Felicitous labeling in circulant graphs

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Abstract- A labeling of a graph G with q edges is said to be felicitous if there is an injective f : V (G) → {0, 1, 2, …, q} and the induced function f * : E(G) →{0, 1, 2,…, (q − 1)} defined as f*(uv) = f(u) + f(v)(mod q) is bijective. In this paper, We shown to be connected three regular circulant graphs are felicitous.

Keywords – harmonious labeling, felicitous labeling, circulant graph.

I. INTRODUCTION

Let G be a simple connected graph with vertex set $V(G)$ and edge set $E(G)$. Let $|V(G)| = n$ and $|E(G)| = q$. A labeling is assignment of integers to the vertices or edges, or both subject to the certain condition. For more about graph labeling, refer the survey [1]. For graph theoretic terminology we refer [4].

Harmonious labeling is introduced by *Graham* and *sloane* [2]. A graph G with q edges to be harmonious if there is an one-to-one function $f : V(G) \to \{0, 1, 2, \ldots, (q-1)\}$ such that for every edge $xy \in E(G)$ is assigned by the label $f(x) + f(y) (mod\ q)$, then the edge labels are distinct.

The generalization of harmonious labeling are felicitous labeling. *Lee, schmeichel, and shee* [3] is defined a graph G with q edges to be *felicitous* if there is an one-to-one function $f : V(G) \to \{0, 1, 2, \ldots, q\}$ and the induced function $f^* : E(G) \to \{0, 1, 2, \ldots, q-1\}$ defined as $f^*(uv) = f(u) + f(v) (mod\ q)$ is bijective. For more details of harmonious and felicitous labeling refer [1].

Let n, m and $a_1, a_2, \ldots, a_m$ be positive integers, $1 \leq a_i \leq \lfloor n/2 \rfloor$ and $a_i \neq a_j$ for all $1 \leq i, j \leq m$. An undirected graph with the set of vertices $V(G) = \{v_1, \ldots, v_n\}$ and the set of edges $E(G) = \{v_i v_{i+aj} : 1 \leq i \leq n, 1 \leq j \leq m\}$, the indices being taken modulo n, is called a circulant graph and it is denoted by $C_n(a_1, a_2, \ldots, a_m)$. The numbers $a_1, a_2, \ldots, a_m$ are called the generators and we say, that the edge $v_i v_{i+a_j}$ is of type a_j. It is easy to see, that the circulant graph $C_n(a_1, a_2, \ldots, a_m)$ is a regular graph of degree r, where

$$r = \begin{cases} 2m-1 & if\ \frac{n}{2} \in \{a_1, a_2, \ldots, a_m\} \\ 2m & otherwise \end{cases}$$

Note that the circulant graph $C_n(a_1, a_2, \ldots, a_m)$ is connected if and only if $gcd(a_1, a_2, \ldots, a_m, n) = 1$.

II. MAIN RESULT

In this section, we show that the circulant graph with specified generating sets admits felicitous labeling. Note that the graph is connected three regular.

Theorem 2.1. *Let $n(\geq 4)$ be an even integer, then the circulant graph $C_n(1, \frac{n}{2}, n-1)$ admits the felicitous labeling.*

Proof **:** Let G = $C_n(1, \frac{n}{2}, n-1)$ be the circulant graph with $n(\geq 4)$, where the vertex set $V(G) = \{v_0, v_1, \ldots, v_{n-1}\}$ and the edge set $E(G)=\{ v_i v_{i+1}, v_i v_{i+\frac{n}{2}} : 0 \leq i \leq n-1\}$, in all the places indices being taken modulo n. Here $q= n+\frac{n}{2}$.

Case(i): If $n \equiv 0, 4, 8 (mod\ 12)$. Note that $\frac{n}{2}$ is even. Let the function $f : V(G) \rightarrow \{0, 1, 2, \ldots, q\}$ defined by

$$f(v_i) = \begin{cases} i & \text{for } i = 0, 2, 4, \ldots, \frac{n}{2} \\ \frac{2n+1}{2} & \text{for } i = \left(\frac{n}{2}\right)+2, \left(\frac{n}{2}\right)+4, \ldots, n-2 \\ \frac{3+n+i}{2} & \text{for } i = 1, 3, \ldots, \left(\frac{n}{2}\right)-1 \\ \frac{-n+2i}{2} & \text{for } i = \left(\frac{n}{2}\right)+1, \left(\frac{n}{2}\right)+3, \ldots, n-1 \end{cases}$$

By the above function f, we get the following induced function $f^* : E(G) \rightarrow \{0, 1, 2, \ldots, q-1\}$ defined as $f^*(uv) = f(u) + f(v)(mod\ q)$, It should include important findings discussed briefly.

$f^*(v_i v_{i+1})=$

$$\begin{cases} \left(\frac{4+n+3i}{2}\right) mod\ q & \text{for } i = 0, 2, 4, \ldots, \left(\frac{n}{2}\right)+1 \\ \left(\frac{2+n}{2}\right) mod\ q & \text{for } i = \frac{n}{2} \\ \left(\frac{2+n+3i}{2}\right) mod\ q & \text{for } i = \left(\frac{n}{2}\right)+2, \left(\frac{n}{2}\right)+4, \ldots, n-2 \\ \left(\frac{5+n+3i}{2}\right) mod\ q & \text{for } i = 1, 3, \ldots, \left(\frac{n}{2}\right)-1 \\ \left(\frac{1+n+3i}{2}\right) mod\ q & \text{for } i = \left(\frac{n}{2}\right)+1, \left(\frac{n}{2}\right)+3, \ldots, n-3 \\ \left(\frac{-n+2i}{2}\right) mod\ q & \text{for } i = n-1 \end{cases}$$

$f^*(v_i v_{i+(n/2)})=$

$$\begin{cases} \left(\frac{n}{2}\right) mod\ q & \text{for } i = 0 \\ \left(\frac{5n+6i}{4}\right) mod\ q & \text{for } i = 2, 4, \ldots, \left(\frac{n}{2}\right)-2 \\ \left(\frac{3+n+3i}{2}\right) mod\ q & \text{for } i = 1, 3, \ldots, \left(\frac{n}{2}\right)-1 \end{cases}$$

From the above function, one can easily check the q number of edge labels form a sequence as follows $(\frac{n}{2})$-1, $\frac{n}{2}$, $\frac{n}{2}$+1,…, $\frac{n}{2}$+ $(\frac{3n-6}{2})$. Taking modulo q for each term of the sequence we get, q distinct edge labels. Hence f^* is bijective.

Case(i): If $n \equiv 2(mod\ 12)$. Note that, $\frac{n}{2}$ is odd. Let the function $f : V(G) \rightarrow \{0, 1, 2, . . . , q\}$ defined by

$f(v_i)$=

$$\begin{cases} i & \text{for } i = 0, 2, 4, \dots, \left(\frac{n}{2}\right) - 1 \\ \frac{-n+2i}{2} & \text{for } i = \left(\frac{n}{2}\right)+1, \quad \left(\frac{n}{2}\right)+3, \dots, n-2 \\ \frac{5+n+i}{2} & \text{for } i = 1, \quad 3, \dots, \left(\frac{n-8}{6}\right) \\ \frac{4+5n+2i}{4} & \text{for } i = \left(\frac{n-8}{2}\right)+2, \left(\frac{n-8}{2}\right)+4, \dots, \left(\frac{n}{2}\right)-2 \\ i & \text{for } i = \left(\frac{n}{2}\right) \\ \frac{1+n+i}{2} & \text{for } i = \left(\frac{n}{2}\right)+2, \left(\frac{n}{2}\right)+4, \dots, \left(\frac{2n-7}{3}\right) \\ \frac{7+n+i}{2} & \text{for } i = \left(\frac{2n-7}{3}\right)+1, \left(\frac{2n-7}{3}\right)+3, \dots, n-3 \\ \frac{4+n+i}{2} & \text{for } i = n-1 \end{cases}$$

The Assignment of the labeling to the vertices by the above function f, we obtain the edge labels by $f^*: E(G) \rightarrow \{0, 1, . . . , q - 1\}$ as follows:

$f^*(v_iv_{i+1})$=

$$\begin{cases} \left(\frac{6+n+i}{2}\right) mod\ q & \text{for } i = 0 \\ \left(\frac{6+n+3i}{2}\right) mod\ q & \text{for } i = 2, 4, \dots, \frac{n-14}{6} \\ \left(\frac{6+5n+6i}{4}\right) mod\ q & \text{for } i = \left(\frac{n-14}{6}\right)+2, \left(\frac{n-14}{6}\right)+4, \dots, \left(\frac{n}{2}\right)-3 \\ (1+2i) mod\ q & \text{for } i = \left(\frac{n}{2}\right)-1 \\ \left(\frac{2+3i}{2}\right) mod\ q & \text{for } i = \left(\frac{n}{2}\right)+1, \left(\frac{n}{2}\right)+3, \dots, \left(\frac{2n-10}{3}\right) \\ \left(\frac{8+3i}{2}\right) mod\ q & \text{for } i = \left(\frac{2n-10}{3}\right)+2, \left(\frac{2n-10}{3}\right)+4, \dots, n-4 \\ \left(\frac{4+2i}{2}\right) mod\ q & \text{for } i = n-2 \\ \left(\frac{7+n+3i}{2}\right) mod\ q & \text{for } i = 1, 3, \dots, \left(\frac{n-8}{6}\right) \\ \left(\frac{8+5n+6i}{4}\right) mod\ q & \text{for } i = \left(\frac{n-8}{6}\right)+2, \left(\frac{n-8}{6}\right)+4, \dots, \left(\frac{n}{2}\right)-2 \\ \left(\frac{2-n+4i}{2}\right) mod\ q & \text{for } i = \left(\frac{n}{2}\right) \\ \left(\frac{3+3i}{2}\right) mod\ q & \text{for } i = \left(\frac{n}{2}\right)+2, \left(\frac{n}{2}\right)+4, \dots, \left(\frac{2n-7}{3}\right) \\ \left(\frac{9+3i}{2}\right) mod\ q & \text{for } i = \left(\frac{2n-7}{3}\right)+2, \left(\frac{2n-7}{3}\right)+4, \dots, n-3 \\ \left(\frac{4+n}{2}\right) mod\ q & \text{for } i = n-1 \end{cases}$$

$f^*(v_iv_{i+(n/2)})$=

$$\begin{cases} \left(\frac{n}{2}\right) mod\ q & \text{for } i=0 \\ \left(\frac{2+3n+6i}{4}\right) mod\ q & \text{for } i=2,4,\ldots,\left(\frac{n-14}{6}\right) \\ \left(\frac{14+3n+6i}{4}\right) mod\ q & \text{for } i=\left(\frac{n-14}{6}\right)+2,\left(\frac{n-14}{6}\right)+2,\ldots,\left(\frac{n}{2}\right)-3 \\ \left(\frac{4+n+2i}{4}\right) mod\ q & \text{for } i=\left(\frac{n}{2}\right)-1 \\ \left(\frac{5+n+3i}{2}\right) mod\ q & \text{for } i=1,3,\ldots,\left(\frac{n-8}{6}\right) \\ \left(\frac{14+3n+6i}{4}\right) mod\ q & \text{for } i=\left(\frac{n-8}{6}\right)+2,\left(\frac{n-8}{6}\right)+2,\ldots,\left(\frac{n}{2}\right)-2 \end{cases}$$

From the above function, one can easily check the q number of edge labels form a sequence as follows $(\frac{n}{2})$, $\frac{n}{2}+1$, $\frac{n}{2}+2$,…, $\frac{n}{2}+(\frac{3n-4}{2})$. Taking modulo q for each term of the sequence we get, q distinct edge labels. Hence f^* is bijective.

Case(iii) : If $n \equiv 6(mod\ 12)$. Here $\frac{n}{2}$ is odd. Let the function $f:V(G) \rightarrow \{0,1,2,\ldots,q\}$ defined by

$f(v_i)$=

$$\begin{cases} i & \text{for } i=0,2,4,\ldots,\left(\frac{n}{2}\right)-1 \\ \frac{-n+2i}{2} & \text{for } i=\left(\frac{n}{2}\right)+1,\left(\frac{n}{2}\right)+1,\ldots,n-2 \\ \frac{-4+5n+2i}{4} & \text{for } i=1,3,\ldots,\frac{n-12}{6} \\ \frac{4+5n+2i}{4} & \text{for } i=\left(\frac{n-12}{6}\right)+2,\left(\frac{n-12}{6}\right)+4,\ldots,\left(\frac{n}{2}\right) \\ \frac{n+2i}{2} & \text{for } i=\frac{n}{2},\left(\frac{n}{2}\right)+2,\ldots,\frac{5n-12}{6} \\ \frac{8+n+2i}{4} & \text{for } i=\left(\frac{5n-12}{6}\right)+2,\left(\frac{5n-12}{6}\right)+4,\ldots,n \\ n & \text{for } i=n-1 \end{cases}$$

The assignment of the labeling to the vertices by the above function f, we obtain the edge labels by $f^*:E(G) \rightarrow \{0,1,2,\ldots,q-1\}$ as follows.

$f^*(v_iv_{i+1})=$

$$\begin{cases} \left(\frac{-2+5n+2i}{4}\right) \, mod \, q & for \; i=0 \\ \left(\frac{6+5n+6i}{4}\right) \, mod \, q & \text{for } i=2,4,\dots,\left(\frac{n}{2}\right)-3 \\ \left(\frac{2+n+6i}{4}\right) \, mod \, q & \text{for } i=\left(\frac{n}{2}\right)-1 \\ \left(\frac{2-n+6i}{4}\right) \, mod \, q & \text{for } i=\left(\frac{n}{2}\right)+1,\left(\frac{n}{2}\right)+3,\dots,\frac{5n-18}{6} \\ \left(\frac{10-n+6i}{4}\right) \, mod \, q & \text{for } i=\left(\frac{5n-18}{6}\right)+2,\left(\frac{5n-18}{6}\right)+4,\dots,n-4 \\ \left(\frac{n+2i}{2}\right) \, mod \, q & \text{for } i=n-2 \\ \left(\frac{5n+6i}{4}\right) \, mod \, q & \text{for } i=1,3,\dots,\frac{n-12}{6} \\ \left(\frac{8+5n+6i}{4}\right) \, mod \, q & \text{for } i=\left(\frac{n-12}{6}\right)+2,\left(\frac{n-12}{6}\right)+2,\dots\left(\frac{n}{2}\right)-2 \\ \left(\frac{4-n+6i}{4}\right) \, mod \, q & \text{for } i=\left(\frac{n}{2}\right),\left(\frac{n}{2}\right)+2,\dots,\frac{5n-12}{6} \\ \left(\frac{12-n+6i}{4}\right) \, mod \, q & \text{for } i=\left(\frac{5n-12}{6}\right)+2,\left(\frac{5n-12}{6}\right)+4,\dots,n-3 \\ (n) \, mod \, q & \text{for } i=n-1 \end{cases}$$

$f^*\left(v_iv_{(i+\frac{n}{2})}\right)=$

$$\begin{cases} \left(\frac{n}{2}\right) mod \, q & \text{for } i=0 \\ \left(\frac{n+3i}{2}\right) mod \, q & \text{for } i=2,4,\dots,\left(\frac{n-6}{3}\right) \\ \left(\frac{-4+5n+3i}{2}\right) mod \, q & \text{for } i=\left(\frac{n-6}{3}\right)+2,\left(\frac{n-6}{3}\right)+4,\dots,\left(\frac{n}{2}\right)-3 \\ (n+i) mod \, q & \text{for } i=\left(\frac{n}{2}\right)-1 \\ \left(\frac{-4+5n+6i}{2}\right) mod \, q & \text{for } i=1,3,\dots,\frac{n-12}{6} \\ \left(\frac{4+5n+6i}{2}\right) mod \, q & \text{for } i=\left(\frac{n-12}{6}\right)+2,\left(\frac{n-12}{6}\right)+4,\dots,\left(\frac{n}{2}\right)-2 \end{cases}$$

From the above function, one can easily check the q number of edge labels form a sequence as follows $\left(\frac{n}{2}\right),\left(\frac{n}{2}\right)+1,\left(\frac{n}{2}\right)+2,\dots,\left(\frac{n}{2}\right)+\left(\frac{2n-4}{2}\right)$. Taking modulo q for each term of the sequence, we get f^* is bijective.

Case(iv): If $n \equiv 10 (mod \; 12)$. Here $\frac{n}{2}$ is odd. Let the function $f: V(G) \to \{0,1,2,\dots,q\}$ defined by

$f(v_i)=$

$$\begin{cases} i & \text{for } i=0,2,4,\dots,\left(\frac{n}{2}\right)-1 \\ \frac{-n+2i}{2} & \text{for } i=\left(\frac{n}{2}\right)+1,\left(\frac{n}{2}\right)+1,\dots,n-2 \\ \frac{n}{2} & \text{for } i=1 \\ \frac{4+5n+2i}{4} & \text{for } i=1,3,\dots,\left(\frac{n}{2}\right)-2 \\ \frac{8+n}{2} & \text{for } i=\left(\frac{n}{2}\right) \\ \frac{16+n+2i}{4} & \text{for } i=\left(\frac{n}{2}\right)+2,\left(\frac{n}{2}\right)+4,\dots,\frac{5n-12}{6} \\ \frac{28+n+2i}{4} & \text{for } i=\left(\frac{5n-20}{6}\right)+2,\left(\frac{5n-12}{6}\right)+4,\dots,n-3 \\ \frac{6+n}{2} & \text{for } i=n-1 \end{cases}$$

$f^*(v_iv_{i+1})=$

$$\begin{cases} \left(\frac{n}{2}\right) \bmod q & \text{for } i = 0 \\ \left(\frac{6+5n+6i}{4}\right) \bmod q & \text{for } i = 2,4,\dots,\left(\frac{n}{2}\right)-3 \\ \left(\frac{8+n+2i}{2}\right) \bmod q & \text{for } i = \left(\frac{n}{2}\right)-1 \\ \left(\frac{18-n+6i}{4}\right) \bmod q & \text{for } i = \left(\frac{n}{2}\right)+1, \left(\frac{n}{2}\right)+3,\dots,\frac{5n-26}{6} \\ \left(\frac{30-n+6i}{4}\right) \bmod q & \text{for } i = \left(\frac{5n-26}{6}\right)+2, \left(\frac{5n-26}{6}\right)+4,\dots,n-4 \\ \left(\frac{6+2i}{2}\right) \bmod q & \text{for } i = n-2 \\ \left(\frac{2+n+2i}{2}\right) \bmod q & \text{for } i = 1 \\ \left(\frac{8+5n+6i}{4}\right) \bmod q & \text{for } i = 3,5,\dots,\left(\frac{n}{2}\right)-2 \\ \left(\frac{10+n}{2}\right) \bmod q & \text{for } i = \left(\frac{n}{2}\right) \\ \left(\frac{20-n+6i}{4}\right) \bmod q & \text{for } i = \left(\frac{n}{2}\right)+2, \left(\frac{n}{2}\right)+4,\dots,\frac{5n-20}{6} \\ \left(\frac{32-n+6i}{4}\right) \bmod q & \text{for } i = \left(\frac{5n-20}{6}\right)+2, \left(\frac{5n-20}{6}\right)+4,\dots,n-3 \\ \left(\frac{6+n}{2}\right) \bmod q & \text{for } i = n-1 \end{cases}$$

$f^*\left(v_i v_{(i+\frac{n}{2})}\right) =$

$$\begin{cases} \left(\frac{8+n}{2}\right) \bmod q & \text{for } i = 0 \\ \left(\frac{8+n+3i}{2}\right) \bmod q & \text{for } i = 2,4,\dots,\left(\frac{n-10}{3}\right) \\ \left(\frac{14+n+3i}{2}\right) \bmod q & \text{for } i = \left(\frac{n-10}{3}\right)+2, \left(\frac{n-10}{3}\right)+4,\dots,\left(\frac{n}{2}\right)-3 \\ \left(\frac{6+n+2i}{2}\right) \bmod q & \text{for } i = \left(\frac{n}{2}\right)-1 \\ \left(\frac{n+2i}{2}\right) \bmod q & \text{for } i = 1 \\ \left(\frac{4+5n+6i}{2}\right) \bmod q & \text{for } i = 3,5,\dots,\left(\frac{n}{2}\right)-2 \end{cases}$$

From the above function, one can easily check the q number of edge labels form a sequence as follows $\left(\frac{n}{2}\right), \left(\frac{n}{2}\right)+1, \left(\frac{n}{2}\right)+2, \dots, \left(\frac{n}{2}\right)+\left(\frac{3n-4}{2}\right)$. Taking modulo q for each term of the sequence, we get f^* is bijective.

Theorem. 2. 2: *Let $n(\geq 6)$ be an even integer, then the circulant graph $C_n\left(\frac{n}{2}-1, \frac{n}{2}, \frac{n}{2}+1\right)$ admits the felicitous labeling.*

Proof : Let $C_n\left(\frac{n}{2}-1, \frac{n}{2}, \frac{n}{2}+1\right)$ be the circulant graph with $n(\geq 6)$. where the vertex set $V(G) = \{v_0, v_1, \dots, v_{n-1}\}$ and the edge set $E(G)=\{ v_i v_{i+(\frac{n}{2})-1}, v_i v_{i+\frac{n}{2}} : 0 \leq i \leq n-1\}$, in all the places indices being taken modulo n. Here $q= n+\frac{n}{2}$ .Let the function $f: V(G) \rightarrow \{0,1,2,\dots,q\}$ defined by

$$f(v_i)=\begin{cases} i & \text{for } i = 0,1,2,\dots,\left(\frac{n}{2}\right) \\ \frac{-n+4i}{2} & \text{for } i = \left(\frac{n}{2}\right)+1, \left(\frac{n}{2}\right)+2,\dots,n-1 \end{cases}$$

The assignment of the labeling to the vertices by the above function f, we obtain the edge labels by $f^*: E(G) \rightarrow \{0,1,2,\dots,q-1\}$ as follows

$$f^*\left(v_i v_{(i+\frac{n}{2}-1)}\right)=$$

$$\begin{cases} \left(\frac{n-2}{2}\right) mod\, q & \text{for } i=0 \\ \left(\frac{n-2+4i}{2}\right) mod\, q & \text{for } i=1 \\ \left(\frac{n-4+6i}{2}\right) mod\, q & \text{for } i=2,3,4,\dots,\left(\frac{n}{2}\right) \\ (-n-1+3i)\, mod\, q & \text{for } i=\left(\frac{n}{2}\right)+1,\left(\frac{n}{2}\right)+2,\dots,n-1 \end{cases}$$

$$f^*\left(v_i v_{(i+\frac{n}{2})}\right)=\begin{cases} \left(\frac{n-2}{2}\right) mod\, q & \text{for } i=0 \\ \left(\frac{n-2}{2}\right) mod\, q & \text{for } i=1,2,\dots,\left(\frac{n}{2}\right)-1 \end{cases}$$

From the above function, one can easily check the q number of edge labels form a sequence as follows $\left(\frac{n}{2}\right)-1, \left(\frac{n}{2}\right), \left(\frac{n}{2}\right)+1, \left(\frac{n}{2}\right)+2, \dots, \left(\frac{n}{2}\right)+\left(\frac{3n-4}{2}\right)$. Taking modulo q for each term of the sequence, we get f^* is bijective.

Important notes:

The graphs wherein Case(i) of the Theorem 2.1 and Theorem 2.2 admits the harmonious labeling. Since the assignment of labels to the vertices of the graphs in Case(i) of the Theorem 2.1 and Theorem 2.2 are $0, 1, \dots, q-1$.

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III. CONCLUSION

In this paper, we proved that the particular class of three regular circulant graphs admits felicitous labeling. Also, we identified some of the three regular circulant felicitous graphs are harmonious. However, there remains proving the existence of felicitous labeling for the arbitrary circulant graphs is still open. Another future work is to address the existence of felicitous labeling for Cayley graphs.

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