Weighted Sharing and Uniqueness of Differential-Difference Polynomials of L-functions

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Abstract- In this paper, we mainly investigate the uniqueness problems of differential-difference polynomials of L-functions. Here we prove uniqueness theorems on differential-difference polynomials of L-functions concerning weighted sharing of small or rational functions. The results of this paper improve and generalize some recent results due to Hao, Chen [4], Zhu, Chen [17], Mandal, Datta [11] and Datta, Mandal [2].

Keywords – L-function, Meromorphic function, Small function, Uniqueness, Weighted sharing

I. Introduction

For the last 150 years the most important open problem in pure mathematics is considered to be the Riemann hypothesis and its extension to the general classes of L-functions. L-functions play most important role in the modern number theory. The L-functions in the Selberg class is defined by the Dirichlet series $L(z) = \sum_{n=1}^{\infty} \frac{a(n)}{n^z}$ satisfying the hypothesis (i) $a(n) \ll n^{\epsilon}$ for every $\epsilon > 0$. (ii) There exists an integer $k \geq 0$ such that $(z-1)^k L(z)$ is a finite order entire function (iii) Every L-function satisfies the functional equation $\lambda_L(z) = \omega \overline{\lambda_L(1-\bar{z})}$, where $\lambda_L(z) = L(z) Q^z \prod_{i=1}^{k} \Gamma(\lambda_i z + \nu_i)$ with positive real numbers Q, λ_i and complex numbers ν_i, ω with $Re\, \nu_i \geq 0$ and $|\omega| = 1$ and (iv) $L(z)$ satisfies $L(z) = \prod_p L_p(z)$, where $L_p(z) = \exp(\sum_{k=1}^{\infty} \frac{b(p^k)}{p^{kz}})$ with coefficients $b(p^k)$ satisfying $b(p^k) \ll p^{k\theta}$ for some $\theta < \frac{1}{2}$ and p denotes prime number.

L-functions satisfying the first three hypothesis only are said L-functions in the extended Selberg class. In this paper by an L-function we always mean an L-function in the extended Selberg class with $a_1(1) = 1$. In this paper, we mainly deal with the question how a differential-difference polynomial of L-functions is uniquely determine in terms of sharing values. We adopt the standard definitions and notations of the Value Distribution Theory [5].

Let $\alpha \in \mathbb{C} \cup \{\infty\}$ and ξ, ψ be meromorphic functions in the complex plane. We say that ξ and ψ share α CM if $\xi - \alpha$ and $\psi - \alpha$ have the same set of zeros with counting multiplicities and if we do not consider the multiplicities then we say that ξ and ψ share α IM. The set of all the zeros of $\xi - \alpha$ with multiplicities $\leq l$ is denoted by $E_{l)}(\alpha; \xi)$, where $l \geq 1$ is an integer and the zeros are counted according to their multiplicities. If we do not consider the multiplicities then we denote it by $\bar{E}_{l)}(\alpha; \xi)$. We denote by $S(r, \xi)$ any function satisfying $S(r, \xi) = o(T(r, \xi))$ as $r \to \infty$ outside a possible exceptional set of finite linear measure. A meromorphic function ρ is said to be a small function of ξ if $T(r, \rho) = S(r, \xi)$. We define the hyper order $\rho_2(\xi)$ of ξ by $\rho_2(\xi) = \limsup_{r \to \infty} \frac{\log \log T(r, \xi)}{\log r}$.

Definition 1.1. [8]. Let ξ be a meromorphic function defined in the complex plane. Let $k \geq 1$ be an integer and $\alpha \in \mathbb{C} \cup \{\infty\}$. By $N(r, \alpha; \xi| \leq k)$ we denote the counting function of the α points of ξ with multiplicity not greater than k and by $\overline{N}(r, \alpha; \xi| \leq k)$ the reduced counting function. Also by $N(r, \alpha; \xi| \geq k)$ we denote the counting function of the α points of ξ with multiplicity not less than k and by $\overline{N}(r, \alpha; \xi| \geq k)$ the reduced counting function. We define $N_k(r, \alpha; \xi) = \overline{N}(r, \alpha; \xi| \geq 1) + \overline{N}(r, \alpha; \xi| \geq 2) + \cdots \overline{N}(r, \alpha; \xi| \geq k)$.

Definition 1.2. [8]. Let ξ be a meromorphic function defined in the complex plane and P be a small function of ξ or a rational function. Then we denote by $N(r,P;\ \xi| \le k)$, $N(r,P;\ \xi| \ge k)$, $\overline{N}(r,P;\ \xi| \le k)$, $\overline{N}(r,P;\ \xi| \ge k)$, $N_k(r,P;\ \xi)$ etc. the counting functions $N(r,0;\ \xi - P| \le k)$, $N(r,0;\ \xi - P| \ge k)$, $\overline{N}(r,0;\ \xi - P| \le k)$, $\overline{N}(r,0;\ \xi - P| \ge k)$, $N_k(r,0;\ \xi - P)$ etc. respectively.

Definition 1.3. [6, 7]. Let ξ and ψ be two meromorphic functions defined in the complex plane and n be an integer (≥ 0) or infinity . For $\alpha \in \mathbb{C} \cup \{\infty\}$ we denote by $E_n(\alpha;\xi)$ the set of all zeros of $\xi - \alpha$ where a zero of multiplicity k is counted k times if $k \le n$ and $n+1$ times if $k > n$. If $E_n(\alpha;\xi) = E_n(\alpha;\psi)$ we say that ξ , ψ share the value α with weight n.

We write ξ , ψ share (α, n) to mean that ξ, ψ share the value α with weight n. Clearly if ξ, ψ share (α, n) then ξ , ψ share (α, m) for all integers $m, 0 \le m < n$. Also we note that ξ, ψ share a value α IM or CM if and only if ξ , ψ share $(\alpha, 0)$ or (α, ∞) respectively.

In 2010 Li [9] study the uniqueness problems of meromorphic functions and L-functions and proved the following theorem.

Theorem 1.1. [9]. Let ξ be a non-constant meromorphic function having finitely many poles and L be a non-constant L-function. If ξ and L share (α,∞) and $(\beta, 0)$ then $L \equiv \xi$, where α and β are two distinct finite values.

In 2017, Liu, Li and Yi [10] proved the following uniqueness theorems of L-functions.

Theorem 1.2. [10]. Let $j \ge 1$ and $k \ge 1$ be integers such that $j > 3k + 6$. Also let L be an L-function and ξ be a non-constant meromorphic function. If $\{\xi^j\}^{(k)}$ and $\{L^j\}^{(k)}$ share $(1,\infty)$ then $\xi \equiv \alpha L$ for some constant α satisfying $\alpha^j = 1$.

Theorem 1.3. [10]. Let $j \ge 1$ and $k \ge 1$ be integers such that $j > 3k + 6$. Also let L be an L-function and ξ be a non-constant meromorphic function. If $\{\xi^j\}^{(k)}(z) - z$ and $\{L^j\}^{(k)}(z) - z$ share $(0,\infty)$ then $\xi \equiv \alpha L$ for some constant α satisfying $\alpha^j = 1$.

Definition 1.4. [11]. Let ξ be a meromorphic function defined in the complex plane and P be a small function of ξ or a rational function. Then we denote by $E_{m)}(P;\xi)$, $\bar{E}_{m)}(P;\ \xi)$ and $E_m(P;\ \xi)$ the sets $E_{m)}(0;\xi - P), \bar{E}_{m)}(0;\ \xi - P)$ and $E_m(0;\ \xi - P)$ respectively.

We write ξ , ψ share (P, n) to mean that $-P$, $\psi - P$ share the value 0 with weight n. Clearly if ξ , ψ share (P,n) then ξ, ψ share (P, m) for all integers m $(0 \le m \le n)$. Also we note that ξ, ψ share P IM or CM if and only if ξ , ψ share $(P, 0)$ or (P,∞) respectively.

In 2018, Hao and Chen [4] obtain the following uniqueness results when certain differential polynomials share a single value.

Theorem 1.4. [4] Let ξ be a non-constant meromorphic function and L be a L-function such that $[\xi^n(\xi-1)^m]^{(\tau)}$ and $L^n(L-1)^m]^{(\tau)}$ share $(1,\infty)$, where $n,\ m,\ \tau \in Z^+$. If $n > m + 3\tau + 6$ and $\tau \ge 2$, then, $\xi \equiv L$ or, $\xi^n(\xi - 1)^m \equiv L^n(L-1)^m$.

Theorem 1.5. [4] Let ξ be a non-constant meromorphic function and L be a L-function such that $[\xi^n(\xi-1)^m]^{(\tau)}$ and $L^n(L-1)^m]^{(\tau)}$ share $(1,0)$, where $n,\ m,\ \tau \in Z^+$. If $n > 4m + 7\tau + 11$ and $\tau \ge 2$, then, $\xi \equiv L$ or, $\xi^n(\xi - 1)^m \equiv L^n(L-1)^m$.

In 2019 W. Q. Zhu and J. F. Chen [17] proved the following uniqueness theorem.

Theorem 1.6. [17] Let L be an L-function and ξ be a transcendental meromorphic function defined in the complex plane $\mathbb{C}$. Also let $n,\ \tau(\geq\ 2),\ l(\geq 2)$ be positive integers such that $n\ \geq\ 7k\ +\ 17$. If $\bar{E}_{l)}(1;\ [\xi^n(\xi-1)]^{(\tau)})\ =\ \bar{E}_{l)}(1;\ [L^n(L-1)]^{(\tau)})$ then $f\ \equiv\ L$.

Considering truncated sharing of small function in 2020 Mandal and Datta [11] proved the following theorem.

Theorem 1.7. [11]. Let L be a non-constant L-function and ρ be a small function of L such that $\rho \not\equiv 0, \infty$. If $\bar{E}_{4)}(\rho; L)\ =\ \bar{E}_{4)}(\rho;\ (L^m)^{(k)}),\ E_{2)}(\rho; L)\ =\ E_{2)}(\rho;\ (L^m)^{(k)})$ and $2N_{2+k}(r,0;L^m) \leq \big(\sigma\ +\ o(1)\big)T(r,L)$, where $m\ \geq 1,\ k\ \geq\ 1$ are integers and $0\ <\ \sigma\ <\ 1$, then $L\ \equiv (L^m)^{(k)}$.

Now the following questions come naturally.

Considering weighted sharing in 2020 Datta and Mandal [2] proved the following uniqueness theorem.

Theorem 1.8. [2] Let ξ be a non-constant meromorphic function and L be a non-constant L-function. If $\mathrm{E_0}(0;\ \xi) = \mathrm{E_0}(0;\ \mathrm{L}),\ \mathrm{E_1}(1;\ \xi)\ =\ \mathrm{E_1}(1;\ \mathrm{L})$ and $\mathrm{N}(\mathrm{r},0;\ \xi)\ + \mathrm{N}(\mathrm{r},1;\ \xi)\ =\ \mathrm{S}(\mathrm{r},\xi)$ then either $\mathrm{L} \equiv \mathrm{f}$ or $\mathrm{T}(\mathrm{r},\mathrm{L}) = \mathrm{N}(\mathrm{r},0;\ \mathrm{L}\ \leq 2)\ +\ \mathrm{S}(\mathrm{r},\mathrm{L})$ and $\mathrm{T}(\mathrm{r},\xi)\ =\ \mathrm{N}(\mathrm{r},0;\ \mathrm{L}'\leq\ 1)\ +\ \mathrm{S}(\mathrm{r},\mathrm{L})$.

Question 1.1. Can we take differential-difference polynomials in place of differential polynomials in theorem 1.4, 1.5, 1.6, 1.7 and 1.8?

Question 1.2. Is it possible to consider small function or rational function sharing in theorem 1.4, 1.5, 1.6 and 1.8?

Definition 1.5. [6]. Let two non-constant meromorphic functions ξ and ψ share a value α IM. We denote by $\overline{N_*}(r,\alpha;\xi,\psi)$ the counting function of the α -points of ξ and ψ with different multiplicities, where each α -point is counted only once.

Definition 1.6. Let two non-constant meromorphic functions ξ and ψ share a value α IM. We denote by $\overline{N}(r,\alpha;\xi| > \psi)$ the counting function of those α -points of ξ and ψ whose multiplicities with respect to ξ is greater than the multiplicities with respect to ψ, where each α -point is counted once only.

Definition 1.7. Let two non-constant meromorphic functions ξ and ψ share a value α IM. We denote by $\overline{N}_E(r,\alpha;\xi,\psi| >\ m)$ the counting function of those α -points of ξ and ψ whose multiplicities greater than m and the multiplicities with respect to ξ is equal to the multiplicities with respect to ψ, where each α -point is counted once only.

Using the concept of weighted sharing we try to solve Questions 1.1, 1.2 and prove the following theorems.

Theorem 1.9. Let L be a non-constant L-function and ξ be a transcendental meromorphic function. Let $\tau, n, \eta, \mu_j (j =, 2, \dots, \eta), \lambda = \sum_{j=1}^{\eta} \mu_j$ be positive integers and $\omega_j \in \mathbb{C} - \{0\}\ (j = 1, 2, \dots, \eta)$ be distinct constants. Also let $\rho_2(L) < 1,\ \rho_2(\xi) < 1,\ L(z)^n[\prod_{j=1}^{\eta} L(z+\omega_j)^{\mu_j}]^{(\tau)}$ and $\xi(z)^n[\prod_{j=1}^{\eta} \xi(z+\omega_j)^{\mu_j}]^{(\tau)}$ share (ρ, l) and ξ, L share $(\infty, 0)$, where $0\ \leq l\ <\ \infty$ and ρ is a small function of ξ and L, then one of the following holds

(i) $L(z)^n[\prod_{j=1}^{\eta} L(z+\omega_j)^{\mu_j}]^{(\tau)} \equiv\ \xi(z)^n[\prod_{j=1}^{\eta} \xi(z+\omega_j)^{\mu_j}]^{(\tau)}$

(ii) $\{L(z)^n[\prod_{j=1}^{\eta} L(z+\omega_j)^{\mu_j}]^{(\tau)}\}\{\ \xi(z)^n\big[\prod_{j=1}^{\eta} \xi\big(z+\omega_j\big)^{\mu_j}\big]^{(\tau)}\} \equiv \rho(z)^2$

if

(i) $l\ =\ 0$ and $n\ > \lambda\ +\ \eta(5\tau\ +\ 7)\ +\ 7$

(ii) $l\ =\ 1$ and $n\ >\ \lambda\ +\ \frac{1}{2}(\eta(5\tau\ +\ 9)\ +\ 7)$

(iii) $l\ \geq 2$ and $n\ > \lambda\ +\ \eta(2\tau\ +\ 4)\ +\ 4$.

Theorem 1.10. Let L be a non-constant L-function and ξ be a transcendental meromorphic function. Let $\tau,\ n,\ \eta,\ \mu_j (j =, 2, \dots, \eta),\ \lambda = \sum_{j=1}^{\eta} \mu_j$ be positive integers and $\omega_j \in \mathbb{C} - \{0\}\ (j = 1, 2, \dots, \eta)$ be distinct constants. Also let

$\rho_2(L) < 1$, $\rho_2(\xi) < 1$, $L(z)^n[\prod_{j=1}^{\eta} L(z+\omega_j)^{\mu_j}]^{(\tau)}$ and $\xi(z)^n[\prod_{j=1}^{\eta} \xi(z+\omega_j)^{\mu_j}]^{(\tau)}$ share (Q,l) and ξ, L share $(\infty, 0)$, where $0 \leq l < \infty$ and $Q(z)$ is a rational function, then one of the following holds

(i) $L(z)^n[\prod_{j=1}^{\eta} L(z+\omega_j)^{\mu_j}]^{(\tau)} \equiv \xi(z)^n[\prod_{j=1}^{\eta} \xi(z+\omega_j)^{\mu_j}]^{(\tau)}$

(ii) $\{L(z)^n[\prod_{j=1}^{\eta} L(z+\omega_j)^{\mu_j}]^{(\tau)}\}\{\xi(z)^n[\prod_{j=1}^{\eta} \xi(z+\omega_j)^{\mu_j}]^{(\tau)}\} \equiv Q(z)^2$

if

(i) $l = 0$ and $n > \lambda + \eta(5\tau + 7) + 7$

(ii) $l = 1$ and $n > \lambda + \frac{1}{2}(\eta(5\tau + 9) + 7)$

(iii) $l \geq 2$ and $n > \lambda + \eta(2\tau + 4) + 4$.

II. LEMMAS

In this section we present some lemmas which will be needed in the proof of our results.
Henceforth we denote by Ω the function defined by

$$\Omega = \left(\frac{\Phi''}{\Phi'} - \frac{\Phi'}{\Phi - 1}\right) - \left(\frac{\Psi''}{\Psi'} - \frac{\Psi'}{\Psi - 1}\right)$$

Lemma 2.1. [13]. Let L be an L-function with degree q. Then $T(r, L) = \frac{q}{\pi}\log r + O(r)$.

Lemma 2.2. [11]. Let L be an L-function. Then $N(r, \infty; L) = S(r, L) = O(\log r)$.

Lemma 2.3. Let ξ be a non-constant meromorphic function and L be an L-function. If ξ and L share $(\infty, 0)$ then $N(r, \infty; \xi) = S(r, L) = O(\log r)$.

Proof. Since ξ and L share $(\infty; 0)$ therefore by lemma 4.2 we have $N(r, \infty; \xi) = N(r, \infty; L) = S(r, L) = O(\log r)$. This completes the proof.

Lemma 2.4. [16]. Let $\xi(z) = \frac{\alpha_0 + \alpha_1(z) + \cdots + \alpha_n z^n}{\beta_0 + \beta_1(z) + \cdots + \beta_m z^m}$ be a non-constant rational function defined in the complex plane $\mathbb{C}$, where $\alpha_0, \alpha_1, \ldots, \alpha_n (\neq 0)$ and $\beta_0, \beta_1, \ldots, \beta_m (\neq 0)$ are complex constants. Then $T(r, \xi) = \max\{m, n\} \log r + O(1)$.

Lemma 2.5. [14]. Let ξ be a transcendental meromorphic function of hyper order $\rho_2(\xi) < 1$. Then for any $\alpha \in \mathbb{C} - \{0\}$

$$T(r, \xi(z+\alpha)) = T(r, \xi(z)) + S(r, \xi(z))$$
$$N(r, \infty; \xi(z+\alpha)) = N(r, \infty; \xi(z)) + S(r, \xi(z))$$
$$N(r; 0; \xi(z+\alpha)) = N(r, 0; \xi(z)) + S(r, \xi(z)).$$

Lemma 2.6. [1] Let Φ and Ψ be two non-constant meromorphic functions sharing $(1, l)$ and $(\infty, 0)$ where $2 \leq l < \infty$ and $\Omega \not\equiv 0$. Then

$$T(r, \Phi) \leq N_2(r, 0; \Phi) + N_2(r, 0; \Psi) + \overline{N}(r, \infty; \Phi) + \overline{N}(r, \infty; \Psi) + \overline{N}_*(r, \infty; \Phi, \Psi) - m(r, 1; \Psi) - N_E(r, 1; \Phi| > 3) - \overline{N}(r, 1; \Psi > \Phi) + S(r, \Phi) + S(r, \Psi)$$
$$T(r, \Psi) \leq N_2(r, 0; \Phi) + N_2(r, 0; \Psi) + \overline{N}(r, \infty; \Phi) + \overline{N}(r, \infty; \Psi) + \overline{N}_*(r, \infty; \Phi, \Psi) - m(r, 1; \Phi) - N_E(r, 1; \Psi| > 3) - \overline{N}(r, 1; \Phi > \Psi) + S(r, \Phi) + S(r, \Psi).$$

Lemma 2.7. [12] Let Φ and Ψ be two non-constant meromorphic functions sharing (1,1) and $(\infty, 0)$ and $\Omega \not\equiv 0$. Then

$$T(r,\Phi) \le N_2(r,0;\Phi) + N_2(r,0;\Psi) + \frac{3}{2}\overline{N}(r,\infty;\Phi) + \overline{N}(r,\infty;\Psi) + \overline{N}_*(r,\infty;\Phi,\Psi) + \frac{1}{2}\overline{N}(r,0;\Phi) + S(r,\phi) + S(r,\Psi)$$
$$T(r,\Psi) \le N_2(r,0;\Phi) + N_2(r,0;\Psi) + \overline{N}(r,\infty;\Phi) + \frac{3}{2}\overline{N}(r,\infty;\Psi) + \overline{N}_*(r,\infty;\Phi,\Psi) + \frac{1}{2}\overline{N}(r,0;\Psi) + S(r,\Phi) + S(r,\Psi).$$

Lemma 2.8. [12] Let Φ and Ψ be two non-constant meromorphic functions sharing $(1,0)$ and $(\infty,0)$ and $\Omega \not\equiv 0$. Then

$$T(r,\Phi) \le N_2(r,0;\Phi) + N_2(r,0;\Psi) + 3\overline{N}(r,\infty;\Phi) + 2\overline{N}(r,\infty;\Psi) + \overline{N}_*(r,\infty;\Phi,\Psi) + 2\,\overline{N}(r,0;\Phi) + \overline{N}(r,0;\Psi) + S(r,\Phi) + S(r,\Psi)$$
$$T(r,\Psi) \le N_2(r,0;\Phi) + N_2(r,0;\Psi) + 2\,\overline{N}(r,\infty;\Phi) + 3\overline{N}(r,\infty;\Psi) + \overline{N}_*(r,\infty;\Phi,\Psi) + \overline{N}(r,0;\Phi) + 2\,\overline{N}(r,0;\Psi) + S(r,\Phi) + S(r,\Psi).$$

Lemma 2.9. [15] Let ϕ be a non-constant meromorphic function and k,p are two positive integers. Then

$$T(r,\phi^{(k)}) \le T(r,\phi) + k\overline{N}(r,\infty;\phi) + S(r,\phi)$$
$$N_p\left(r,0;\phi^{(k)}\right) \le T\left(r,\phi^{(k)}\right) - T(r,\phi) + N_{p+k}(r,0;\phi) + S(r,\phi)$$
$$N_p\left(r,0;\phi^{(k)}\right) \le N_{p+k}(r,0;\phi) + k\overline{N}(r,\infty;\phi) + S(r,\phi)$$
$$N\left(r,0;\phi^{(k)}\right) \le N(r,0;\phi) + k\overline{N}(r,\infty;\phi) + S(r,\phi).$$

Lemma 2.10. [3] Let ξ be a transcendental meromorphic function of hyper order $\rho_2(\xi) < 1$ and $\phi(z) = \prod_{j=1}^{\eta}\xi(z+\omega_j)^{\mu_j}$, where $n,\eta,\mu_j(j=,2,\dots,\eta), \lambda = \sum_{j=1}^{\eta}\mu_j$ be positive integers and $\omega_j \in \mathbb{C}-\{0\}$ $(j=1,2,\dots,\eta)$ are distinct constants. Then

$$(n-\lambda)T(r,\xi) + S(r,\xi) \le T(r,\xi^n\phi) \le (n+\lambda)T(r,\xi) + S(r,\xi).$$

Lemma 2.11. Let ξ be a transcendental meromorphic function of hyper order $\rho_2(\xi) < 1$ and $\phi(z) = \prod_{j=1}^{\eta}\xi(z+\omega_j)^{\mu_j}$, where $k,\ n,\ \eta,\ \mu_j(j=,2,\dots,\eta),\ \lambda = \sum_{j=1}^{\eta}\mu_j$ be positive integers and $\omega_j \in \mathbb{C}-\{0\}$ $(j=1,2,\dots,\eta)$ are distinct constants. If $\overline{N}(r,\infty;\xi) = S(r,\xi)$, then

$$(n-\lambda)T(r,\xi) + S(r,\xi) \le T\left(r,\xi^n\phi^{(k)}\right) \le (n+\lambda)T(r,\xi) + S(r,\xi).$$

Proof. Since $\overline{N}(r,\infty;\xi) = S(r,\xi)$; therefore by lemma 4.5 we have

$$\begin{aligned} nT(r,\xi) = T(r,\xi^n) + S(r,\xi) &\le T\left(r,\frac{\xi^n\phi^{(k)}}{\phi^{(k)}}\right) + S(r,\xi) \le T\left(r,\xi^n\phi^{(k)}\right) + T\left(r,\phi^{(k)}\right) + S(r,\xi) \\ &\le T\left(r,\xi^n\phi^{(k)}\right) + T(r,\phi) + k\overline{N}(r,\infty;\phi) + S(r,\xi) \\ &\le T\left(r,\xi^n\phi^{(k)}\right) + T(r,\phi) + k\overline{N}(r,\infty;\xi) + S(r,\xi) \\ &\le T\left(r,\xi^n\phi^{(k)}\right) + \lambda T(r,\xi) + S(r,\xi). \end{aligned} \quad (2.1)$$

From (2.1) we have

$$(n-\lambda)T(r,\xi) \le T\left(r,\xi^n\phi^{(k)}\right) + S(r,\xi). \quad (2.2)$$

Since $\overline{N}(r,\infty;\xi) = S(r,\xi)$ therefore by lemma 2.5 we have

$$\begin{aligned} T\left(r,\xi^n\phi^{(k)}\right) &\le T(r,\xi^n) + T\left(r,\phi^{(k)}\right) + S(r,\xi) \le nT(r,\xi) + T(r,\phi) + k\overline{N}(r,\infty;\phi) + S(r,\xi) \\ &\le nT(r,\xi) + T(r,\xi) + k\overline{N}(r,\infty;\xi) + S(r,\xi) \\ &\le (n+\lambda)T(r,\xi) + S(r,\xi). \end{aligned} \quad (2.3)$$

From (2.2) and (2.3) we have

$$(n-\lambda)T(r,\xi)+S(r,\xi)\leq T\left(r,\xi^n\phi^{(k)}\right)\leq (n+\lambda)T(r,\xi)+S(r,\xi).$$

This completes the proof of the lemma.

III. Proof of The Main Results

Proof of Theorem 1.9.

Let $\phi(z)=\prod_{j=1}^{\eta}\xi(z+\omega_j)^{\mu_j}$, $\psi(z)=\prod_{j=1}^{\eta}L(z+\omega_j)^{\mu_j}$, $\Phi=\frac{\xi^n\phi^{(\tau)}}{\rho}$ and $\Psi=\frac{L^n\psi^{(\tau)}}{\rho}$. Then Φ and Ψ share $(1,l)$ and $(\infty,0)$ except zeros and poles of $\rho(z)$. By lemma 2.1 it is clear that L is a transcendental meromorphic function. Now we have to consider the following two cases.

Case 1. Let $\Omega\not\equiv 0$. In this case we have to consider the following three subcases.

Subcase 1.1. Let $l=0$. Hence by lemma 2.2, lemma 2.3 and lemma 2.8 we have

$$\begin{aligned}
T\left(r,L^n\psi^{(\tau)}\right) &= T(r,\Psi)+S(r,L)\\
&\leq N_2(r,0;\Phi)+N_2(r,0;\Psi)+3\overline{N}(r,\infty;\Psi)+2\overline{N}(r,\infty:\Phi)+\overline{N}_*(r,\infty;\Phi,\Psi)+2\overline{N}(r,0;\Psi)\\
&\quad+\overline{N}(r,0;\Phi)+S(r,\Phi)+S(r,L)\\
&\leq N_2\left(r,0;\xi^n\phi^{(\tau)}\right)+N_2\left(r,0;L^n\psi^{(\tau)}\right)+2\overline{N}\left(r,0;L^n\psi^{(\tau)}\right)+\overline{N}\left(r,0;\xi^n\phi^{(\tau)}\right)+S(r,\xi)+S(r,L)\\
&\leq N_2(r,0;\xi^n)+N_2\left(r,0;\phi^{(\tau)}\right)+N_2(r,0;L^n)+N_2\left(r,0;\psi^{(\tau)}\right)+2\overline{N}(r,0;L^n)+2\overline{N}\left(r,0;\psi^{(\tau)}\right)\\
&\quad+\overline{N}(r,0;\xi^n)+\overline{N}\left(r,0;\phi^{(\tau)}\right)+S(r,\xi)+S(r,L)\\
&\leq 2T(r,L)+N_2\left(r,0;\phi^{(\tau)}\right)+N_2\left(r,0;\psi^{(\tau)}\right)+2\overline{N}(r,0;L^n)+2\overline{N}\left(r,0;\psi^{(\tau)}\right)+2T(r,\xi)\\
&\quad+\overline{N}(r,0;\xi^n)+\overline{N}\left(r,0;\phi^{(\tau)}\right)+S(r,\xi)+S(r,L)\\
&\leq 2T(r,L)+T\left(r,\psi^{(\tau)}\right)-T(r,\psi)+N_{2+\tau}(r,0;\psi)+2\overline{N}(r,0;L^n)+2\overline{N}\left(r,0;\psi^{(\tau)}\right)+2T(r,\xi)\\
&\quad+N_2\left(r,0;\phi^{(\tau)}\right)+\overline{N}(r,0;\xi^n)+\overline{N}\left(r,0;\phi^{(\tau)}\right)+S(r,\xi)+S(r,L)\\
&\leq 2T(r,L)+T\left(r,L^n\psi^{(\tau)}\right)-T(r,L^n\psi)+\eta(2+\tau)T(r,L)+2\overline{N}(r,0;L^n)+2\overline{N}\left(r,0;\psi^{(\tau)}\right)\\
&\quad+2T(r,\xi)+\eta(2+\tau)T(r,\xi)+\overline{N}(r,0;\xi^n)+\overline{N}\left(r,0;\phi^{(\tau)}\right)+S(r,\xi)+S(r,L)\\
&\leq 2T(r,L)+T\left(r,L^n\psi^{(\tau)}\right)-T(r,L^n\psi)+\eta(2+\tau)T(r,L)+2T(r,L)+2N_{\tau+1}(r,0;\psi)\\
&\quad+2T(r,\xi)+\eta(2+\tau)T(r,\xi)+T(r,\xi)+N_{\tau+1}(r,0;\phi)+S(r,\xi)+S(r,L)\\
&\leq 2T(r,L)+T\left(r,L^n\psi^{(\tau)}\right)-T(r,L^n\psi)+\eta(2+\tau)T(r,L)+2T(r,L)+2\eta(1+\tau)T(r,L)\\
&\quad+2T(r,\xi)+\eta(2+\tau)T(r,\xi)+T(r,\xi)+\eta(1+\tau)T(r,\xi)+S(r,\xi)+S(r,L)\\
&\leq (\eta(3\tau+4)+4)T(r,L)+(\eta(2\tau+3)+3)T(r,\xi)+T\left(r,L^n\psi^{(\tau)}\right)-T(r,L^n\psi)+S(r,\xi)\\
&\quad+S(r,L)
\end{aligned}\tag{3.1}$$

Hence from (3.1) we have

$$T(r,L^n\psi)\leq(\eta(3\tau+4)+4)T(r,L)+(\eta(2\tau+3)+3)T(r,\xi)+S(r,\xi)+S(r,L)\tag{3.2}$$

Using lemma 2.10 we have from (5.2)

$$(n-\lambda)T(r,L)\leq(\eta(3\tau+4)+4)T(r,L)+(\eta(2\tau+3)+3)T(r,\xi)+S(r,\xi)+S(r,L)\tag{3.3}$$

Similarly as above we have

$$(n-\lambda)T(r,\xi)\leq(\eta(3\tau+4)+4)T(r,\xi)+(\eta(2\tau+3)+3)T(r,L)+S(r,\xi)+S(r,L)\tag{3.4}$$

Combining (3.3) and (3.4) we have

$$(n-(\lambda+\eta(5\tau+7)+7))(T(r,L)+T(r,\xi))\leq S(r,\xi)+S(r,L)\tag{3.5}$$

From (3.5) we arrive at a contradiction since $n>\lambda+\eta(5\tau+7)+7$.

Subcase 1.2. Let $l = 1$. Hence by lemma 2.2, lemma 2.3 and lemma 2.7 we have

$$T\big(r, L^n\psi^{(\tau)}\big) = T(r,\Psi) + S(r,L)$$
$$\leq N_2(r,0;\Phi) + N_2(r,0;\Psi) + \frac{3}{2}\overline{N}(r,\infty;\Psi) + \overline{N}(r,\infty:\Phi) + \overline{N}_*(r,\infty;\Phi,\Psi) + \frac{1}{2}\overline{N}(r,0;\Psi)$$
$$+ S(r,\Phi) + S(r,L)$$
$$\leq N_2\big(r,0;\xi^n\phi^{(\tau)}\big) + N_2\big(r,0;L^n\psi^{(\tau)}\big) + \frac{1}{2}\overline{N}\big(r,0;L^n\psi^{(\tau)}\big) + S(r,\xi) + S(r,L)$$
$$\leq N_2(r,0;\xi^n) + N_2\big(r,0;\phi^{(\tau)}\big) + N_2(r,0;L^n) + N_2\big(r,0;\psi^{(\tau)}\big) + \frac{1}{2}\overline{N}(r,0;L^n) + \frac{1}{2}\overline{N}\big(r,0;\psi^{(\tau)}\big)$$
$$+ S(r,\xi) + S(r,L)$$
$$\leq 2T(r,L) + N_2\big(r,0;\psi^{(\tau)}\big) + \frac{1}{2}\overline{N}(r,0;L^n) + \frac{1}{2}\overline{N}\big(r,0;\psi^{(\tau)}\big) + 2T(r,\xi) + N_2\big(r,0;\phi^{(\tau)}\big)$$
$$+ S(r,\xi) + S(r,L)$$
$$\leq 2T(r,L) + T\big(r,\psi^{(\tau)}\big) - T(r,\psi) + N_{2+\tau}(r,0;\psi) + \frac{1}{2}\overline{N}(r,0;L^n) + \frac{1}{2}\overline{N}\big(r,0;\psi^{(\tau)}\big) + 2T(r,\xi)$$
$$+ N_2\big(r,0;\phi^{(\tau)}\big) + S(r,\xi) + S(r,L)$$
$$\leq 2T(r,L) + T\big(r,L^n\psi^{(\tau)}\big) - T(r,L^n\psi) + \eta(2+\tau)T(r,L) + \frac{1}{2}\overline{N}(r,0;L^n) + \frac{1}{2}\overline{N}\big(r,0;\psi^{(\tau)}\big)$$
$$+ 2T(r,\xi) + \eta(2+\tau)T(r,\xi) + S(r,\xi) + S(r,L)$$
$$\leq 2T(r,L) + T\big(r,L^n\psi^{(\tau)}\big) - T(r,L^n\psi) + \eta(2+\tau)T(r,L) + \frac{1}{2}T(r,L) + \frac{1}{2}N_{\tau+1}(r,0;\psi)$$
$$+ 2T(r,\xi) + \eta(2+\tau)T(r,\xi) + S(r,\xi) + S(r,L)$$
$$\leq 2T(r,L) + T\big(r,L^n\psi^{(\tau)}\big) - T(r,L^n\psi) + \eta(2+\tau)T(r,L) + \frac{1}{2}T(r,L) + \frac{1}{2}\eta(1+\tau)T(r,L)$$
$$+ 2T(r,\xi) + \eta(2+\tau)T(r,\xi) + S(r,\xi) + S(r,L)$$
$$\leq \left(\frac{1}{2}\eta(3\tau+5) + 3\right)T(r,L) + (\eta(\tau+2)+2)T(r,\xi) + T\big(r,L^n\psi^{(\tau)}\big) - T(r,L^n\psi) + S(r,\xi)$$
$$+ S(r,L) \qquad (3.6)$$

Hence from (3.6) we have

$$T(r,L^n\psi) \leq \left(\frac{1}{2}\eta(3\tau+5) + 3\right)T(r,L) + (\eta(\tau+2)+2)T(r,\xi) + S(r,\xi) + S(r,L) \qquad (3.7)$$

Using lemma 2.10 we have from (3.7)

$$(n-\lambda)T(r,L) \leq \left(\frac{1}{2}\eta(3\tau+5) + 3\right)T(r,L) + (\eta(\tau+2)+2)T(r,\xi) + S(r,\xi) + S(r,L) \qquad (3.8)$$

Similarly as above we have

$$(n-\lambda)T(r,\xi) \leq \left(\frac{1}{2}\eta(3\tau+5) + 3\right)T(r,\xi) + (\eta(\tau+2)+2)T(r,L) + S(r,\xi) + S(r,L) \qquad (3.9)$$

Combining (3.8) and (3.9) we have

$$(n - \left(\lambda + \frac{1}{2}\eta(5\tau+9) + 7\right))(T(r,L) + T(r,\xi)) \leq S(r,\xi) + S(r,L) \qquad (3.10)$$

From (3.10) we arrive at a contradiction since $n > \lambda + \frac{1}{2}\eta(5\tau+9) + 7$.

Subcase 1.3. Let $2 \leq l < 1$. Hence by lemma 2.2, lemma 2.3 and lemma 2.6 we have

$$
\begin{aligned}
T\left(r, L^n\psi^{(\tau)}\right) &= T(r,\Psi) + S(r,L) \\
&\le N_2(r,0;\Phi) + N_2(r,0;\Psi) + \overline{N}(r,\infty;\Psi) + \overline{N}(r,\infty:\Phi) + \overline{N}_*(r,\infty;\Phi,\Psi) - m(r,1;\Phi) \\
&- N_E(r,1;\Psi| > 3) - \overline{N}(r,1;\Phi| > \Psi) + S(r,\Phi) + S(r,L) \\
&\le N_2\left(r,0;\xi^n\phi^{(\tau)}\right) + N_2\left(r,0;L^n\psi^{(\tau)}\right) + S(r,\xi) + S(r,L) \\
&\le N_2(r,0;\xi^n) + N_2\left(r,0;\phi^{(\tau)}\right) + N_2(r,0;L^n) + N_2\left(r,0;\psi^{(\tau)}\right) + S(r,\xi) + S(r,L) \\
&\le 2T(r,L) + N_2\left(r,0;\psi^{(\tau)}\right) + 2T(r,\xi) + N_2\left(r,0;\phi^{(\tau)}\right) + S(r,\xi) + S(r,L) \\
&\le 2T(r,L) + T\left(r,\psi^{(\tau)}\right) - T(r,\psi) + N_{2+\tau}(r,0;\psi) + 2T(r,\xi) + N_2\left(r,0;\phi^{(\tau)}\right) + S(r,\xi) \\
&+ S(r,L) \\
&\le 2T(r,L) + T\left(r,L^n\psi^{(\tau)}\right) - T(r,L^n\psi) + \eta(2+\tau)T(r,L) + 2T(r,\xi) + \eta(2+\tau)T(r,\xi) \\
&+ S(r,\xi) + S(r,L) \\
&\le (\eta(\tau+2)+2)T(r,L) + (\eta(\tau+2)+2)T(r,\xi) + T\left(r,L^n\psi^{(\tau)}\right) - T(r,L^n\psi) + S(r,\xi) \\
&+ S(r,L) \qquad (3.11)
\end{aligned}
$$

Hence from (3.11) we have

$$T(r,L^n\psi) \le (\eta(\tau+2)+2)T(r,L) + (\eta(\tau+2)+2)T(r,\xi) + S(r,\xi) + S(r,L) \qquad (3.12)$$

Using lemma 2.10 we have from (3.12)

$$(n-\lambda)T(r,L) \le (\eta(\tau+2)+2)T(r,L) + (\eta(\tau+2)+2)T(r,\xi) + S(r,\xi) + S(r,L) \qquad (3.13)$$

Similarly as above we have

$$(n-\lambda)T(r,\xi) \le (\eta(\tau+2)+2)T(r,\xi) + (\eta(\tau+2)+2)T(r,L) + S(r,\xi) + S(r,L) \qquad (3.14)$$

Combining (3.13) and (3.14) we have

$$(n-(\lambda+\eta(2\tau+4)+4))(T(r,L)+T(r,\xi)) \le S(r,\xi) + S(r,L) \qquad (3.15)$$

From (3.15) we arrive at a contradiction since $n > \lambda + \eta(2\tau+4) + 4$.

Case 2 Let $\Omega \equiv 0$. Then $\left(\frac{\Phi''}{\Phi'} - \frac{\Phi'}{\Phi-1}\right) - \left(\frac{\Psi''}{\Psi'} - \frac{\Psi'}{\Psi-1}\right) \equiv 0$. Hence by integration we get

$$\frac{1}{1-\Phi} \equiv \frac{D(\Psi-1)-C}{1-\Psi}, \qquad (3.16)$$

Where $C \ne 0$ and D are constants.

Now we have to consider the following two subcases.

Subcase 2.1. Let $D = 0$. Then from (3.16) we have

$$\frac{1}{1-\Phi} \equiv \frac{C}{1-\Psi}. \qquad (3.17)$$

If possible let $C \ne 1$, then from (3.17) we have

$$\overline{N}(r,0;\Phi) = \overline{N}(r,1-C;\Psi). \qquad (3.18)$$

Hence using second fundamental theorem we get by lemma 2.2, lemma 2.3 and lemma 4.9

$$
\begin{aligned}
T\left(r,L^n\psi^{(\tau)}\right) &= T(r,\Psi) + S(r,L) \le \overline{N}(r,0;\Psi) + \overline{N}(r,1-C;\Psi) + \overline{N}(r,\infty;\Psi) + S(r,L) \\
&\le \overline{N}\left(r,0;\xi^n\phi^{(\tau)}\right) + \overline{N}\left(r,0;L^n\psi^{(\tau)}\right) + S(r,L) \\
&\le \overline{N}(r,0;\xi^n) + \overline{N}\left(r,0;\phi^{(\tau)}\right) + \overline{N}(r,0;L^n) + \overline{N}\left(r,0;\psi^{(\tau)}\right) + S(r,L) \\
&\le T(r,L) + \tau\overline{N}(r,\infty;\psi) + N_{\tau+1}(r,0;\psi) + T(r,\xi) + \tau\overline{N}(r,0;\phi) + N_{\tau+1}(r,0;\phi) + S(r,\xi) \\
&+ S(r,L) \le T(r,L) + \eta(\tau+1)T(r,L) + T(r,\xi) + \eta(\tau+1)T(r,\xi) + S(r,\xi) + S(r,L) \\
&\le (\eta(\tau+1)+1)T(r,L) + (\eta(\tau+1)+1)T(r,\xi) + S(r,\xi) + S(r,L) \qquad (3.19)
\end{aligned}
$$

Hence from (3.19) we have

$$T\left(r, L^{n}\psi^{(\tau)}\right) \leq (\eta(\tau+1)+1)T(r,L) + (\eta(\tau+1)+1)T(r,\xi) + S(r,\xi) + S(r,L) \tag{3.20}$$

Using lemma 2.11 we have from (3.20)

$$(n-\lambda)T(r,L) \leq (\eta(\tau+1)+1)T(r,L) + (\eta(\tau+1)+1)T(r,\xi) + S(r,\xi) + S(r,L) \tag{3.21}$$

Similarly as above we have

$$(n-\lambda)T(r,\xi) \leq (\eta(\tau+1)+1)T(r,\xi) + (\eta(\tau+1)+1)T(r,L) + S(r,\xi) + S(r,L) \tag{3.22}$$

Combining (3.21) and (3.22) we have

$$(n-(\lambda+\eta(2\tau+2)+2))(T(r,L)+T(r,\xi)) \leq S(r,\xi) + S(r,L) \tag{3.23}$$

From (3.23) we arrive at a contradiction since $n > \lambda + \eta(2\tau+2) + 2$.

Hence $C = 1$ and therefore we get from (3.17)

$$L(z)^{n}[\prod_{j=1}^{\eta} L(z+\omega_j)^{\mu_j}]^{(\tau)} \equiv \xi(z)^{n}\left[\prod_{j=1}^{\eta} \xi\left(z+\omega_j\right)^{\mu_j}\right]^{(\tau)}.$$

Subcase 2.2. Let $D \neq 0$.

If possible let $C \neq -D$.

If $D = 1$, then from (3.17) we have

$$\frac{1}{\Phi} \equiv \frac{1}{C}(1 + C - \Psi). \tag{3.24}$$

Using lemma 2.3 we have from (3.24)

$$\overline{N}(r, C+1; \Psi) = \overline{N}(r, \infty; \Phi) = \overline{N}(r, \infty; \xi) + S(r, L) = S(r, L).$$

Now proceeding as in subcase 2.1 we arrive at a contradiction.
If $D \neq 1$, then from (3.17) we have

$$\frac{1}{\Phi - (1 - \frac{1}{D})} \equiv \frac{D^2}{C}\left(\frac{C+D}{D} - \Psi\right).$$

Hence we get by lemma 2.3

$$\overline{N}\left(r, \frac{C+D}{D}; \Psi\right) = \overline{N}(r, \infty; \Phi) = \overline{N}(r, \infty; \xi) + S(r, L) = S(r, L).$$

Again proceeding as in the subcase 2.1 we arrive at a contradiction.
Hence $C = -D$.
If $D = 1$, then from (3.17) we have

$$\{L(z)^{n}[\prod_{j=1}^{\eta} L(z+\omega_j)^{\mu_j}]^{(\tau)}\}\left\{\xi(z)^{n}\left[\prod_{j=1}^{\eta} \xi\left(z+\omega_j\right)^{\mu_j}\right]^{(\tau)}\right\} \equiv \rho(z)^2.$$

If $D \neq 1$, then from (3.17) we have $\frac{1}{\Phi} \equiv \frac{-D\Psi}{(1-D)\Psi - 1}$.

Hence $N(r,0;\Phi)=N(r,\frac{1}{1-D};\Psi)$.
Now proceeding as in the subcase 2.1 we arrive at a contradiction.
This completes the proof of the theorem.

Proof of Theorem 1.10.
Since ξ and L are transcendental meromorphic functions therefore by lemma 2.4 Q is a small function of ξ and L.
Hence the result follows by theorem 1.9.

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