A note on growth properties of composite $p-$adic entire functions on the basis of slowly changing functions

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Abstract

Let K be a complete ultametric algebraically closed field and $A(K)$ be the K -algebra of entire functions on K . For f , g in $A(K)$,in this paper we introduce the notation of L $(p,q)^{th}$ Ψ-order (L^* $(p,q)^{th}$ Ψ-order) and L $(p,q)^{th}$ Ψ-lower order (L^* $(p,q)^{th}$ Ψ-order) of composition of entire function using the concept of central index. Hence after proving some basic results, in this paper, we study some growth properties of composite p-adic entire functions according to their L-order(L^*-order) and L-lower order(L^*-lower order).

Keywords and Phrases: p-adic entire functions, growth, relative L-order, relative L-lower order, relative L^*-order, relative L^*-lower order, composition, central index.

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1 INTRODUCTION

1.1 Introduction, Definitions and Notations

Let us consider K to be an algebrically closed field of charecteristic 0 , complete with respect to a p-adic absolute value $|.|$ (example $\mathbb{C}_p$). For any $\alpha \in K$ and $R \in]0, +\infty[$,

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the closed disc$\{x \in K : |x-a| \leq R\}$ and the open disc $\{x \in K : |x-a| < R\}$ are denoted by $d(\alpha, R)$ and $d(\alpha, R^-)$, respectively. Also $C(\alpha, R)$ denotes the circle $\{x \in K : |x-a| = R\}$. More over, $A(K)$ represents the K-algebra of analytic functions in K i.e. the set of power series with an infinite radius of convergence. For the most comprehensive study of analytic functions inside a disc or in the whole field K, we refer the reader to the books [9, 14, 16] During the last several years , the idea of p-adic analysis have been studied from different aspect and many important result were gained [2, 3, 4, 6, 8, 11, 16, 17]

Let $f \in A(K)$ and $r > 0$, then we denote by $|f|(r)$ the number $\sup\{|f(x)| : |x| = r\}$, where $|.|(r)$ is a multiplicative norm on $A(K)$. More over if f is not constant the $|f|(r)$ is a strictly increasing function of r and tends to$+\infty$ with r.So there exists its inverse function

$$\widehat{|f|} : (|f(0)|, \infty) \to (0, \infty)$$

with

$$\lim_{s\to\infty} \widehat{|f|}(s) = \infty$$

For $x \in [0, \infty)$ and $k \in \mathbb{N}$, we define

$$\log^{[k]} x = \log(\log^{[k-1]} x)$$

and

$$\exp^{[k]} x = \exp(\exp^{[k-1]} x),$$

where $\mathbb{N}$ be the set of all positive integers. We also denote $\log^{[0]} x = x$ and $\exp^{[0]} x = x$. Throughout the paper,log denotes the Neperian logarithm.

Let p be a prime number, let Q_p be the field of p-adic numbers and let $\mathbb{C}_p$ be the p-adic completion of the algebric closure of Q_p.The absolute value $|\cdot|_p$in $\mathbb{C}_p$ is normalized so that $|\cdot|_p = p^{-1}$. We furthur use the notation ord_p for additive valuation on $\mathbb{C}_p$.

We recall that in a complete metric space whose metric comes from a non archimedean norm,an infinite sum converges if and only if its general term approaches zero. The expression of the form

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, (a_n \in \mathbb{C}_p)$$

is well defined whenever $|a_n z^n|_p \to 0$.

Yang and Hu[7] defined the " radius of convergence " by

$$\frac{1}{\rho} = \limsup |a_n|_p^{\frac{1}{n}} .$$

Then the series converges if $|z|_p < \rho$ and dieverges if $|z|_p > \rho$. Also the function is said to be p-adic analytic on $B(\rho)$, where

$$B(\rho) = \left\{z \in \mathbb{C}_p : |z|_p < \rho\right\}$$

If $\rho = \infty$, the function is said to be entire on $\mathbb{C}_p$.

Let $f\in A(K)$,we consider a non constant p-adic analytic function

$$f\left(z\right)=\sum_{n=0}^{\infty}a_{n}z^{n},\left(a_{n}\in\mathbb{C}_{p}\right)\text{ on }B\left(\rho\right),\left(0<\rho\leq\infty\right).$$

The essence of Wiman- Valiron methode is the analysis of behaviour of the function by means of the maximum term :

$$\widehat{\mu}\left(r,f\right)=\max_{n\geq0}\left|a_{n}\right|_{p}r^{n},\left(0<r=\left|z\right|_{p}<\rho\right)$$

together with the central index

$$\widehat{\nu}\left(r,f\right)=\max_{n\geq0}\left\{n:\left|a_{n}\right|_{p}r^{n}=\widehat{\mu}\left(r,f\right)\right\}$$

Define

$$\widehat{\nu}\left(0,f\right)=\lim_{r\rightarrow0}\widehat{\nu}(r,f).$$

Obviously $\left|f\right|(r),\widehat{\mu}\left(r,f\right)$ and $\widehat{\nu}\left(r,f\right)$ are real and increasing function of r.For another entire function $g\in A(K)$ the ratio $\frac{|f|(r)}{|g|(r)}$ as $r\rightarrow\infty$ as well as $\frac{\widehat{\mu}(r,f)}{\widehat{\mu}(r,g)}$ as $r\rightarrow\infty$ is called the growth of f with respect to g in terms of their multiplicative norm and maximum term respectively.Though $\widehat{\nu}\left(r,f\right)$ is very much weaker than $\left|f\right|(r)$ and $\widehat{\mu}\left(r,f\right)$ in some sense, from another angle of view $\frac{\widehat{\nu}(r,f)}{\widehat{\nu}(r,g)}$ as $r\rightarrow\infty$ is called the growth of f with respect g where $\widehat{\nu}\left(r,g\right)$ denote the central index of entire function g.

However the notation of relative order was first introduced by *Bernal [1],[2]* .In order to make some progress in the study of p-adic analysis , now we introduce the definition of relative orderof entire function $f\in A(K)$ with respect to another $f\in A(K)$ entire function $g\in A(K)$ in the following way:

$$\rho_{g}\left(f\right)=\inf\left\{\mu>0:\left|f\right|(r)<\left|g\right|\left(r^{\mu}\right)\text{ for all }r>r_{0}\left(\mu\right)>0\right\}$$

Like wise to complex analysis [17] we can say that for any entire function f of positive finite order $\log\widehat{\mu}\left(r,f\right)$, $\log\left|f\right|(r)$, $\widehat{\nu}\left(r,f\right)$ both have same order.

Definition 1.1 *[4, 5] The order$\rho(f)$ (resp. lower order $\lambda(f)$) of an entire function $f\in A(K)$ as*

$$\rho(f)=\lim_{r\rightarrow\infty}\sup\frac{\log^{[2]}\left|f\right|(r)}{\log r}$$

and

$$\lambda(f)=\lim_{r\rightarrow\infty}\inf\frac{\log^{[2]}\left|f\right|(r)}{\log r}.$$

We defined the order $\rho(f)$ (resp. lower order $\lambda(f)$) of an entire function $f\in A(K)$ in terms of central index as

$$\rho(f)=\lim_{r\rightarrow\infty}\sup\frac{\log\widehat{\nu}\left(r,f\right)}{\log r}$$

and

$$\lambda(f)=\lim_{r\rightarrow\infty}\inf\frac{\log\widehat{\nu}\left(r,f\right)}{\log r}$$

Definition 1.2 *The L- order and L- lower order of an p-adic entire function $f \in A(K)$ is defined as*

$$\rho_f^L = \lim_{r\to\infty} \sup \frac{\log^{[2]} |f|(r)}{\log rL(r)}$$

and

$$\lambda_f^L = \lim_{r\to\infty} \inf \frac{\log^{[2]} |f|(r)}{\log rL(r)}.$$

Let $f \in A(K)$,we define the L- order and L- lower order of an p-adic entire function f using the concept of central index are respectively as

$$\rho_f^L = \lim_{r\to\infty} \sup \frac{\log \widehat{\nu}(r,f)}{\log rL(r)}$$

and

$$\lambda_f^L = \lim_{r\to\infty} \inf \frac{\log \widehat{\nu}(r,f)}{\log rL(r)}.$$

Where $L \equiv L(r)$ is a positive continuous function increasing slowly i.e.,

$$L(ar) \sim L(r) \ \ as\ r \to \infty$$

for every positive constant $'a'$.The more generalized concept for L-order and L- lower order for entire function are L^*-order and L^*- lower order.

Definition 1.3 *[5],[6]The $(p,q)^{th}$ order and $(p,q)^{th}$lower order of an entire function $f \in A(K)$ is defined as*

$$\rho_f^{(p,q)} = \lim_{r\to\infty} \sup \frac{\log^{[p]} |f|(r)}{\log^{[q]} r}$$

and

$$\lambda_f^{(p,q)} = \lim_{r\to\infty} \inf \frac{\log^{[p]} |f|(r)}{\log^{[q]} r}.$$

Similar to the comlex analysis we say that for $0 \le r < R$,

$$\widehat{\nu}(r,f) \le |f|(r) \le \frac{R}{R-r}\widehat{\nu}(r,f).$$

So,it is easy to see that

$$\rho_f^{(p,q)} = \lim_{r\to\infty} \sup \frac{\log^{[p]} \widehat{\nu}(r,f)}{\log^{[q]} r}$$

and

$$\lambda_f^{(p,q)} = \lim_{r\to\infty} \inf \frac{\log^{[p]} \widehat{\nu}(r,f)}{\log^{[q]} r}$$

Definition 1.4 *The* $L\ (p,q)^{th}$ Ψ*-order and relative* $L\ (p,q)^{th}$ Ψ*- lower order of an entire function* $f\in A(K)$ *is defined as*

$$\rho_{f,\Psi}^{L}=\lim_{r\rightarrow\infty}\sup\frac{\log^{[p]}|f|(r)}{\log^{[q]}[\Psi(r)\,L(r)]}$$

and

$$\lambda_{f,\Psi}^{L}=\lim_{r\rightarrow\infty}\inf\frac{\log^{[p]}|f|(r)}{\log^{[q]}[\Psi(r)\,L(r)]}$$

We define the $L\ (p,q)^{th}$ Ψ*-order and relative* $L\ (p,q)^{th}$ Ψ*- lower order of an entire function* f *using the concept of central index as*

$$\rho_{f,\Psi}^{L}=\lim_{r\rightarrow\infty}\sup\frac{\log^{[p]}\widehat{\nu}(r,f)}{\log^{[q]}[\Psi(r)\,L(r)]}$$

and

$$\lambda_{f,\Psi}^{L}=\lim_{r\rightarrow\infty}\inf\frac{\log^{[p]}\widehat{\nu}(r,f)}{\log^{[q]}[\Psi(r)\,L(r)]}.,$$

where p,q *are integers and* $p>q$ *and* $\Psi:[0,\infty)\rightarrow(0,\infty)$ *is a non-decreasing unbounded function, satisfying the following two conditions:*

$$(i)\lim_{r\rightarrow\infty}\frac{\log^{[p]}r}{\log^{[q]}\Psi(r)}=0$$

and

$$(ii)\lim_{r\rightarrow\infty}\frac{\log^{[q]}\Psi(\alpha r)}{\log^{[q]}\Psi(r)}=1,$$

for $\alpha>1$.

Definition 1.5 *The* $L^{*}\ (p,q)^{th}$ Ψ*-order and relative* $L^{*}\ (p,q)^{th}$ Ψ*- lower order of an entire function* $f\in A(K)$ *is defined as*

$$\rho_{f,\Psi}^{L^{*}}=\lim_{r\rightarrow\infty}\sup\frac{\log^{[p]}|f|(r)}{\log^{[q]}[\Psi(r)\,e^{L(r)}]}$$

and

$$\rho_{f,\Psi}^{L^{*}}=\lim_{r\rightarrow\infty}\inf\frac{\log^{[p]}|f|(r)}{\log^{[q]}[\Psi(r)\,e^{L(r)}]}$$

Let $f\in A(K)$*, we define the* $L^{*}\ (p,q)^{th}$ Ψ*-order and relative* $L^{*}\ (p,q)^{th}$ Ψ*- lower order of an entire function* $f\in A(K)$ *using the concept of central index as*

$$\rho_{f,\Psi}^{L^{*}}=\lim_{r\rightarrow\infty}\sup\frac{\log^{[p]}\widehat{\nu}(r,f)}{\log^{[q]}[\Psi(r)\,e^{L(r)}]}$$

and

$$\lambda_{f,\Psi}^{L^{*}}=\lim_{r\rightarrow\infty}\inf\frac{\log^{[p]}\widehat{\nu}(r,f)}{\log^{[q]}[\Psi(r)\,e^{L(r)}]}.,$$

where p,q *are integers and* $p>q$.

2 Lemmas.

In this section we present some lemmas which will be needed in the sequel.

Lemma 2.1 *[18] Let* $f(z) = \sum_{n=0}^{\infty} a_n z^n$ *be a p-adic entire function,* $\widehat{\mu}(r, f)$ *be the maximum term that is*

$$\widehat{\mu}(r, f) = \max_{n\geq 0} |a_n|_p r^n, \left(0 < r = |z|_p < \rho\right)$$

and $\widehat{\nu}(r, f)$ *be the central index of* f*. Then* (i) *for* $a_0 \neq 0$

$$\log \widehat{\mu}(r, f) = \log |a_0| + \int_0^r \frac{\widehat{\nu}(t, f)}{t} dt$$

(ii) *for* $r < R$

$$|f|(r) < \widehat{\mu}(r, f) \left\{\widehat{\nu}(r, f) + \frac{R}{R - r}\right\}$$

Lemma 2.2 *The definitions of order* $\rho(f)$ *(resp. lower order* $\lambda(f)$*) of an entire function* $f \in A(K)$ *defined in Definition (1.1) in terms of multiplicative norm and in terms of central index are equivalent.*

Proof. We set

$$f(z) = \sum_{n=0}^{\infty} a_n z^n.$$

Without loss of generality We can assume that $|a_0| \neq 0$.By (i) of Lemma (2.1), we have

$$\log \widehat{\mu}(2r, f) = \log |a_0| + \int_0^{2r} \frac{\widehat{\nu}(t, f)}{t} dt \geq \widehat{\nu}(r, f) \log 2$$

Using Cauchy inequality,it is easy to see that

$$\widehat{\mu}(2r, f) \leq |f|(2r).$$

Hence

$$\widehat{\nu}(r, f) \log 2 \leq \log |f|(2r)$$

So,

$$\limsup_{r\to\infty} \frac{\log \widehat{\nu}(r, f)}{\log r} \leq \limsup_{r\to\infty} \frac{\log^{[2]} |f|(r)}{\log r} = \rho(f). \quad (1)$$

On the other hand by (ii) of Lemma (2.1), we have

$$|f|(r) < \widehat{\mu}(r, f) \{\widehat{\nu}(2r, f) + 2\} = \left|a_{\widehat{\nu}(r,f)}\right| r^{\widehat{\nu}(r,f)} \{\widehat{\nu}(2r, f) + 2\}.$$

Since $\{|a_n|\}$ is a bounded sequence , we have

$$\log |f|(r) \leq \widehat{\nu}(r, f) \log r + \log \widehat{\nu}(2r, f) + C,$$

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where C is a constant.So,

$$\log^{[2]}|f|(r) \leq \log\widehat{\nu}(r,f) + \log^{[2]}\widehat{\nu}(2r,f) + \log^{[2]} r + C_1$$

i.e,

$$\limsup_{r\to\infty} \frac{\log^{[2]}|f|(r)}{\log r} \leq \limsup_{r\to\infty} \frac{\log\widehat{\nu}(r,f)}{\log r}. \qquad (2)$$

Combining equation (1) and equation (2) we say that the two definition are equivalent.

Similarly we can prove the same for lower order. ■

3 Main Results.

In this section we state the main results of the paper

Theorem 3.1 *.Let f and $g \in A(K)$ be three p-adic entire functions such that*

$$0 < \lambda^L_{f\circ g,\Psi}(p,q) \leq \rho^L_{f\circ g,\Psi}(p,q) < \infty$$

and

$$0 < \rho^L_{g,\Psi}(n,q) < \infty,$$

where p,q,n are positive integers such that $q < \min\{p,n\}$ .Then

$$(i) \quad \liminf_{r\to\infty} \frac{\log^{[p]}\widehat{\nu}(r,f\circ g)}{\log^{[n]}\widehat{\nu}(r,g)} \leq \frac{\rho^L_{f\circ g,\Psi}(p,q)}{\rho^L_{g,\Psi}(n,q)} \leq \limsup_{r\to\infty} \frac{\log^{[p]}\widehat{\nu}(r,f\circ g)}{\log^{[n]}\widehat{\nu}(r,g)}.$$

Further if $\lambda^L_{g,\Psi}(n,q) > 0$ then

$$\begin{aligned}(ii) \quad \frac{\lambda^L_{f\circ g,\Psi}(p,q)}{\rho^L_{g,\Psi}(n,q)} &\leq \liminf_{r\to\infty} \frac{\log^{[p]}\widehat{\nu}(r,f\circ g)}{\log^{[n]}\widehat{\nu}(r,g)} \leq \frac{\lambda^L_{f\circ g,\Psi}(p,q)}{\lambda^L_{g,\Psi}(n,q)} \\ &\leq \limsup_{r\to\infty} \frac{\log^{[p]}\widehat{\nu}(r,f\circ g)}{\log^{[n]}\widehat{\nu}(r,g)} \\ &\leq \frac{\rho^L_{f\circ g,\Psi}(p,q)}{\lambda^L_{g,\Psi}(n,q)}\end{aligned}$$

and

$$(iii) \quad \liminf_{r\to\infty} \frac{\log^{[p]}\widehat{\nu}(r,f\circ g)}{\log^{[n]}\widehat{\nu}(r,g)} \leq \min\left\{\frac{\lambda^L_{f\circ g,\Psi}(p,q)}{\lambda^L_{g,\Psi}(n,q)}, \frac{\rho^L_{f\circ g,\Psi}(p,q)}{\rho^L_{g,\Psi}(n,q)}\right\}$$

$$\leq \max\left\{\frac{\lambda^L_{f\circ g,\Psi}(p,q)}{\lambda^L_{g,\Psi}(n,q)}, \frac{\rho^L_{f\circ g,\Psi}(p,q)}{\rho^L_{g,\Psi}(n,q)}\right\} \leq \limsup_{r\to\infty} \frac{\log^{[p]}\widehat{\nu}(r,f\circ g)}{\log^{[n]}\widehat{\nu}(r,g)}.$$

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Proof. (i) From the definition of L-$(p,q)th$ Ψ-order we have for arbitrary positive ϵ and for all large values of r

$$\begin{aligned} \frac{\log^{[p]}\widehat{\nu}(r,\ f\circ g)}{\log^{[q]}\left[\Psi\left(r\right)L\left(r\right)\right]} &\leq \rho^{L}_{f\circ g,\Psi}(p,q)+\epsilon \\ \log^{[p]}\widehat{\nu}(r,\ f\circ g) &\leq \left(\rho^{L}_{f\circ g,\Psi}(p,q)+\epsilon\right)\log^{[q]}\left[\Psi\left(r\right)L\left(r\right)\right] \end{aligned} \tag{3}$$

and for a sequence of values of r tending to infinity,

$$\begin{aligned} \frac{\log^{[n]}\widehat{\nu}\left(r,g\right)}{\log^{[q]}\left[\Psi\left(r\right)L\left(r\right)\right]} &\geq \rho^{L}_{g,\Psi}(n,q)-\epsilon \\ \log^{[n]}\widehat{\nu}(r,g) &\geq \left(\rho^{L}_{g,\Psi}(n,q)-\epsilon\right)\log^{[q]}\left[\Psi\left(r\right)L(r)\right] \end{aligned} \tag{4}$$

Now from Equation (3) and Equation (4) it follows for a sequence of values of r tending to infinity,

$$\frac{\log^{[p]}\widehat{\nu}(r,f\circ g)}{\log^{[n]}\widehat{\nu}(r,g)}\leq\frac{\rho^{L}_{f\circ g,\Psi}(p,q)+\epsilon}{\left(\rho^{L}_{g,\Psi}(n,q)-\epsilon\right)}$$

As ϵ (>0) is arbitrary, we obtain that

$$\liminf_{r\to\infty}\frac{\log^{[p]}\widehat{\nu}(r,f\circ g)}{\log^{[n]}\widehat{\nu}(r,g)}\leq\frac{\rho^{L}_{f\circ g,\Psi}(p,q)}{\rho^{L}_{g,\Psi}(n,q)} \tag{5}$$

Again for a sequence of values of r tending to infinity,

$$\begin{aligned} \frac{\log^{[p]}\widehat{\nu}(r,\ f\circ g)}{\log^{[q]}\left[\Psi\left(r\right)L\left(r\right)\right]} &\geq \rho^{L}_{f\circ g,\Psi}(p,q)-\epsilon \\ \log^{[p]}\widehat{\nu}(r,\ f\circ g) &\geq \left(\rho^{L}_{f\circ g,\Psi}(p,q)-\epsilon\right)\log^{[q]}\left[\Psi\left(r\right)L\left(r\right)\right] \end{aligned} \tag{6}$$

Also for all sufficiently large values of r,

$$\begin{aligned} \frac{\log^{[n]}\widehat{\nu}\left(r,g\right)}{\log^{[q]}\left[\Psi\left(r\right)L\left(r\right)\right]} &\leq \rho^{L}_{g,\Psi}(n,q)+\epsilon \\ \log^{[n]}\widehat{\nu}(r,g) &\leq \left(\rho^{L}_{g,\Psi}(n,q)+\epsilon\right)\log^{[q]}\left[\Psi\left(r\right)L(r)\right] \end{aligned} \tag{7}$$

So combining Equation (6) and Equation (7), we get for a sequence of values of r tending to infinity,

$$\frac{\log^{[p]}\widehat{\nu}(r,f\circ g)}{\log^{[n]}\widehat{\nu}(r,g)}\geq\frac{\rho^{L}_{f\circ g,\Psi}(p,q)-\epsilon}{\left(\rho^{L}_{g,\Psi}(n,q)+\epsilon\right)}$$

Since ϵ (>0) is arbitrary,we get that

$$\limsup_{r\to\infty}\frac{\log^{[p]}\widehat{\nu}(r,f\circ g)}{\log^{[n]}\widehat{\nu}(r,g)}\geq\frac{\rho^{L}_{f\circ g,\Psi}(p,q)}{\rho^{L}_{g,\Psi}(n,q)} \tag{8}$$

Combining Equation (5) and Equation (8) we get (i)

(ii) From the definition of L-(p,q)-th lower order, we have for arbitrary positive ϵ and for all large values of r,

$$\begin{aligned} \frac{\log^{[p]}\widehat{\nu}(r,\ f\circ g)}{\log^{[q]}\left[\Psi\left(r\right)L\left(r\right)\right]} &\geq \lambda^{L}_{f\circ g,\Psi}(p,q)-\epsilon \\ \log^{[p]}\widehat{\nu}(r,f\circ g) &\geq \left(\lambda^{L}_{f\circ g,\Psi}(p,q)-\epsilon\right)\log^{[q]}\left[\Psi\left(r\right)L(r)\right] \end{aligned} \tag{9}$$

Now from Equation (9) and Equation (7) , it follows for all large values of r,

$$\frac{\log^{[p]}\widehat{\nu}(r,f\circ g)}{\log^{[n]}\widehat{\nu}(r,g)} \geq \frac{\lambda^{L}_{f\circ g,\Psi}(p,q)-\epsilon}{\rho^{L}_{g,\Psi}(n,q)+\epsilon}$$

As ϵ (>0) is arbitrary ,we obtain that

$$\liminf_{r\to\infty}\frac{\log^{[p]}\widehat{\nu}(r,f\circ g)}{\log^{[n]}\widehat{\nu}(r,g)} \geq \frac{\lambda^{L}_{f\circ g,\Psi}(p,q)}{\rho^{L}_{g,\Psi}(n,q)}. \tag{10}$$

Again for a sequence of values of r tending to infinity,

$$\begin{aligned} \frac{\log^{[n]}\widehat{\nu}\left(r,f\circ g\right)}{\log^{[q]}\left[\Psi\left(r\right)L\left(r\right)\right]} &\leq \lambda^{L}_{f\circ g,\Psi}(n,q)+\epsilon \\ \log^{[n]}\widehat{\nu}(r,f\circ g) &\leq \left(\lambda^{L}_{f\circ g,\Psi}(n,q)+\epsilon\right)\log^{[q]}\left[\Psi\left(r\right)L(r)\right] \end{aligned} \tag{11}$$

and for all large values of r,

$$\log^{[n]}\widehat{\nu}(r,g) \geq \left(\lambda^{L}_{g,\Psi}(n,q)-\epsilon\right)\log^{[q]}\left[\Psi\left(r\right)L(r)\right]. \tag{12}$$

So combining Equation (11) and Equation (12) we get for a sequence of values of r tending to infinity,

$$\frac{\log^{[p]}\widehat{\nu}(r,f\circ g)}{\log^{[n]}\widehat{\nu}(r,g)} \leq \frac{\lambda^{L}_{f\circ g,\Psi}(p,q)+\epsilon}{\lambda^{L}_{g,\Psi}(n,q)-\epsilon}$$

Since ϵ (>0) is arbitrary, it follows that

$$\liminf_{r\to\infty}\frac{\log^{[p]}\widehat{\nu}(r,f\circ g)}{\log^{[n]}\widehat{\nu}(r,g)} \leq \frac{\lambda^{L}_{f\circ g,\Psi}(p,q)}{\lambda^{L}_{g,\Psi}(n,q)}. \tag{13}$$

Also for a sequence of values of r tending to infinity,

$$\begin{aligned} \frac{\log^{[n]}\widehat{\nu}\left(r,g\right)}{\log^{[q]}\left[\Psi\left(r\right)L\left(r\right)\right]} &\leq \lambda^{L}_{g,\Psi}(n,q)+\epsilon. \\ \log^{[n]}\widehat{\nu}(r,g) &\leq \left(\lambda^{L}_{g,\Psi}(n,q)+\epsilon\right)\log^{[q]}\left[\Psi\left(r\right)L(r)\right] \end{aligned} \tag{14}$$

Now from Equation (9) and Equation (14) we obtain for a sequence of values of r tending to infinity,

$$\frac{\log^{[p]}\widehat{\nu}(r,f\circ g)}{\log^{[n]}\widehat{\nu}(r,g)} \geq \frac{\lambda^{L}_{f\circ g,\Psi}(p,q)-\epsilon}{\lambda^{L}_{g,\Psi}(n,q)+\epsilon}$$

As ϵ (>0) is arbitrary ,we obtained

$$\limsup_{r\to\infty}\frac{\log^{[p]}\widehat{\nu}(r,f\circ g)}{\log^{[n]}\widehat{\nu}(r,g)}\geq\frac{\lambda^{L}_{f\circ g,\Psi}(p,q)}{\lambda^{L}_{g,\Psi}(n,q)}. \quad (15)$$

Again from Equation (3) and Equation (12) , it follows for all large values of r,

$$\frac{\log^{[p]}\widehat{\nu}(r,f\circ g)}{\log^{[n]}\widehat{\nu}(r,g)}\leq\frac{\rho^{L}_{f\circ g,\Psi}(p,q)+\epsilon}{\lambda^{L}_{g,\Psi}(n,q)-\epsilon}$$

As ϵ (>0) is arbitrary ,we obtain that,

$$\limsup_{r\to\infty}\frac{\log^{[p]}\widehat{\nu}(r,f\circ g)}{\log^{[n]}\widehat{\nu}(r,g)}\leq\frac{\rho^{L}_{f\circ g,\Psi}(p,q)}{\lambda^{L}_{g,\Psi}(n,q)} \quad (16)$$

From Equation (10) , (13) , (15)and (16) we get (ii).

Combining (i) and (ii) of the Theorem (3.1) we get (iii)

Remark 3.1 *The condition* $q<\min\{p,n\}$ *in theorem* (3.1) *is essential as we see in the following example.*

Example 1 *Let* $f=z^2, g=\log z, p=1, q=1, n=1, \Psi=z^2, L=\log r$ *then,*

$$\begin{aligned}\rho^{L}_{f\circ g,\Psi}(p,q) &= \limsup_{r\to\infty}\frac{\log\widehat{\nu}(r,(\log z)^2)}{\log\left[r^2\log r\right]}=\limsup_{r\to\infty}\frac{[2\log(\log r)]}{2\log r+\log^{[2]}(r)}\\ &= \limsup_{r\to\infty}\frac{2\log^{[2]}(r)}{2\log r+\log^{[2]}(r)}=0\\ \rho^{L}_{g,\Psi}(n,q) &= \limsup_{r\to\infty}\frac{\log\widehat{\nu}(r,\log z)}{\log\left[r^2\log(r)\right]}=\limsup_{r\to\infty}\frac{\log[\log r]}{2\log r+\log^{[2]}(r)}=0\end{aligned}$$

This leads to a contradriction. So $q<\min\{p,n\}$ *is essential for Theorem* (3.1)

Theorem 3.2 *If* f *and* $g\in A(K)$ *be three p-adic entire functions with* $\rho^{L}_{g,\Psi}(n,q)<\infty$ *and* $\rho^{L}_{f\circ g,\Psi}(p,q)=\infty$, *then,*

$$\limsup_{r\to\infty}\frac{\log^{[p]}\widehat{\nu}(r,f\circ g)}{\log^{[n]}\widehat{\nu}(r,g)}=\infty$$

where p,q,n *are positive integers with* $q<\min\{p,n\}$.

Proof. Let us assume that the conclusion of the theorem do not hold. Then there exists a constant $A_1>0$ such that for all sufficiently large values of r,

$$\begin{aligned}\frac{\log^{[p]}\widehat{\nu}(r,f\circ g)}{\log^{[n]}\widehat{\nu}(r,g)} &\leq A_1\\ \log^{[p]}\widehat{\nu}(r,f\circ g) &\leq A_1\log^{[n]}\widehat{\nu}(r,g) \quad (17)\end{aligned}$$

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Again from the definition of $\rho^{L}_{g,\Psi}(n,q)$, it follows that for all large values of r,

$$\begin{aligned}\frac{\log^{[n]}\widehat{\nu}\left(r,g\right)}{\log^{[q]}\left[\Psi\left(r\right)L\left(r\right)\right]} &\leq \rho^{L}_{g,\Psi}(n,q)+\epsilon \\ \log^{[n]}\widehat{\nu}\left(r,g\right) &\leq \left[\rho^{L}_{g,\Psi}(n,q)+\epsilon\right]\log^{[q]}\left[\Psi\left(r\right)L\left(r\right)\right] \qquad (18)\end{aligned}$$

So from Equation (17) and Equation (18), we obtain for all sufficiently large values of r,

$$\begin{aligned}\log^{[p]}\widehat{\nu}(r,f\circ g) &\leq A_1(\rho^{L}_{g,\Psi}(n,q)+\epsilon)\log^{[q]}\left[\Psi\left(r\right)L(r)\right] \\ \frac{\log^{[p]}\widehat{\nu}\left(r,f\circ g\right)}{\log^{[q]}\left[\Psi\left(r\right)L\left(r\right)\right]} &\leq A_1(\rho^{L}_{g,\Psi}(n,q)+\epsilon) \\ i.e.\limsup_{r\to\infty}\frac{\log^{[p]}\widehat{\nu}\left(r,f\circ g\right)}{\log^{[q]}\left[\Psi\left(r\right)L\left(r\right)\right]} &\leq A_1(\rho^{L}_{g,\Psi}(n,q)+\epsilon). \qquad (19)\end{aligned}$$

From Equation (19) it follows that $\rho^{L}_{f\circ g,\Psi}(p,q)<\infty$. So we arrive at a contradiction. Thus our assumption is wrong.

This proves the theorem.

Remark 3.2 *Similar to the Theorem (3.2) we can prove the same for L $(p,q)^{th}$ Ψ-lower order and L^{*} $(p,q)^{th}$ Ψ- lower order.*

Theorem 3.3 *If f and $g\in A\left(K\right)$ be three p-adic entire functions with $\lambda^{L}_{g,\Psi}(n,q)<\infty$ and $\lambda^{L}_{f\circ g,\Psi}(p,q)<\infty$, then,*

$$\liminf_{r\to\infty}\frac{\log^{[p]}\widehat{\nu}(r,f\circ g)}{\log^{[n]}\widehat{\nu}(r,g)}<\infty$$

where p,q,n are positive integers with $q<\min\left\{p,n\right\}$.

Proof. Let us assume that the conclusion of the theorem do not hold. Then there exists a large number $G>0$ such that for all sufficiently large values of r,

$$\begin{aligned}\frac{\log^{[p]}\widehat{\nu}(r,f\circ g)}{\log^{[n]}\widehat{\nu}(r,g)} &> G \\ \log^{[p]}\widehat{\nu}(r,f\circ g) &> G\log^{[n]}\widehat{\nu}(r,g) \qquad (20)\end{aligned}$$

Again from the definition of $\lambda^{L}_{g\circ h,\Psi}(n,q)$, it follows that for all large values of $r\to\infty$.

$$\begin{aligned}\frac{\log^{[n]}\widehat{\nu}\left(r,g\right)}{\log^{[q]}\left[\Psi\left(r\right)L\left(r\right)\right]} &\geq \lambda^{L}_{g,\Psi}(n,q)-\epsilon \\ \log^{[n]}\widehat{\nu}\left(r,g\right) &\geq \left[\lambda^{L}_{g,\Psi}(n,q)-\epsilon\right]\log^{[q]}\left[\Psi\left(r\right)L\left(r\right)\right] \qquad (21)\end{aligned}$$

So from Equation (20) and Equation (21), we obtain for all sufficiently large values of r,

$$\begin{aligned}\log^{[p]}\widehat{\nu}(r,f\circ g) &> G(\lambda^{L}_{g,\Psi}(n,q)-\epsilon)\log^{[q]}\left[\Psi\left(r\right)L(r)\right] \\ \frac{\log^{[p]}\widehat{\nu}\left(r,f\circ g\right)}{\log^{[q]}\left[\Psi\left(r\right)L\left(r\right)\right]} &\leq G(\lambda^{L}_{g,\Psi}(n,q)-\epsilon)\end{aligned}$$

i.e.,

$$\liminf_{r\to\infty}\frac{\log^{[p]}\widehat{\nu}\left(r,f\circ g\right)}{\log^{[q]}\left[\Psi\left(r\right)L\left(r\right)\right]}>G(\lambda^{L}_{g,\Psi}(n,q)-\epsilon)$$

So, from the above we see that

$$\lambda^{L}_{f\circ g,\Psi}(p,q)=\infty.$$

Hence we arrive at a contradiction. Thus our assumption is wrong.

Hence the theorem is proved. ■

Remark 3.3 *The following example justifies the validity of Theorem* (3.3)*.*

Example 2 *Let* $f=z^2, g=z, p=2, q=1, n=2, \Psi=z^2, L=\log r$ *then,*

$$\begin{aligned}\lambda^{L}_{f\circ g,\Psi}(p,q) &= \liminf_{r\to\infty}\frac{\log^{[2]}\widehat{\nu}\left(r,z^2\right)}{\log^{[1]}\left[r^2\log r\right]}\\ &= \liminf_{r\to\infty}\frac{\log 2+\log^{[2]}r}{2\log r+\log^{[2]}r}\\ &= 0<\infty\end{aligned}$$

and

$$\begin{aligned}\lambda^{L}_{g,\Psi}(n,q) &= \liminf_{r\to\infty}\frac{\log^{[2]}\widehat{\nu}\left(r,z\right)}{\log^{[1]}\left[r^2\log r\right]}\\ &= \liminf_{r\to\infty}\frac{\log^{[2]}r}{2\log r+\log^{[2]}r}\\ &= 0<\infty.\end{aligned}$$

So,

$$\begin{aligned}\liminf_{r\to\infty}\frac{\log^{[p]}\widehat{\nu}(r,f\circ g)}{\log^{[n]}\widehat{\nu}(r,g)} &= \liminf_{r\to\infty}\frac{\log^{[2]}r^2}{\log^{[2]}r}\\ &= \liminf_{r\to\infty}\frac{\log 2+\log^{[2]}r}{\log^{[2]}r}\\ &= 1<\infty.\end{aligned}$$

Remark 3.4 *Similar to the Theorem* (3.3) *we can prove the same for* L $(p,q)^{th}$ Ψ*-order and* L^* $(p,q)^{th}$ Ψ*-order*

Theorem 3.4 *.Let* f, g *and* $h\in A(K)$ *be three p-adic entire functions such that* $0<\lambda^{L^*}_{f\circ g,\Psi}(p,q)\leq\rho^{L^*}_{f\circ g,\Psi}(p,q)<\infty$ *and* $0<\rho^{L^*}_{g,\Psi}(n,q)<\infty$ *where* p,q,n *are positive integers such that* $q<\min\{p,n\}$*.Then*

$$(i)\ \liminf_{r\to\infty}\frac{\log^{[p]}\widehat{\nu}(r,f\circ g)}{\log^{[n]}\widehat{\nu}(r,g)}\leq\frac{\rho^{L^*}_{f\circ g,\Psi}(p,q)}{\rho^{L^*}_{g,\Psi}(n,q)}\leq\limsup_{r\to\infty}\frac{\log^{[p]}\widehat{\nu}(r,f\circ g)}{\log^{[n]}\widehat{\nu}(r,g)}.$$

Further, if $\lambda^{L^*}_{g\circ h,\Psi}(n,q) > 0$ then

$$(ii) \quad \frac{\lambda^{L^*}_{f\circ g,\Psi}(p,q)}{\rho^{L^*}_{g,\Psi}(n,q)} \leq \liminf_{r\to\infty} \frac{\log^{[p]} \widehat{\nu}(r, f\circ g)}{\log^{[n]} \widehat{\nu}(r,g)} \leq \frac{\rho^{L^*}_{f\circ g,\Psi}(p,q)}{\rho^{L^*}_{g,\Psi}(n,q)}$$

and

$$(iii) \liminf_{r\to\infty} \frac{\log^{[p]} \widehat{\nu}(r, f\circ g)}{\log^{[n]} \widehat{\nu}(r,g)} \leq \frac{\lambda^{L^*}_{f\circ g,\Psi}(p,q)}{\lambda^{L^*}_{g,\Psi}(n,q)} \leq \limsup_{r\to\infty} \frac{\log^{[p]} \widehat{\nu}(r, f\circ g)}{\log^{[n]} \widehat{\nu}(r,g)}$$

$$(iv) \frac{\lambda^{L^*}_{f\circ g,\Psi}(p,q)}{\lambda^{L^*}_{g,\Psi}(n,q)} \leq \limsup_{r\to\infty} \frac{\log^{[p]} \widehat{\nu}(r, f\circ g)}{\log^{[n]} \widehat{\nu}(r,g)} \leq \frac{\rho^{L^*}_{f\circ g,\Psi}(p,q)}{\lambda^{L^*}_{g,\Psi}(n,q)}$$

$$(v) \quad \liminf_{r\to\infty} \frac{\log^{[p]} \widehat{\nu}(r, f\circ g)}{\log^{[n]} \widehat{\nu}(r,g)} \leq \min\left\{ \frac{\lambda^{L^*}_{f\circ g,\Psi}(p,q)}{\lambda^{L^*}_{g,\Psi}(n,q)}, \frac{\rho^{L^*}_{f\circ g,\Psi}(p,q)}{\rho^{L^*}_{g,\Psi}(n,q)} \right\}$$

$$\leq \max\left\{ \frac{\lambda^{L^*}_{f\circ g,\Psi}(p,q)}{\lambda^{L^*}_{g,\Psi}(n,q)}, \frac{\rho^{L^*}_{f\circ g,\Psi}(p,q)}{\rho^{L^*}_{g,\Psi}(n,q)} \right\} \leq \limsup_{r\to\infty} \frac{\log^{[p]} \widehat{\nu}(r, f\circ g)}{\log^{[n]} \widehat{\nu}(r,g)}.$$

Proof. (i) From the definition of L^*-(p,q)-th order ,we have for arbitrary positive ϵ and for all large values of r,

$$\begin{aligned} \frac{\log^{[p]} \widehat{\nu}(r, \ f\circ g)}{\log^{[q]} \left[\Psi\left(r\right) e^{L(r)}\right]} &\leq \rho^{L^*}_{f\circ g,\Psi}(p,q) + \epsilon. \\ \log^{[p]} \widehat{\nu}(r, \ f\circ g) &\leq \left(\rho^{L^*}_{f\circ g,\Psi}(p,q) + \epsilon\right) \log^{[q]} \left[\Psi\left(r\right) e^{L(r)}\right] \end{aligned} \quad (22)$$

and for a sequence of values of r tending to infinity,

$$\begin{aligned} \frac{\log^{[n]} \widehat{\nu}\left(r,g\right)}{\log^{[q]} \left[\Psi\left(r\right) e^{L(r)}\right]} &\geq \rho^{L^*}_{g,\Psi}(n,q) - \epsilon \\ \log^{[n]} \widehat{\nu}(r,g) &\geq \left(\rho^{L^*}_{g,\Psi}(n,q) - \epsilon\right) \log^{[q]} \left[\Psi\left(r\right) e^{L(r)}\right] \end{aligned} \quad (23)$$

Now from Equation (22) and Equation (23) it follows for a sequence of values of r tending to infinity,

$$\frac{\log^{[p]} \widehat{\nu}(r, f\circ g)}{\log^{[n]} \widehat{\nu}(r,g)} \leq \frac{\rho^{L^*}_{f\circ g,\Psi}(p,q) + \epsilon}{\left(\rho^{L^*}_{g,\Psi}(n,q) - \epsilon\right)}.$$

As ϵ (>0) is arbitrary, we obtain that,

$$\liminf_{r\to\infty} \frac{\log^{[p]} \widehat{\nu}(r, f\circ g)}{\log^{[n]} \widehat{\nu}(r,g)} \leq \frac{\rho^{L^*}_{f\circ g,\Psi}(p,q)}{\rho^{L^*}_{g,\Psi}(n,q)} \quad (24)$$

Again from the definition of L^*-(p,q)-th order for a sequence of values of r tending to infinity,

$$\begin{aligned} \frac{\log^{[p]} \widehat{\nu}(r, \ f\circ g)}{\log^{[q]} \left[\Psi\left(r\right) e^{L(r)}\right]} &\geq \rho^{L^*}_{f\circ g,\Psi}(p,q) - \epsilon \\ \log^{[p]} \widehat{\nu}(r, \ f\circ g) &\geq \left(\rho^{L^*}_{f\circ g,\Psi}(p,q) - \epsilon\right) \log^{[q]} \left[\Psi\left(r\right) e^{L(r)}\right] \end{aligned} \quad (25)$$

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Also for all sufficiently large values of r,

$$\begin{aligned}\frac{\log^{[n]}\widehat{\nu}\left(r,g\right)}{\log^{[q]}\left[\Psi\left(r\right)e^{L(r)}\right]} &\leq \rho_{g,\Psi}^{L^*}(n,q)+\epsilon \\ \log^{[n]}\widehat{\nu}(r,g) &\leq \left(\rho_{g,\Psi}^{L^*}(n,q)+\epsilon\right)\log^{[q]}\left[\Psi\left(r\right)e^{L(r)}\right] \end{aligned} \tag{26}$$

So combining Equation (25) and Equation (26), we get for a sequence of values of r tending to infinity,

$$\frac{\log^{[p]}\widehat{\nu}(r,f\circ g)}{\log^{[n]}\widehat{\nu}(r,g)} \geq \frac{\rho_{f\circ g,\Psi}^{L^*}(p,q)-\epsilon}{\left(\rho_{g,\Psi}^{L^*}(n,q)+\epsilon\right)}$$

Since $\epsilon\ (>0)$ is arbitrary,we get that

$$\limsup_{r\to\infty}\frac{\log^{[p]}\widehat{\nu}(r,f\circ g)}{\log^{[n]}\widehat{\nu}(r,g)} \geq \frac{\rho_{f\circ g,\Psi}^{L^*}(p,q)}{\rho_{g,\Psi}^{L^*}(n,q)}. \tag{27}$$

Combining Equation (24) and Equation (27) we get (i).

(ii)From the definition of L^*-(p,q)-th lower order, we have for arbitrary positive ϵ and for all large values of r,

$$\begin{aligned}\frac{\log^{[p]}\widehat{\nu}(r,\ f\circ g)}{\log^{[q]}\left[\Psi\left(r\right)e^{L(r)}\right]} &\geq \lambda_{f\circ g,\Psi}^{L^*}(p,q)-\epsilon \\ \log^{[p]}\widehat{\nu}(r,f\circ g) &\geq \left(\lambda_{f\circ g,\Psi}^{L^*}(p,q)-\epsilon\right)\log^{[q]}\left[\Psi\left(r\right)e^{L(r)}\right] \end{aligned} \tag{28}$$

Now from Equation (26) and Equation (28) it follows for all large values of r,

$$\frac{\log^{[p]}\widehat{\nu}(r,f\circ g)}{\log^{[n]}\widehat{\nu}(r,g)} \geq \frac{\lambda_{f\circ g,\Psi}^{L^*}(p,q)-\epsilon}{\rho_{g,\Psi}^{L^*}(n,q)+\epsilon}$$

since $\epsilon\ (>0)$ is arbitrary, it follows that

$$\liminf_{r\to\infty}\frac{\log^{[p]}\widehat{\nu}(r,f\circ g)}{\log^{[n]}\widehat{\nu}(r,g)} \geq \frac{\lambda_{f\circ g,\Psi}^{L^*}(p,q)}{\rho_{g,\Psi}^{L^*}(n,q)}. \tag{29}$$

Combining (24) and (29) we get (ii).

(iii) Also for a sequence of values of r tending to infinity,

$$\begin{aligned}\frac{\log^{[p]}\widehat{\nu}\left(r,f\circ g\right)}{\log^{[q]}\left[\Psi\left(r\right)e^{L(r)}\right]} &\leq \lambda_{f\circ g,\Psi}^{L^*}(p,q)+\epsilon \\ \log^{[p]}\widehat{\nu}(r,f\circ g) &\leq \left(\lambda_{f\circ g,\Psi}^{L^*}(p,q)+\epsilon\right)\log^{[q]}\left[\Psi\left(r\right)e^{L(r)}\right] \end{aligned} \tag{30}$$

Again for all large values of r,

$$\begin{aligned}\frac{\log^{[n]}\widehat{\nu}\left(r,g\right)}{\log^{[q]}\left[\Psi\left(r\right)e^{L(r)}\right]} &\geq \lambda_{g,\Psi}^{L^*}(n,q)-\epsilon \\ \log^{[n]}\widehat{\nu}(r,g) &\geq \left(\lambda_{g,\Psi}^{L^*}(n,q)-\epsilon\right)\log^{[q]}\left[\Psi\left(r\right)e^{L(r)}\right] \end{aligned} \tag{31}$$

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Now from Equation (30) and Equation (31) we obtain for a sequence of values of r tending to infinity,

$$\frac{\log^{[p]}\widehat{\nu}(r,f\circ g)}{\log^{[n]}\widehat{\nu}(r,g)}\leq\frac{\lambda^{L^*}_{f\circ g,\Psi}(p,q)+\epsilon}{\lambda^{L^*}_{g,\Psi}(n,q)-\epsilon}$$

As $\epsilon\ (>0)$ is arbitrary we obtained,

$$\liminf_{r\to\infty}\frac{\log^{[p]}\widehat{\nu}(r,f\circ g)}{\log^{[n]}\widehat{\nu}(r,g)}\leq\frac{\lambda^{L^*}_{f\circ g,\Psi}(p,q)}{\lambda^{L^*}_{g,\Psi}(n,q)}. \tag{32}$$

Again for a sequence of values of r tending to infinity,

$$\begin{aligned}\frac{\log^{[n]}\widehat{\nu}\left(r,g\right)}{\log^{[q]}\left[\Psi\left(r\right)e^{L(r)}\right]} &\leq \lambda^{L^*}_{g,\Psi}(n,q)+\epsilon\\ \log^{[n]}\widehat{\nu}(r,g) &\leq \left(\lambda^{L^*}_{g,\Psi}(n,q)+\epsilon\right)\log^{[q]}\left[\Psi\left(r\right)e^{L(r)}\right]\end{aligned} \tag{33}$$

From Equation (28) and Equation (33) we obtain for a sequence of values of r tending to infinity,

$$\frac{\log^{[p]}\widehat{\nu}(r,f\circ g)}{\log^{[n]}\widehat{\nu}(r,g)}\geq\frac{\lambda^{L^*}_{f\circ g,\Psi}(p,q)-\epsilon}{\lambda^{L^*}_{g,\Psi}(n,q)+\epsilon}$$

As $\epsilon\ (>0)$ is arbitrary

$$\limsup_{r\to\infty}\frac{\log^{[p]}\widehat{\nu}(r,f\circ g)}{\log^{[n]}\widehat{\nu}(r,g)}\geq\frac{\lambda^{L^*}_{f\circ g,\Psi}(p,q)}{\lambda^{L^*}_{g,\Psi}(n,q)}. \tag{34}$$

So, combining Equation (32) and Equation (34) we get (iii)

(iv) Again from Equation (22) and Equation (31) we get for all large value of r

$$\frac{\log^{[p]}\widehat{\nu}(r,f\circ g)}{\log^{[n]}\widehat{\nu}(r,g)}\leq\frac{\rho^{L^*}_{f\circ g,\Psi}(p,q)+\epsilon}{\lambda^{L^*}_{g,\Psi}(n,q)-\epsilon}$$

As $\epsilon\ (>0)$ is arbitrary, we get that

$$\limsup_{r\to\infty}\frac{\log^{[p]}\widehat{\nu}(r,f\circ g)}{\log^{[n]}\widehat{\nu}(r,g)}\leq\frac{\rho^{L^*}_{f\circ g,\Psi}(p,q)}{\lambda^{L^*}_{g,\Psi}(n,q)}. \tag{35}$$

Combining Equation (34) and Equation (35) we get (iv).

Combining (i) , (ii) , (iii) and (iv) of the Theorem (3.4) we get (v) ■

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