formulation of problems, normally such felids are dealing with varying systems, such system may change with some parameters i.e. time, temperature, frequency.

$$\frac{dv}{dt} = g - \frac{c}{m}v$$

Where;

v: is a dependent variable;

t: is an independent variable;

Hence, if a function containing a single variable such as only time or only temperature is known as an ordinary differential equation.

Partial Differential Equations-

The short form PDE is representing the partial differential equations and this type of equations normally contain more than the single independent variable. Another categorizing of the DE may refer the order of derivation of the function in DE and this can be stated like first-order DE and second-order DE etc. the first order refers to the first derivative of the main function and so on the higher order of derivation existing with its equation in the second and third-order DE. Such orders can be minimized back to the basic order by a redefinition of the variables.

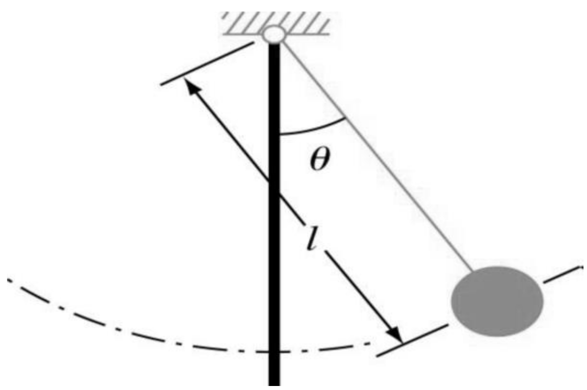

Figure 1: physics exercise describing the ordinary differential equations

Runga-Kutta (R K) Method-

These methods have been developed after Euler's method (EM) and represented as RKM [8]. So the general concept of EM is used here and produced a modification in the method of origin. The modification is presented by changing the slope in the solution curve to be approximated with the slopes at the end simulation period (solution). The general concept of this type method is finding the slope by means of weight averaging of selective slopes in every solution period [10]. However, the explicitly naturalized aspects in the developed RK algorithm is considered as the unique identity of RK as compare to Euler's method. The final figures of this method can be written as:

$M_{i+1} = M_i +$ (weighted and average selective slops).

i: is an integer having of value 1 and above.

$$\frac{dM}{dt} = f(t, M)$$

Euler's Method (EM)-

In order to get a good approximation, any order from Tylor's method (TM) can be employed to generate the initial value problem [1] [3]. TM has some disadvantages such as the existence of higher-order derivation of the equation, such existence can be assumed when actually it's not required i.e. solution of first-order problems. This higher derivative may difficult to compute for any function given in the problem, and also the computer programs can not be designed in that smooth manner for finding the solutions of the given problem in the case of higher derivative existence. Such methods like TM are more suitable for manual solution. Euler's method was found to overcome such problems. This method scheme was developed by approximation of higher-order derivative. This scheme can be stated as the following expressions:

The first order derivation of a function called Y is $y' = f(x, y)$, $y(x_0) = y_0$

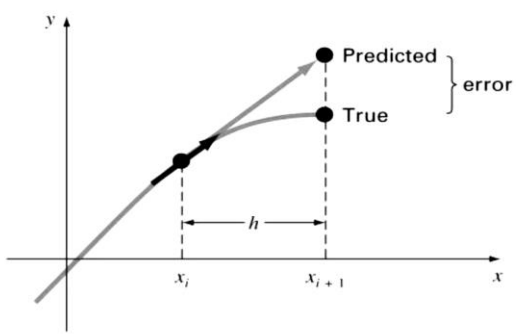

Figure 2: Euler theorem

This first order of derivative produces the slope estimation directly at x_e .

$$\theta = f(x_e, y_e)$$

The $f(x_e, y_e)$ termed to the DE calculated at x_e and y_e variables, and by substitution of this in the equation we can get the following:

$$y_{e+1} = y_e + f(x_e, y_e)H$$

Fresh y value can be generated by the slope to the extrapolate linearly over the step size h;

$$\frac{dy}{dx} = f(x, y) = -2x^3 + 12x^2 - 20x + 8.5$$

Beginning point is

$$x_e = 0 \text{ and } y_e = 1$$

$$h = 1 + 8.5 * 0.5 = 5.25$$

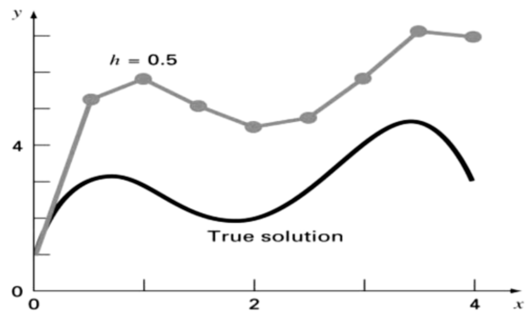

Figure 3: error estimation

III. Analyzing of Error in Euler's Method

The solution of an ordinary differential equation can produce two error classes like truncation and local truncation [4]. The first appears at the time of truncating the infinite approximation and summation by means of finite summation. For example the first two terms of Tylor's series of a sin the function is get approximated. Then the error resulted from that is truncating error.

The second model of error (local truncation error) can be found when the function is undergoing incrementing. That causes a single iteration. Suppose that knowledge of the previous iterations is presented we can write the following:

$$E_a = \frac{f'(x_i, y_i)}{2!} h^2$$

$$E_a = O(h^2)$$

For quantifying the error whining Euler's method solution, Tylor's series is used. The last is providing local truncation error estimations only, which is the error found during the step of the method. In real-world solutions, the functions are more than simple polynomials actually, it is more complicated. Tylor's series is evaluated base on the requirement of the derivations and that is considered a complication.

IV. The problem of Initial Value

The linear issue and the initial value issue can be divided from the problems of boundary value, in the literature [6] [9], there are many ways to develop a solution for the initial value problem and below are some suggested methods:

Method 1*:* considering the problem of the initial value

$$x'' + M(t)\,x' + Z(t)\,x = S(t)$$

$$x(t_0) = q\,;$$

$$x'(t_0) = w;$$

Where M and Z and S are always continuing functions in the interval of I (the open interval) in which involving all point of (t_0). $X = \emptyset$, (t) is the single exact solution available of this problem which is existing only by the I interval. Generally, the ordinary differential equations (non-linear type) are having solution existing on the intervals containing the initial value. This theory is revealing about the interval that is containing the solution and also the by default it is stating the solution existence.

Method 2: this method is highlighting the difference of the ordinary differential equations the non-linear and linear type.

Suppose that x_1 and x_2 are a couple of solutions for known differential equations.

$L\{x\} = x'' + M(t)\,x' + Z(t)\,x$; That all equal to zero.

Other solution can be defined by the linear explanation $a_1\,x_1\, +\, a_2\,x_2$; which is also solutions for the values of a_1 anda_2. Actually, this theory is not standing for the non-linear equations and this comprising the differences between linear and non-linear problems.

The Wronskian matrix is also developed from the assumption that x_1 and x_2 are the solutions for the provided differential equations [11].

Let $W = Wronskian$;

$$W = \text{Matrix of solution } (x_1 \text{ and } x_2)$$

$$W = \begin{bmatrix} x_1 & x_2 \\ x_1{}', & x_2{}' \end{bmatrix}$$

$$\text{Wronskian} = x_1 x_2{}' - x_2 x_1{}'$$

Method 3*:* form the definition of Wronskian matrix we can state this theory x_1 and x_2 are solutions of new differential equations so;

$L\{x\} = x'' + M(t)\,x' + Z(t)\,x$; all equal to zero.

From the Wronskian matrix we got that;

$$\text{Wronskian} = x_1 x_2{}' - x_2 x_1{}'$$

” W ” equation is non zero at the initial value t_0.

Then the following values are decided:

$$x(t_0) = b_1\ ;\ \ x'(t_0) = b_2$$

This is following by choosing the values of b_1 and b_2 as in the below equation:

$$X = b_1x_1(t_0) + b_2x_2(t_0)$$

This final formula is defining the differential equation and the initial value of it. This can be used to simplify the problem with such initial value differential equation.

Method 4: by applying the same assumption that x_1 and x_2 are both solutions for the differential type equations.

$L\{x\} = x'' + M(t)\,x' + Z(t)\,x$; all equal to zero.

The solution family can be expressed as X

Where $X = b_1x_1(t_0) + b_2x_2(t_0)$. With the arbitrary b_1 and b_2 are representing the solution of every differential equation.

When the (t_0) is a point where [W matrix] of x_1 and x_2 is acting as non zero.

From those four theories, we can notice that the fourth one is handling the linear differential equations only. It is not involving the initial value problem. Some facts can be states from the above derivation as follow:

- There are two solutions for each second-order ordinary differential equation
- Those solutions are not dependent on linear conditions
- All the solutions of differential equations can be expressed as a combination of two or more variables of linear form.

V. The Problem of Boundary Value

By assuming the following expression is a form of the differential equation:

$$x'' + M(n)x' + Z(n)x = 0$$

and the conditions of the boundary are looking like:

$$X\,(v) = a \;;\; X\,(w) = b \;;$$

For solving the boundary value in differential equation, it is must find the function $x = \emptyset$; this final formula is satisfying the differential question on the given limits (v, w); and producing both values of (a) and (b) at the last point of its interval. In order to find the value of $x = \emptyset$; the first step is examining the ordinary differential equation’s solution and the other important step is employing the conditions of boundary to solve the problem by finding the values of the constants.

In spite of the fact telling that initial value problem and boundary value problem linear solution is straight forward; the noticeable point is the solutions highly independent.

Many types of solutions can be assigned to the boundary value problem such as a unique solution, the non-finite solution and no solution. That can be decided on the basis that how is the problem nature.

$$x'' + x = 0$$

When the boundaries are given as below:

$$x\,(0) = c_1 \;;$$

$$x\left(\frac{\pi}{2}\right) = c_2 \;;$$

If the boundaries are changed into:

$$x\,(0) = c_1 \;;\text{and}\;\; x\,(2\pi) = c_2 \;;$$

We can achieve a single solution existence. The rest derivation can be given as below:

$$C * y = D$$

C: is $n \times n$ matrix; y: is $n \times 1$ vector and D: is $n \times 1$ vector;

$$C\,y = 0;$$

$$x'' + M(y)\,x' + Z(y)\,x = 0$$

with $x(v) = 0$ and $x(m) = 0$

$$\begin{bmatrix} x_1(v) & x_2(v) \\ x_1(m) & x_2(m) \end{bmatrix} * \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$C * y = \gamma * y$$

$$x'' + \gamma\, x = 0$$

Finally:

$$x\,(v) = 0 \;\;,\;\; x(m) = 0$$

VI. CONCLUSION

By forming a step size reduction we can minimize the error. If the solution to the differential equation is linear, the method will provide error-free predictions as for a straight line the 2nd derivative would be zero. Because we can choose an infinite number of values for x_1 and x_2, there is an infinite number of second-order Runge-Kutta methods. Every version would yield exactly the same results if the solution to Ordinary Differential Equations were quadratic, linear, or a constant. However, they yield different results if the solution is more complicated (typically the case).

REFERENCES

[1] Abraham O., Improving the modified Euler method, Leonardo J. Sci, 2007, 10, p. 1-8.

[2] Rattenbury N., Almost Runge-Kutta methods for stiff and non-stiff problems, PhD dissertation, The University of Auckland, New Zealand, 2005.

[3] Julyan, E. H. C., Piro O., The dynamics of Runge-Kutta methods, Int'l J. Bifur. and Chaos, 1992, 2. 1 – 8.

[4] Butcher J. C., Numerical methods for ordinary differential equations in the 20th, Century, J. Comput. Appl. Math., 2000, 125, p. 1-29.

[5] Euler H., Institutions calculi integrals, Volume Premium (1768), Opera Omnia, Vol. XI, B. G. Teubneri Lipsiae et Berolini, MCMXIII, 1768.

[6] Butcher J. C., General linear methods: A survey, Appl. Number. Math., 1985, 1. 107 – 108.

[7] Runge D., Ueber die numerische Auflösung von differential gleichungen, Math. Ann. 1985, 46, p. 167 – 178.

[8] Butcher J. C., and Wanner G., Runge-Kutta methods: some historical notes, Appl. Number. Math., 1996, 22. 115 – 116.

[9] Fatunla S. O., Numerical methods for initial value problems in ordinary differential equations, Academic Press Inc. (London) Ltd. p. 48-51.

[10] Abraham O., A fifth-order six-stage explicit Runge-Kutta method for the solution of initial value problems, M. Tech Dissertation, Federal University of Technology, Minna, Nigeria, 2004.

[11] Coddington E.A., *Ordinarly Linear Differential Equations*, Krieger Pub Co,(1984).