On Eccentricity Sum Energy of Some Graphs

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Abstract- Let G be a simple graph with n vertices and m edges. For a vertex v_i, its eccentricity e_i is the largest distance from v_i to any other vertices of G. The eccentricity sum matrix is defined by $ES(G) = [a_{ij}]$, where a_{ij} is equal to sum of the eccentricities of v_i and v_j if $i \neq j$, otherwise it is zero. The eccentricity sum energy is defined as sum of the absolute values of the eigenvalues of the eccentricity sum matrix. G^+ is the graph obtained from G by adding n new vertices v_i' and joining v_i' to v_i an edge. We have obtained eccentricity sum energy of S_n^+, K_n^+ and coalescence of graphs.

Keywords: Eccentricity, Eigenvalues, Eccentricity sum energy, coalescence
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I. INTRODUCTION

Let G be a simple connected graph with n vertices and m edges. Let the vertices of G be labeled as $v_1, v_2, ..., v_n$. The degree of a vertex v in the graph G, denoted by $deg(v)$, is the number of edges incident to v. The distance between the vertices v_i and v_j is the length of the shortest path joining v_i and v_j in G. For a vertex v_i its eccentricity, e_i is the largest distance from v_i to any other vertices of G. The adjacency matrix $A(G)$ of a graph G is the square matrix of order n, whose (i, j)-entry is equal to one, if the vertex v_i and v_j are adjacent, otherwise it is equal to zero. The eigenvalues of the adjacency matrix $A(G)$ are denoted by $\lambda_1, \lambda_2, ..., \lambda_n$ and since they are real it can be ordered as $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. [3,4]

The energy of a graph G is defined as [1,2], $E = E_\pi(G) = \sum_{i=1}^{n} |\lambda_i|$

Motivated by this definition of energy for the Huckel molecular orbital total π-electron energy [1], many researchers have defined energy of a graph, based on distance, eccentricity, degree etc. parameters and studied their spectra [6 – 12, 14 – 20]. The eccentricity sum matrix and eccentricity sum energy associated with a simple graph are introduced and studied their bounds in the paper [10].

Let G be a simple graph with n vertices. The n vertices of a simple graph G be labeled as $v_1, v_2, ..., v_n$. For a vertex v_i, its eccentricity e_i is the largest distance from v_i to any other vertices of G. Let $ecc(v_i) = e_i$ be the eccentricity of v_i, $i =1, 2, …, n$. Then $ES(G) = [a_{ij}]$ is called the eccentricity sum matrix of a graph G, where $a_{ij} = e_i + e_j$, when $i \neq j$ and 0 otherwise.

Let the characteristic equation of the eccentricity sum matrix is called the eccentricity sum polynomial and is defined by $\phi(G: \xi) = det(\xi I - ES(G))=0$

where I is the identity matrix of order n. The roots of the eccentricity sum matrix be ordered as $\xi_1 \geq \xi_2 \geq \cdots \geq \xi_n$, where ξ_1 is largest and ξ_n is smallest eigenvalues.

Then the eccentricity sum energy of a graph G is defined as [10],

$$E_{ES}(G) = \sum_{i=1}^{n} |\xi_i|$$

Coalescence [11]: Let H_1 and H_2 be two graphs on disjoint sets of vertices. Suppose $U = \{u_1, u_2, …., u_t\}$ is a clique in H_1 and $W =\{w_1, w_2, ….., w_t\}$ is a clique in H_2. Let G be a graph obtained from H_1 and H_2 by identifying (coalescence into a single vertex) u_i and w_i, $1 \leq i \leq t$. Then G is an overlap of H_1 and H_2 in K_t. It may be considered as generalized coalescence denoted by $H_1 \text{ o } H_2$.

II. Bounds for the Eccentricity Sum Energy

Since trace $ES(G) = 0$, the eigenvalues *of* $ES(G)$ satisfies the relations

$$\sum_{i=1}^{n} \xi_i = 0 \text{ and } \sum_{i=1}^{n} {\xi_i}^2 = 2M. \text{ Where, } M = \sum_{1 \leq i < j \leq n} \left(e_i + e_j\right)^2$$

Theorem 2.1[10]: If G is a connected graph with $ecc(v_i)= e_i = e$, $i =1, 2, …, n$, then G has only one positive eigenvalue equal to $2(n – 1)e$.

Theorem 2.2[10]: If G is a connected graph with $ecc(v_i)= e_i = e$, $i =1, 2, …, n$, then $E_{ES}(G) = 4(n – 1) e$.

Theorem 2.3[10]: If G is any graph with n vertices, then $\xi_1 \leq \sqrt{\frac{2M(n-1)}{n}}$. Equality holds if $ecc(v_i)= e_i = e$, $i =1, 2, …, n$.

Theorem 2.4[10]: Let G be any graph with n vertices and let Δ be the absolute value of the determinant of the eccentricity sum matrix $ES(G)$. Then

$$\sqrt{2M + n(n-1)\Delta^{\frac{2}{n}}} \leq E_{ES}(G) \leq \sqrt{2(n-1)M + n\Delta^{\frac{2}{n}}}$$

Theorem 2.5[11]: The degree sum energy of the vertex coalescence of two r-regular graphs of G_1 & G_2 of order n_1 & n_2 respectively is given by,

$$E_{DS}(G_1 o_v G_1) = 2r\left[n_1 + n_2 - 3 + \sqrt{n_1^2 + n_1^2 + 3n_2 + 2n_1n_2 - 6n_1 - 9}\right]$$

Corollary 2.6[10]: If $G = K_n$ is a complete graph, then $spectra\left(ES(K_n)\right) = \begin{pmatrix} 2(n-1) & -2 \\ 1 & n-1 \end{pmatrix}$.

Corollary 2.7[10]: If K_n is a complete graph, then $E_{ES}(K_n) = 4(n – 1)$.

Lemma 2.8[13]: If a and b are scalars then,

$$\left|aI + b(J-I)\right|_{(n\times n)} = \begin{vmatrix} a & b & \cdots & b \\ b & a & \cdots & b \\ \vdots & \vdots & \ddots & \vdots \\ b & b & \cdots & a \end{vmatrix}_{n\times n} = (a-b)^{n-1}[a+(n-1)b]$$

III. MAIN RESULTS

In this section we have obtained eccentricity sum energy of S_n^+, K_n^+ and coalescence of graphs.

Theorem 3.1: Let K_n be a complete graph, then $E_{ES}(K_n^+) = 10(n-1) + \sqrt{4(n-1)^2 + 100n^2}$.

Proof: Consider a complete graph K_n with vertices v_i, i= 1, 2, …, n. The graph K_n^+ is obtained by adding n new vertices v_i' to K_n by joining v_i' to v_i through an edge. In K_n^+, $ecc(v_i) = 2$ and $ecc(v_i') = 3$.

Then the eccentricity sum polynomial of K_n^+ is given by, $|\lambda I - ES(K_n^+)| = 0$.

$$|\lambda I - ES(K_n^+)| = \begin{vmatrix} \lambda & -4 & \cdots & -4 & -5 & -5 & \cdots & -5 \\ -4 & \lambda & \cdots & -4 & -5 & -5 & \cdots & -5 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -4 & -4 & \cdots & \lambda & -5 & -5 & \cdots & -5 \\ -5 & -5 & \cdots & -5 & \lambda & -6 & \cdots & -6 \\ -5 & -5 & \cdots & -5 & -6 & \lambda & \cdots & -6 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -5 & -5 & \cdots & -5 & -6 & -6 & \cdots & \lambda \end{vmatrix} \quad \ldots (1)$$

Operating $R_i + \frac{5\sum_{j=1}^{n} R_j}{\lambda - 4(n-1)}$ for $i = n+1, n+2, \ldots, 2n$.

Expanding the resulting determinant, we obtain,

$$|\lambda I - ES(K_n^+)| = (\lambda + 4)^{n-1}(\lambda + 6)^{n-1}[\lambda^2 - 10(n-1)\lambda + 24(n-1)^2 - 25n^2]$$

$$spectra\big(ES(K_n^+)\big) = \begin{pmatrix} -4 & -6 & 5(n-1) - \sqrt{(n-1)^2 + 25n^2} & 5(n-1) + \sqrt{(n-1)^2 + 25n^2} \\ n-1 & n-1 & 1 & 1 \end{pmatrix}$$

Further $25(n-1)^2 < 25(n-1)^2 + (n-1)^2 < 25n^2 + (n-1)^2$

$$\Rightarrow 5(n-1) < \sqrt{25n^2 + (n-1)^2}$$
$$\Rightarrow 5(n-1) - \sqrt{25n^2 + (n-1)^2} < 0$$

Therefore $5(n-1) + \sqrt{25n^2 + (n-1)^2} > 0$ and is the only one positive largest eigenvalue.

Hence $E_{ES}(K_n^+) = 10(n-1) + \sqrt{4(n-1)^2 + 100n^2}$.

Theorem 3.2: If S_n is a Star graph, then $E_{ES}(S_n) = 4(n-2) + 2\sqrt{4n^2 - 7n + 7}$.

Proof: Consider star graph S_n having n vertices. Let v_i, i= 1, 2, …, n be the vertices of S_n. Then $ecc(v_1) = 1$ and $ecc(v_i) = 2$ for i= 2, …, n

Let the eccentricity sum polynomial of S_n is $|\lambda I - ES(S_n)| = 0$.

Now,
$$|\lambda I - ES(S_n)| = \begin{vmatrix} \lambda & -3 & -3 & \ldots & -3 \\ -3 & \lambda & -4 & \ldots & -4 \\ -3 & -4 & \lambda & \cdots & -4 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -3 & -4 & -4 & \ldots & \lambda \end{vmatrix} \quad \ldots (2)$$

Operating $R_i + \frac{3R_1}{\lambda}$ for i = 2, 3, …, n and expanding the resulting determinant, we obtain,

$$|\lambda I - ES(S_n)| = (\lambda + 4)^{n-2}[\lambda^2 - 4(n-2)\lambda - 9(n-1)]$$

$$spectra\big(ES(S_n)\big) = \begin{pmatrix} -4 & 2(n-2) - \sqrt{4(n-2)^2 + 9(n-1)} & 2(n-2) + \sqrt{4(n-2)^2 + 9(n-1)} \\ n-2 & 1 & 1 \end{pmatrix}$$

Further, $2(n-2) + \sqrt{4(n-2)^2 + 9(n-1)} > 0$ is the only one positive largest eigenvalue.
Hence, $E_{ES}(S_n) = 4(n-2) + \sqrt{16(n-2)^2 + 36(n-1)}$.
Therefore, $E_{ES}(S_n) = 4(n-2) + 2\sqrt{4n^2 - 7n + 7}$

Corollary 3.3: If W_n is a Wheel graph having n vertices then, $E_{ES}(W_n) = 4(n-2) + 2\sqrt{4n^2 - 7n + 7}$.
Proof: Consider a wheel graph W_n having n vertices. Let v_i, i= 1, 2, …, n be the vertices of W_n. Then $ecc(v_1) = 1$ and $ecc(v_i) = 2$ for i= 2, …, n. But we know that $ecc(v_1) = 1$ and $ecc(v_i) = 2$ for i= 2, …, n for the star graph S_n.
Therefore, from Theorem 3.2,

$$Spectra\ (ES(W_n)) = Spectra\ (ES(S_n))$$

Hence $E_{ES}(S_n) = E_{ES}(W_n) = 4(n-2) + 2\sqrt{4n^2 - 7n + 7}$

Theorem 3.4: If S_n is a Star graph, then $E_{ES}(S_n^+) = 6(n-1) + 8(n-2) + |K_1| + |K_2| + |K_3|$.
Where K_1, K_2 and K_3 are the roots of the polynomial $\lambda^3 - (14n-22)\lambda^2 - (n^2 + 156n - 132)\lambda - (4n^2 + 412n - 216) = 0$
Proof: Consider the star graph S_n having vertices v_i, i= 1, 2, …, n. The graph S_n^+ is obtained by adding n new vertices v_i' and joining v_i' to v_i an edge. In S_n^+, $ecc(v_1) = 2$, $ecc(v_i) = 3$, i= 2, 3, …, n, n +1 and $ecc(v_j') = 4$, $j = n$ +2, n +3, …, $2n$.
Consider the eccentricity sum polynomial of S_n^+ as $|\lambda I - ES(S_n^+)| = 0$.

$$|\lambda I - ES(S_n^+)| = \begin{vmatrix} \lambda & -5 & -5 & \cdots & -5 & -6 & -6 & \cdots & -6 \\ -5 & \lambda & -6 & \cdots & -6 & -7 & -7 & \cdots & -7 \\ -5 & -6 & \lambda & \cdots & -6 & -7 & -7 & \cdots & -7 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -5 & -6 & -6 & \cdots & \lambda & -7 & -7 & \cdots & -7 \\ -6 & -7 & -7 & \cdots & -7 & \lambda & -8 & \cdots & -8 \\ -6 & -7 & -7 & \cdots & -7 & -8 & \lambda & \cdots & -8 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -6 & -7 & -7 & \cdots & -7 & -8 & -8 & \cdots & \lambda \end{vmatrix}_{(2n\times 2n)} \quad \ldots (3)$$

Operating $R_i + \frac{5R_1}{\lambda}$ and $R_j + \frac{6R_1}{\lambda}$ for i= 2, 3, …, n, n +1 and $j = n$ +2, n +3, …, $2n$.
The resulting determinant is,

$$|\lambda I - ES(S_n^+)| = \begin{vmatrix} \lambda & -5 & -5 & \cdots & -5 & -6 & \cdots & -6 \\ 0 & \lambda - \frac{25}{\lambda} & -6 - \frac{25}{\lambda} & \cdots & -6 - \frac{25}{\lambda} & -7 - \frac{30}{\lambda} & \cdots & -7 - \frac{30}{\lambda} \\ 0 & -6 - \frac{25}{\lambda} & \lambda - \frac{25}{\lambda} & \cdots & -6 - \frac{25}{\lambda} & -7 - \frac{30}{\lambda} & \cdots & -7 - \frac{30}{\lambda} \\ 0 & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & -6 - \frac{25}{\lambda} & -6 - \frac{25}{\lambda} & \cdots & \lambda - \frac{25}{\lambda} & -7 - \frac{30}{\lambda} & \cdots & -7 - \frac{30}{\lambda} \\ 0 & -7 - \frac{30}{\lambda} & -7 - \frac{30}{\lambda} & \cdots & -7 - \frac{30}{\lambda} & \lambda - \frac{36}{\lambda} & \cdots & -8 - \frac{36}{\lambda} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & -7 - \frac{30}{\lambda} & -7 - \frac{30}{\lambda} & \cdots & -7 - \frac{30}{\lambda} & -8 - \frac{36}{\lambda} & \cdots & \lambda - \frac{36}{\lambda} \end{vmatrix} \quad \ldots (4)$$

Operating $C_i + \left(\left(7 + \frac{30}{\lambda}\right) \frac{\sum_{j=n+2}^{2n} C_j}{\left(\lambda - \frac{36}{\lambda}\right) - (n-2)\left(8 + \frac{36}{\lambda}\right)}\right)$ for i= 2, 3, …, n, n +1 on Eq. (4),
The determinant in Eq.(4) reduces to,

$$|\lambda I - ES(S_n^+)| = \lambda \begin{vmatrix} a & b & \cdots & b \\ b & a & \cdots & b \\ \vdots & \vdots & \ddots & \vdots \\ b & b & \cdots & a \end{vmatrix}_{n\times n} \begin{vmatrix} 7-\frac{36}{\lambda} & -8-\frac{36}{\lambda} & \cdots & -8-\frac{36}{\lambda} \\ -8-\frac{36}{\lambda} & 7-\frac{36}{\lambda} & \cdots & -8-\frac{36}{\lambda} \\ \vdots & \vdots & \ddots & \vdots \\ -8-\frac{36}{\lambda} & -8-\frac{36}{\lambda} & \cdots & 7-\frac{36}{\lambda} \end{vmatrix}_{(n-1)\times(n-1)} \quad \ldots (5)$$

In Eq.(5), $a = \lambda - \frac{25}{\lambda} - \left(\frac{(n-1)\left(7+\frac{30}{\lambda}\right)^2}{\left(\left(\lambda-\frac{36}{\lambda}\right)-(n-2)\left(8+\frac{36}{\lambda}\right)\right)}\right)$ and $b = -6 - \frac{25}{\lambda} - \left(\frac{(n-1)\left(7+\frac{30}{\lambda}\right)^2}{\left(\left(\lambda-\frac{36}{\lambda}\right)-(n-2)\left(8+\frac{36}{\lambda}\right)\right)}\right)$.

Using Lemma 2.8, the eccentricity sum polynomial reduces to,

$(\lambda+6)^{n-1}(\lambda+8)^{n-2}[\lambda^3-(14n-22)\lambda^2-(n^2+156n-132)\lambda-(4n^2+412n-216)]=0.$

Hence, $E_{ES}(S_n^+) = 6(n-1)+8(n-2)+|K_1|+|K_2|+|K_3|$.

Where K_1, K_2 and K_3 are the roots of polynomial $\lambda^3-(14n-22)\lambda^2-(n^2+156n-132)\lambda-(4n^2+412n-216)=0$.

Corollary 3.5: If W_n is a Wheel graph having n vertices then,

$$E_{ES}(W_n^+) = 6(n-1)+8(n-2)+|K_1|+|K_2|+|K_3|.$$

Where K_1, K_2 and K_3 are the roots of polynomial $\lambda^3-(14n-22)\lambda^2-(n^2+156n-132)\lambda-(4n^2+412n-216)=0$.

Proof: Consider the Wheel graph W_n having vertices v_i, i= 1, 2, …, n.

The graph W_n^+ is obtained by adding n new vertices v_i' and joining v_i' to v_i an edge. In W_n^+, $ecc(v_1)$ = 2, $ecc(v_i)$ = 3, i= 2, 3, …, n, n +1 and $ecc(v_j')$ = 4, j = n +2, n +3, …, 2n.

But we know that for S_n^+, $ecc(v_1)$ = 2, $ecc(v_i)$ = 3, i= 2, 3, …, n, n +1 and $ecc(v_j')$ = 4, j = n +2, n +3, …, 2n

Therefore, from Theorem 3.4,

$$Spectra(ES(S_n^+) = Spectra(ES(W_n^+)$$

Hence the result follows.

Theorem 3.6: Let G_1 and G_2 be two non-isomorphic graphs having n vertices each. Let $ecc(v_i)$ = 2 for all i = 1, 2, …, n in G_1 and G_2. Let k be the degree of w, where w is the vertex obtained by identifying (coalescence) the vertices $v_1 \epsilon V(G_1)$ and $v_1' \epsilon V(G_2)$. The graph so obtained is denoted by $G_1 o_v G_2$.Then,

$$E_{ES}(G_1 o_v G_2) = 6(k-1)+8(2n-k-3)+|L_1|+|L_2|+|L_3|.$$

Where L_1, L_2 and L_3 are the roots of the polynomial $\lambda^3-[6(k-1)+8(2n-k-3)]\lambda^2-[25k+36(2n-k-2)+49k(2n-k-2)-48(k-1)(2n-k-3)]\lambda+[200k(2n-k-3)+216(k-1)(2n-k-2)-420k(2n-k-2)]=0$.

Proof: Consider two non-isomorphic graphs G_1 and G_2 having n vertices each. Let $ecc(v_i)$ = 2 for all i = 1, 2, …, n in G_1 and G_2. Let k be the degree of w, where w is the vertex obtained by a single vertex coalescence of G_1 and G_2 denoted by $G_1 o_v G_2$.

In graph $G_1 o_v G_1$, $ecc(v_1)$ = 2, $ecc(v_i)$ = 3, i= 2, 3, …, k +1 and $ecc(v_j)$ = 4, j = k +2, k +3, …, 2n – 1.

$$ES(G_1 o_v G_2) = \begin{vmatrix} 0 & 5J_{1\times k} & 6J_{1\times(2n-k-2)} \\ 5J_{k\times 1} & 6(J-I)_{k\times k} & 7J_{k\times(2n-k-2)} \\ 6J_{(2n-k-2)\times 1} & 7J_{(2n-k-2)\times k} & 8(J-I)_{(2n-k-2)\times(2n-k-2)} \end{vmatrix} \quad \ldots (6)$$

Where I is an identity matrix and J is a matrix whose all the entries are 1.

Then the eccentricity sum polynomial is given by $|\lambda I - ES(G_1 o_v G_2)| = 0$.

$$|\lambda I - ES(G_1 o_v G_2)| = \begin{vmatrix} \lambda & -5J_{1\times k} & -6J_{1\times(2n-k-2)} \\ -5J_{k\times 1} & (\lambda I - 6(J-I))_{k\times k} & -7J_{k\times(2n-k-2)} \\ -6J_{(2n-k-2)\times 1} & -7J_{(2n-k-2)\times k} & (\lambda I - 8(J-I))_{(2n-k-2)\times(2n-k-2)} \end{vmatrix} \quad \ldots (7)$$

Operating $R_i + \frac{5R_1}{\lambda}$ for i= 2, 3, …, k +1 and $R_j + \frac{6R_1}{\lambda}$ for j = k +2, k +3, …, 2n – 1.
Then Eq.(7) reduces to
$|\lambda I - ES(G_1 o_v G_2)| =$

$$\begin{vmatrix} \lambda & -5J_{1\times k} & -6_{1\times(2n-k-2)} \\ 0_{k\times 1} & \left(\left(\lambda-\frac{25}{\lambda}\right)I-\left(6+\frac{25}{\lambda}\right)(J-I)\right)_{k\times k} & -\left(7+\frac{30}{\lambda}\right)J_{k\times(2n-k-2)} \\ 0_{(2n-k-2)\times 1} & -\left(7+\frac{30}{\lambda}\right)J_{(2n-k-2)\times k} & \left(\left(\lambda-\frac{36}{\lambda}\right)I-\left(8+\frac{36}{\lambda}\right)(J-I)\right)_{(2n-k-2)\times(2n-k-2)} \end{vmatrix} \quad \dots (8)$$

Operating $C_i + \left(\left(7+\frac{30}{\lambda}\right)\frac{\sum_{j=k+2}^{2n-1} c_j}{\left(\lambda-\frac{36}{\lambda}\right)-(2n-k-3)\left(8+\frac{36}{\lambda}\right)}\right)$ for i= 2, 3, …, k +1 on determinant (8),
The determinant reduces to

$$|\lambda I - ES(G_1 o_v G_2)| = \begin{vmatrix} \lambda & X1_{1\times k} & X3_{1\times(2n-k-2)} \\ 0_{k\times 1} & X2_{(k\times k)} & X4_{k\times(2n-k-2)} \\ 0_{(2n-k-2)\times 1} & 0_{(2n-k-2)\times k} & X5_{(2n-k-2)\times(2n-k-2)} \end{vmatrix}$$

Therefore $|\lambda I - ES(G_1 o_v G_2)| = \lambda \times |X2|_{(k\times k)} \times |X5|_{(2n-k-2)\times(2n-k-2)}$
Where $|X2|_{(k\times k)} = |AI - B(J-I)|_{(k\times k)}$, … (9)

In Eq.(9), $A = \lambda - \frac{25}{\lambda} - \left(\frac{(2n-k-2)\left(7+\frac{30}{\lambda}\right)^2}{\left(\left(\lambda-\frac{36}{\lambda}\right)-(2n-k-3)\left(8+\frac{36}{\lambda}\right)\right)}\right)$ & $B = -6 - \frac{25}{\lambda} - \left(\frac{(2n-k-2)\left(7+\frac{30}{\lambda}\right)^2}{\left(\left(\lambda-\frac{36}{\lambda}\right)-(2n-k-3)\left(8+\frac{36}{\lambda}\right)\right)}\right)$.

And $|X5| = \left|\left(\lambda-\frac{36}{\lambda}\right)I - \left(8+\frac{36}{\lambda}\right)(J-I)\right|_{(2n-k-2)\times(2n-k-2)}$.
Using Lemma 2.8, we get the eccentricity sum polynomial,
$|\lambda I - ES(G_1 o_v G_2)| = \lambda \times |X2|_{(k\times k)} \times |X5|_{(2n-k-2)\times(2n-k-2)} = 0$ reduces to,
$(\lambda+6)^{k-1}(\lambda+8)^{2n-k-3}\lambda^3 - [6(k-1)+8(2n-k-3)]\lambda^2 - [25k+36(2n-k-2)+49k(2n-k-2) - 48(k-1)(2n-k-3)]\lambda + [200k(2n-k-3)+216(k-1)(2n-k-2)-420k(2n-k-2)] = 0.$
Let L_1, L_2 and L_3 are the roots of the polynomial $\lambda^3 - [6(k-1)+8(2n-k-3)]\lambda^2 - [25k+36(2n-k-2) + 49k(2n-k-2) - 48(k-1)(2n-k-3)]\lambda + [200k(2n-k-3)+216(k-1)(2n-k-2)-420k(2n - k-2)] = 0$.
Hence, $E_{ES}(G_1 o_v G_1) = 6(k-1) + 8(2n-k-3) + |L_1| + |L_2| + |L_3|$

Definition: Let G_1 and G_2 be two non-isomorphic graphs having vertex sets $V(G_1) = \{v_i \mid i = 1, 2, \dots, n\}$ and $V(G_2) = \{v_i' \mid i = 1, 2, \dots, n\}$. The edge coalescence graph $G_1 o_e G_2$ is a graph in which w_1 and w_2 are the vertices obtained by coalescence of v_1 with v_1' and v_2 with v_2'. Where edges $v_1v_2 \in E(G_1)$ and $v_1'v_2' \in E(G_2)$.

Theorem 3.7: Let G_1 and G_2 be two non-isomorphic graphs with $|V(G_1)| = |V(G_2)| = n$. Let $ecc(v_i) = ecc(v_i') = 2$. Where $v_i \in V(G_1)$ and $v_i' \in V(G_2)$, for i = 1, 2, …, n. Let w_1 and w_2 be the vertices obtained by edge coalescence of G_1 and G_2. Then,

$$E_{ES}(G_1 o_e G_2) = 4 + 6(d-1) + 8(D-1) + |M_1| + |M_2| + |M_3|.$$

Where M_1, M_2 and M_3 are the roots of polynomial $\lambda^3 - [6\,d + 8D - 10]\lambda^2 - [dD + 74d + 88D + 8]\lambda - [4dD + 208d + 240D + 192] = 0$.
Where $d = deg(w_1)$ + (Number of vertices adjacent to w_2 and are not adjacent to w_1) – 2 and $D = 2n - d - 4$.
Proof: Let G_1 and G_2 be two non-isomorphic graphs having vertex sets $\{v_i\}$ and $\{v_i'\}$, i = 1, 2, …, n. Let $ecc(v_i) = ecc(v_i') = 2$. Let w_1 and w_2 be the points obtained by coalescence of v_1 with v_1' and v_2 with v_2'.
Let the resulting graph $G_1 o_e G_2$ is having vertex set $\{w_i$, i = 1, 2, …, 2n – 2$\}$.

In graph $G_1 o_e G_2$, $ecc(w_1) = ecc(w_2) = 2$ (2-times), $ecc(w_i) = 3$ (d -times), i= 3, 4, …, d +2 and $ecc(w_i) = 4$ (D -times), $i = d$ +3, d +4, …, $2n – 2$. Where $d = deg(w_1)$ + (Number of vertices adjacent to w_2 and are not adjacent to w_1) – 2 and $D = 2n – d – 4$.

$$ES(G_1 o_v G_2) = \begin{vmatrix} 4(J-I)_{2\times2} & 5J_{2\times d} & 6J_{2\times D} \\ 5J_{d\times2} & 6(J-I)_{d\times d} & 7J_{d\times D} \\ 6J_{D\times2} & 7J_{D\times d} & 8(J-I)_{D\times D} \end{vmatrix} \quad \dots (10)$$

Then the eccentricity sum polynomial is given by $|\lambda I - ES(G_1 o_e G_2)| = 0$.

$$\text{Now, } |\lambda I - ES(G_1 o_e G_2)| = \begin{vmatrix} (\lambda I - 4(J-I))_{2\times2} & -5J_{2\times d} & -6J_{2\times D} \\ -5J_{d\times2} & (\lambda I - 6(J-I))_{d\times d} & -7J_{d\times D} \\ -6J_{D\times2} & -7J_{D\times d} & 8(J-I)_{D\times D} \end{vmatrix} \quad \dots (11)$$

Operating $R_i + \frac{5(R_1+R_2)}{\lambda-4}$ for i= 3, 4, …, d +2 and $R_i + \frac{6(R_1+R_2)}{\lambda-4}$ for $i = d$ +3, d +4, …, $2n – 2$, we have,

$$|\lambda I - ES(G_1 o_e G_2)|$$
$$= \begin{vmatrix} (\lambda I - 4(J-I))_{2\times2} & -5J_{2\times d} & -6J_{2\times D} \\ 0_{d\times2} & \left(\left(\lambda - \frac{50}{\lambda-4}\right)I - \left(6 + \frac{50}{\lambda-4}\right)(J-I)\right)_{d\times d} & -\left(7+\frac{60}{\lambda-4}\right)J_{d\times D} \\ 0_{D\times2} & -\left(7+\frac{60}{\lambda-4}\right)J_{D\times d} & \left(\left(\lambda - \frac{72}{\lambda-4}\right)I - \left(8 + \frac{72}{\lambda-4}\right)(J-I)\right)_{D\times D} \end{vmatrix} \quad \dots (12)$$

Operating $C_i + \left(\left(7+\frac{60}{\lambda-4}\right)\frac{\sum_{j=d+3}^{2n-2} C_j}{\left(\lambda-\frac{72}{\lambda-4}\right)-(D-1)\left(8+\frac{72}{\lambda-4}\right)}\right)$ for i= 3, 4, …, d +2 on Eq. (12),

We have the determinant,

$$|\lambda I - ES(G_1 o_e G_2)| = \begin{vmatrix} (\lambda I - 4(J-I))_{2\times2} & X1_{2\times d} & X3_{2\times D} \\ 0_{d\times2} & X2_{d\times d} & X4_{d\times D} \\ 0_{D\times2} & 0_{D\times d} & X5_{D\times D} \end{vmatrix} \quad \dots (13)$$

$$|\lambda I - ES(G_1 o_e G_2)| = (\lambda^2 - 16) \times |X2|_{(d\times d)} \times |X5|_{(D\times D)}$$
$$\text{Where } |X2|_{(d\times d)} = |AI - B(J-I)|_{(d\times d)}. \quad \dots (14)$$

In Eq. (14), $A = \lambda - \frac{50}{\lambda-4} - \left(7+\frac{60}{\lambda-4}\right)^2\left(\frac{D}{\left(\left(\lambda-\frac{72}{\lambda-4}\right)-(D-1)\left(8+\frac{72}{\lambda-4}\right)\right)}\right)$ and

$$B = -6 - \frac{50}{\lambda-4} - \left(7+\frac{60}{\lambda-4}\right)^2\left(\frac{D}{\left(\left(\lambda-\frac{72}{\lambda-4}\right)-(D-1)\left(8+\frac{72}{\lambda-4}\right)\right)}\right).$$

and $|X5| = \left|\left(\lambda - \frac{70}{\lambda-4}\right)I - \left(8+\frac{70}{\lambda-4}\right)(J-I)\right|_{(D\times D2)}$.

Solving $|X2|_{(d\times d)}$ and $|X5|_{(D\times D)}$, by using Lemma 2.8, we have the characteristic polynomial,

$|\lambda I - ES(G_1 o_e G_2)| = (\lambda^2 - 16) \times |X2|_{(d\times d)} \times |X5|_{(D\times D)} = 0$ gives,

$(\lambda+4)(\lambda+6)^{d-1}(\lambda+8)^{D-1}[\lambda^3 - [6\,d + 8D - 10]\lambda^2 - [dD + 74d + 88D + 8]\lambda - [4dD + 208d + 240D + 192] = 0.$

Hence,

$$E_{ES}(G_1 o_e G_2) = 4 + 6(d-1) + 8(D-1) + |M_1| + |M_2| + |M_3|.$$

Where M_1, M_2 and M_3 are the roots of the polynomial $\lambda^3 - [6\,d + 8D - 10]\lambda^2 - [dD + 74d + 88D + 8]\lambda - [4dD + 208d + 240D + 192] = 0$.

IV. CONCLUSION

We have obtained generalized results on eccentricity sum energy of S_n^+, K_n^+ and shown eccentricity sum energy of S_n^+ and W_n^+ are equal. Obtained a vertex and edge coalescence of two non-isomorphic graphs G_1 and G_2 having equal vertices with eccentricity equal to 2. Results can be generalized for vertex and edge coalescence of clique present in G_1 and G_2. Also, for eccentricity is equal to e the results can be generalized.

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