On Pythagorean Fuzzy Ideals in Semigroups

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Abstract: In this paper, we introduce the notion of Pythagorean fuzzy ideals in semigroups and some interesting properties with suitable examples.

Keywords: Pythagorean fuzzy, fuzzy ideals, semigroup.

1. INTRODUCTION

After the introduction of the fuzzy set by [9], several types of research conducted experiments on the generalizations of the notion of fuzzy set. The concept of the intuitionistic fuzzy set was introduced by [1,2] as a generalization of the fuzzy set. [3,4] considered the fuzzification of interior ideals in semigroups and the notion of an intuitionistic fuzzy interior ideal of a semigroup S, and its properties were investigated. [6] discussed some properties of fuzzy ideals and fuzzy bi-ideals in the semigroup. [5] considered the fuzzification of (1,2)-ideals in semigroups and investigated its properties. [7,8] introduced the Pythagorean fuzzy set as a generalization of the fuzzy set. After its existence, several researchers also studied the properties of fuzzy ideals of the semigroup. In this paper, we discuss and investigate the properties of Pythagorean fuzzy ideals in semigroups.

2. PRELIMINARIES

Definition 2.1 [2] Let S be a semigroup. M and N be subsets of S, the product of M and N is defined as $MN = \{mn \in S | m \in Mandn \in\}$ A non- empty subset M of S is called a sub-semigroup of S if $MM \subseteq M$. A non-empty subset M of S is called a left (resp. right) ideal of S if $SM \subseteq M$ (resp.$MS \subseteq M$). A is called a two sided ideal of S if it is both a left ideal and right ideal of S. A sub- semigroup M of S is called a bi-ideal of S if $MSM \subseteq M$. A sub-semigroup M of S is called a (1,2) ideal of S if $MSM^2 \subseteq M$. A semigroup S is said to be (2,2)- regular if $m \in m^2Sm^2$ for any $m \in S$. A semigroup S is called regular if for each element $m \in S$ there exists $x \in S$ such that $m = mxm$. A semigroup S is said to be completely regular if, for any $m \in S$, there exists $x \in S$ such that $m = mxm$ and $mx = xm$. For a semigroup S, is completely regular if and only if(iff) S is a union of groups iff S is (2,2)-regular. By a fuzzy set ϑ in a non-empty set S we mean a function $\vartheta: S \to [0,1]$, and the complement of ϑ, denoted by $\overline{\vartheta}$, is the fuzzy set in S given by $\overline{\vartheta}(x) = 1 - \vartheta(x)$ for all $x \in S$.

Definition 2.2 [7] Let X be a universe of discourse, A **Pythagorean fuzzy set** (PFS) $P = \{z, \vartheta_p(x), \omega_p(x)/z \in X\}$ where $\vartheta: X \to [0,1]$ and $\omega: X \to [0,1]$ represent the degree of membership and non-membership of the object $z \in X$ to the set P subset to the condition $0 \leq \left(\vartheta_p(z)\right)^2 + \left(\omega_p(z)\right)^2 \leq 1$ for all $z \in X$. For the sake of simplicity a PFS is denoted as $P = (\vartheta_p(z), \omega_p(z))$.

3. PYTHAGOREAN FUZZY IDEALS IN SEMIGROUPS

In this section, let S denote a semigroup unless otherwise specified.

Definition 3.1 A Pythagorean fuzzy set (PFS) $P = (\vartheta_p, \omega_p)$ in S is called a Pythagorean fuzzy sub-semigroup of S, if

(i) $\vartheta_p(x_1x_2) \geq min\{\vartheta_p(x_1), \vartheta_p(x_2)\}$
(ii) $\omega_p(x_1x_2) \leq max\{\omega_p(x_1), \omega_p(x_2)\}$ for all $x_1, x_2 \in S$.

Definition 3.2 A PFS $P = (\vartheta_p, \omega_p)$ in S is called a Pythagorean fuzzy left ideal of S, if

(i) $\vartheta_p(x_1x_2) \geq \vartheta_p(x_2)$
(ii) $\omega_p(x_1x_2) \leq \omega_p(x_2)$ $x_1, x_2 \in S$.

A Pythagorean fuzzy right ideal of S is defined in an analogous way.
A PFS $P = (\vartheta_p, \omega_p)$ in S is called a Pythagorean fuzzy ideal of S, if it is both a Pythagorean fuzzy left and Pythagorean right ideal of S.
It is clear that any Pythagorean fuzzy left(resp. right) ideal of S is a Pythagorean fuzzy sub-semigroup of S.

Example 3.1 Let $S = \{u, v, w, x, y\}$ be a semigroup the following cayley table.

Cayley table

•	p	q	r	s
p	p	p	p	p
q	p	p	p	p
r	p	p	q	p
s	p	p	q	q

Define a Pythagorean fuzzy set $P = (\vartheta, \omega)$ in S as follows.

S	$\vartheta(x_1)$	$\omega(x_1)$
p	0.9	0.3
q	0.6	0.5
r	0.4	0.7
s	0.6	0.6

Thus $P = (\vartheta, \omega)$ is a Pythagorean fuzzy sub-semigroup of S.
Definition 3.4 A Pythagorean fuzzy sub-semigroup $P = (\vartheta_p, \omega_p)$ of S is called a Pythagorean fuzzy bi-ideal of S.

(i) $\vartheta_p(x_1ux_2) \geq min\{\vartheta_p(x_1), \vartheta_p(x_2)\}$
(ii) $\omega_p(x_1ux_2) \leq max\{\omega_p(x_1), \omega_p(x_2)\}$ for all $u, x_1, x_2 \in S$.

Example 3.5 Table-1 Define a Pythagorean fuzzy set $P = (\vartheta, \omega)$ in S as follows.

S	$\vartheta(x_1)$	$\omega(x_1)$
p	0.8	0.3
q	0.4	0.5
r	0.3	0.7
s	0.5	0.4

Thus $P = (\vartheta, \omega)$ is a Pythagorean fuzzy bi-ideal of S. But it is not Pythagorean fuzzy ideal of S.
Since (i)$\vartheta_p(x_1x_2) \geq \{\vartheta_p(x_1) \vee \vartheta_p(x_2)\}$ $0.50 \nleq 0.6$
(ii)$\omega_p(x_1x_2) \leq \{\omega_p(x_1) \wedge \omega_p(x_2)\}$ $0.7 \ngeq 0.5$

Theorem 3.6 If $\{P_i\}_{i\in I}$ is a family of PFBI of S, then $\cap P_i$ is a PFBI of S. Where $\cap P_i = (\wedge \vartheta_{p_i}, \vee \omega_{p_i})$ and $\wedge \vartheta_{p_i} = inf\{\vartheta_{p_i}(x_1)|i \in I, x_1 \in S\}$, $\vee \omega_{p_i} = sup\{\omega_{p_i}(x_1)|i \in I, x_1 \in S\}$.

Proof. Let $x_1, x_2 \in S$. Then we have

$$\wedge \vartheta_{p_i}(x_1x_2) \geq \wedge \left\{min\{\vartheta_{p_i}(x_1), \vartheta_{p_i}(x_2)\}\right\}$$
$$= min\left\{min\{\vartheta_{p_i}(x_1), \vartheta_{p_i}(x_2)\}\right\}$$
$$= min\left\{min\{\vartheta_{p_i}(x_1)\}, min\{\vartheta_{p_i}(x_2)\}\right\}$$
$$= min\{\wedge \vartheta_{p_i}(x_1), \wedge \vartheta_{p_i}(x_2)\}$$
$$\vee \omega_{p_i}(x_1x_2) \leq \vee \left\{max\{\omega_{p_i}(x_1), \omega_{p_i}(x_2)\}\right\}$$
$$= max\left\{max\{\omega_{p_i}(x_1), \omega_{p_i}(x_2)\}\right\}$$
$$= max\left\{max\{\omega_{p_i}(x_1)\}, max\{\omega_{p_i}(x_2)\}\right\}$$
$$= max\{\vee \omega_{p_i}(x_1), \vee \omega_{p_i}(x_2)\}.$$

Hence $\cap P_i$ is a Pythagorean fuzzy sub-semigroup of S.

Next for $u, x_1, x_2 \in S$, we obtain

$$\wedge \vartheta_{p_i}(x_1ux_2) \geq \wedge \left\{min\{\vartheta_{p_i}(x_1), \vartheta_{p_i}(x_2)\}\right\}$$
$$= min\left\{min\{\vartheta_{p_i}(x_1), \vartheta_{p_i}(x_2)\}\right\}$$
$$= min\left\{min\{\vartheta_{p_i}(x_1)\}, min\{\vartheta_{p_i}(x_2)\}\right\}$$
$$= min\{\wedge \vartheta_{p_i}(x_1), \wedge \vartheta_{p_i}(x_2)\}$$
$$\vee \omega_{p_i}(x_1ux_2) \leq \vee \left\{max\{\omega_{p_i}(x_1), \omega_{p_i}(x_2)\}\right\}$$
$$= max\left\{max\{\omega_{p_i}(x_1), \omega_{p_i}(x_2)\}\right\}$$
$$= max\left\{max\{\omega_{p_i}(x_1)\}, max\{\omega_{p_i}(x_2)\}\right\}$$
$$= max\{\vee \omega_{p_i}(x_1), \vee \omega_{p_i}(x_2)\}.$$

Hence $\cap P_i$ is a PFBI of S.

This completes the proof.

Theorem 3.7 Every Pythagorean fuzzy left(right) ideal of S is a Pythagorean fuzzy bi-ideal of S.

Proof. Let $P = (\vartheta_p, \omega_p)$ be a Pythagorean fuzzy left ideal of S and $u, x_1, x_2 \in S$.

Then

$$\vartheta_p(x_1ux_2) = \vartheta_p(x_1ux_2)$$
$$\geq \vartheta_p(x_2)$$
$$\vartheta_p(x_1ux_2) \geq min\{\vartheta_p(x_1, \vartheta_p(x_2))\}$$
$$\omega_p(x_1ux_2) = \omega_p(x_1ux_2)$$
$$\leq \omega_p(x_2)$$
$$\omega_p(x_1ux_2) \leq max\{\omega_p(x_1, \omega_p(x_2))\}$$

Thus $P = (\vartheta_p, \omega_p)$ is a PFBI of S.

The right case is provided in an analogous way.

Theorem 3.8 Every Pythagorean fuzzy bi-ideal of a group S is constant.

Proof. Let $P = (\vartheta_p, \omega_p)$ be a PFBI of a group S and let x_1 be any element of S.

Then

$$\vartheta_p(x_1) = \vartheta_p(ex_1e)$$
$$\geq min\{\vartheta_p(e), \vartheta_p(e)\}$$
$$= \vartheta_p(e)$$
$$= \vartheta_p(ee)$$
$$= \vartheta_p(x_1x_1^{-1})(x_1^{-1}x_1)$$
$$= \vartheta_p(x_1(x_1^{-1}x_1^{-1})x_1)$$
$$\geq min\{\vartheta_p(x_1, \vartheta_p(x_1))\}$$

$= \vartheta_p(x_1)$

and

$\omega_p(x_1) = \omega_p(ex_1e)$

$\leq max\{\omega_p(e), \omega_p(e)\}$

$= \omega_p(e)$

$= \omega_p(ee)$

$= \omega_p(x_1x_1^{-1})(x_1^{-1}x_1)$

$= \omega_p(x_1(x_1^{-1}x_1^{-1})x_1)$

$\leq max\{\omega_p(x_1, \omega_p(x_1))\}$

$= \omega_p(x_1)$.

Where e is the identity of S. It follows that $\vartheta_p(x_1) = \vartheta_p(e)$ and $\omega_p(x_1) = \omega_p(e)$ which means that $P = (\vartheta_p, \omega_p)$ is constant.

Theorem 3.9 If PFS $P = (\vartheta_p, \omega_p)$ in S is a PFBI of S, then so is $\Box P = (\vartheta_p, \overline{\vartheta}_p)$.

Proof. It is sufficient to show that $\overline{\vartheta}_p$ satisfies the conditions in Definition 3.1 and 3.3. For any $u, x_1, x_2 \in S$, we have

$\overline{\vartheta}_p(x_1x_2) = 1 - \vartheta_p(x_1x_2)$

$\leq 1 - min\{\vartheta_p(x_1), \vartheta_p(x_2)\}$

$= max\{1 - \vartheta_p(x_1), 1 - \vartheta_p(x_2)\}$

$= max\{\overline{\vartheta}_p(x_1), \overline{\vartheta}_p(x_2)\}$

and

$\overline{\vartheta}_p(x_1ux_2) = 1 - \vartheta_p(x_1ux_2)$

$\leq 1 - min\{\vartheta_p(x_1), \vartheta_p(x_2)\}$

$= max\{1 - \vartheta_p(x_1), 1 - \vartheta_p(x_2)\}$

$= max\{\overline{\vartheta}_p(x_1), \overline{\vartheta}_p(x_2)\}$.

Therefore $\Box P$ is a PFBI of S.

Definition 3.10 A Pythagorean fuzzy sub-semigroup $P = (\vartheta_p, \omega_p)$ of S is called a Pythagorean fuzzy (1,2) ideal of S. If

(i) $\vartheta_p(x_1u(x_2x_3)) \geq min\{\vartheta_p(x_1), \vartheta_p(x_2), \vartheta_p(x_3)\}$

(ii) $\omega_p(x_1u(x_2x_3)) \leq max\{\omega_p(x_1), \omega_p(x_2), \omega_p(x_3)\}$ $u, x_1, x_2, x_3 \in S$.

Theorem 3.11 Every PFBI is a Pythagorean fuzzy (1,2) ideal of S.

Proof. Let PFS $P = (\vartheta_p, \omega_p)$ be a PFBI of S and let $u, x_1, x_2, x_3 \in S$.

Then

$\vartheta_p(x_1u(x_2x_3)) = \vartheta_p((x_1ux_2)x_3)$

$\geq min\{\vartheta_p(x_1ux_2), \vartheta_p(x_3)\}$

$\geq min\{min\{\vartheta_p(x_1), \vartheta_p(x_2)\}, \vartheta_p(x_3)\}$

$= min\{\vartheta_p(x_1), \vartheta_p(x_2), \vartheta_p(x_3)\}$

and

$\omega_p(x_1u(x_2x_3)) = \omega_p((x_1ux_2)x_3)$

$\leq max\{\omega_p(x_1ux_2), \omega_p(x_3)\}$

$\leq max\{max\{\omega_p(x_1), \omega_p(x_2)\}, \omega_p(x_3)\}$

$= max\{\omega_p(x_1), \omega_p(x_2), \omega_p(x_3)\}$.

Hence $P = (\vartheta_p, \omega_p)$ is a Pythagorean fuzzy (1,2) ideal of S.

To consider the converse of theorem next theorem, we need to strengthen the condition of a semigroup S.

Theorem 3.12 If S is a regular semigroup, then every Pythagorean fuzzy (1,2) ideal of S is a PFBI of S.

Proof. Assume that a semigroup S is regular and let $P = (\vartheta_p, \omega_p)$ be a Pythagorean fuzzy (1,2) ideal of S.

Let $u, x_1, x_2, x_3 \in S$. Since S is regular, we have $x_1 u \in (x_1 S x_1) S \subseteq x_1 S x_1$, which implies that $x_1 u = x_1 S x_1$ for some $s \in S$.
Thus

$$\begin{aligned}\vartheta_p(x_1 u x_2) &= \vartheta_p((x_1 s x_1) x_2)\\ &= \vartheta_p(x_1 s(x_1 x_2))\\ &\geq min\{\vartheta_p(x_1), \vartheta_p(x_1), \vartheta_p(x_2)\}\\ &= min\{\vartheta_p(x_1), \vartheta_p(x_2)\}\end{aligned}$$

and

$$\begin{aligned}\omega_p(x_1 u x_2) &= \omega_p((x_1 s x_1) x_2)\\ &= \omega_p(x_1 s(x_1 x_2))\\ &\leq max\{\omega_p(x_1), \omega_p(x_1), \omega_p(x_2)\}\\ &= max\{\omega_p(x_1), \omega_p(x_2)\}.\end{aligned}$$

Therefore $P = (\vartheta_p, \omega_p)$ is a PFBI of S.

Theorem 3.13 A PFS $P = (\vartheta_p, \omega_p)$ is a PFBI of S if and only if the fuzzy sets ϑ_p and $\overline{\omega_p}$ are FBI of S.
Proof. Let $P = (\vartheta_p, \omega_p)$ be a PFBI of S. Then clearly ϑ_p is a FBI of S. Let $u, x_1, x_2 \in S$.
Then

$$\begin{aligned}\overline{\omega_p}(x_1 x_2) &= 1 - \omega_p(x_1 x_2)\\ &\geq 1 - max\{\omega_p(x_1), \omega_p(x_2)\}\\ &= min\{(1 - \omega_p(x_1)), (1 - \omega_p(x_2))\}\\ &= min\{\overline{\omega_p}(x_1), \overline{\omega_p}(x_2)\}\end{aligned}$$

$$\begin{aligned}\overline{\omega_p}(x_1 u x_2) &= 1 - \omega_p(x_1 u x_2)\\ &\geq 1 - max\{\omega_p(x_1), \omega_p(x_2)\}\\ &= min\{(1 - \omega_p(x_1)), (1 - \omega_p(x_2))\}\\ &= min\{\overline{\omega_p}(x_1), \overline{\omega_p}(x_2)\}.\end{aligned}$$

Hence $\overline{\omega_p}$ is a fuzzy bi-ideal of S.
Conversely, suppose that ϑ_p and $\overline{\omega_p}$ are FBI of S. Let $u, x_1, x_2 \in S$.
Then

$$\begin{aligned}1 - \omega_p(x_1 x_2) &= \overline{\omega_p}(x_1 x_2)\\ &\geq min\{\overline{\omega_p}(x_1), \overline{\omega_p}(x_2)\}\\ &= min\{(1 - \omega_p(x_1)), (1 - \omega_p(x_2))\}\\ &= max\{\omega_p(x_1), \omega_p(x_2)\}\end{aligned}$$

$$\begin{aligned}1 - \omega_p(x_1 u x_2) &= \overline{\omega_p}(x_1 u x_2)\\ &\geq min\{\overline{\omega_p}(x_1), \overline{\omega_p}(x_2)\}\\ &= 1 - max\{\omega_p(x_1), \omega_p(x_2)\}.\end{aligned}$$

Which implies that $\omega_p(x_1 x_2) \leq max\{\omega_p(x_1), \omega_p(x_2)\}$ and $\omega_p(x_1 u x_2) \leq max\{\omega_p(x_1), \omega_p(x_2)\}$
This completes the proof.

Definition 3.14 A PFS $P = (\vartheta_p, \omega_p)$ in S is called a Pythagorean fuzzy interior ideal (PFII) of S if it satisfies
(i) $\vartheta_p(x_1 u x_2) \geq \vartheta_p(u)$
(ii) $\omega_p(x_1 u x_2) \leq \omega_p(u)$ $u, x_1, x_2 \in S$.

Example 3.15 Define a Pythagorean fuzzy set $P = (\vartheta, \omega)$ in S as follows.

S	$\vartheta(x_1)$	$\omega(x_1)$
p	0.4	0.3
q	0.2	0.4
r	0.1	0.3
s	0.2	0.6

Thus $P = (\vartheta, \omega)$ is a Pythagorean fuzzy interior ideal of S.

Theorem 3.16 If $\{P_i\}_{i\in I}$ is a family of PFII of S, then $\cap P_i$ is a PFII of S.
Where $\cap P_i = (\wedge \vartheta_{p_i}, \vee \omega_{p_i})$ and
$\wedge \vartheta_{p_i}(x_1) = inf\{\vartheta_{p_i}(x_1) | i \in I, x_1 \in S\}$,
$\vee \omega_{p_i}(x_1) = sup\{\omega_{p_i}(x_1) | i \in I, x_1 \in S\}$.
Proof. Let $u, x_1, x_2 \in S$.
Then

$$\wedge \vartheta_{p_i}(x_1x_2) \geq min\left\{min\{\vartheta_{p_i}(x_1), \vartheta_{p_i}(x_2)\}\right\}$$
$$= \left(\wedge \vartheta_{p_i}(x_1)\right) \wedge \left(\wedge \vartheta_{p_i}(x_2)\right)$$

and

$$\vee \omega_{p_i}(x_1x_2) \leq max\left\{max\{\omega_{p_i}(x_1), \omega_{p_i}(x_2)\}\right\}$$
$$= \left(\vee \omega_{p_i}(x_1)\right) \vee \left(\vee \omega_{p_i}(x_2)\right)$$
$$\wedge \vartheta_{p_i}(x_1ux_2) \geq \wedge \vartheta_{p_i}(u)$$

and

$$\vee \omega_{p_i}(x_1ux_2) \leq \vee \omega_{p_i}(u).$$

Hence $\cap P_i$ is a PFII of S.

Theorem 3.17 If PFS $P = (\vartheta_p, \omega_p)$ in S is a PFII of S, then so is $\Box P = (\vartheta_p, \overline{\vartheta}_p)$, $\overline{\vartheta}_p = 1 - \vartheta_p$.
Proof. It is sufficient to show that $\overline{\vartheta}_p$ satisfies condition of Definition 3.12. For any $u, x_1, x_2 \in S$. We have

$$\overline{\vartheta}_p(x_1x_2) = 1 - \vartheta_p(x_1x_2)$$
$$\leq 1 - \left(\vartheta_p(x_1) \wedge \vartheta_p(x_2)\right)$$
$$= \left(1 - \vartheta_p(x_1)\right) \vee \left(1 - \vartheta_p(x_2)\right)$$
$$= \overline{\vartheta}_p(x_1) \vee \overline{\vartheta}_p(x_2)$$

and

$$\overline{\vartheta}_p(x_1ux_2) = 1 - \vartheta_p(x_1ux_2)$$
$$\leq 1 - \vartheta_p(u)$$
$$= \overline{\vartheta}_p(u)$$

Therefore P is a PFII of S.

Definition 3.18 Let $P = (\vartheta_p, \omega_p)$ be a PFS of S and let $\alpha \in [0,1]$ then the sets.
$\vartheta_{p,\alpha} = \{x_1 \in S: \vartheta_p(x_1)\alpha\}$ and $\omega_{p,\alpha} = \{x_1 \in S: \omega_p(x_1)\alpha\}$ are called a ϑ-level α-cut and ω-level α-cut of K respectively.

Theorem 3.19 If PFS $P = (\vartheta_p, \omega_p)$ in S is a PFII of S, then the ϑ-level α-cut $\vartheta_{p,\alpha}$ and ω-level α-cut $\omega_{p,\alpha}$ of P are interior ideal of S, for every $\alpha \in Im(\vartheta_p) \cap Im(\omega_p) \subseteq [0,1]$.
Proof. Let $\alpha \in Im(\vartheta_p) \cap Im(\omega_p) \subseteq [0,1]$ and let $x_1, x_2 \in \vartheta_{p,\alpha}$ then $\vartheta_p(x_1) \geq \alpha$ and $\vartheta_p(x_2) \geq \alpha$.
It follows from that $\vartheta_p(x_1x_2) \geq \vartheta_p(x_1) \wedge \vartheta_p(x_2) \geq \alpha$. So that $x_1, x_2 \in \vartheta_{p,\alpha}$.
If $x_1, x_2 \in \omega_{p,\alpha}$, then $\omega_p(x_1) \leq \alpha$ and $\omega_p(x_2) \leq \alpha$ and so $\omega_p(x_1x_2) \leq \omega_p(x_1) \vee \omega_p(x_2) \leq \alpha$, that is $x_1, x_2 \in \omega_{p,\alpha}$.
Hence $\vartheta_{p,\alpha}$ and $\omega_{p,\alpha}$ are sub-semigroup of S. Now let $x_1x_2 \in S$ and $u \in \vartheta_{p,\alpha}$.
Then $\vartheta_p(x_1ux_2) \geq \vartheta_p(u) \geq \alpha$ and so $x_1ux_2 \in \vartheta_{p,\alpha}$.
If $u \in \omega_{p,\alpha}$. Then $\omega_p(x_1ux_2) \leq \omega_p(u) \leq \alpha$ thus $x_1ux_2 \in \omega_{p,\alpha}$.
Therefore $\vartheta_{p,\alpha}$ and $\omega_{p,\alpha}$ are interior ideal of S.

Theorem 3.20 A PFS $P = (\vartheta_p, \omega_p)$ is a PFII of S if and only if the fuzzy set $\vartheta_p, \overline{\omega_p}$ are fuzzy interior ideal (FII) of S.
Proof. Let $P = (\vartheta_p, \omega_p)$ be a PFII of S. Then clearly ϑ_p is a FII of S. Let $u, x_1, x_2 \in S$. Then

$\overline{\omega_p}(x_1x_2) = 1 - \omega_p(x_1x_2)$
$\geq 1 - (\omega_p(x_1)) \vee \omega_p(x_2)$
$= (1 - \omega_p(x_1)) \wedge (1 - \omega_p(x_2))$
$= \overline{\omega_k}(x_1) \wedge \overline{\omega_p}(x_2)$
$\overline{\omega_p}(x_1ux_2) = 1 - \omega_p(x_1ux_2)$
$\geq 1 - (\omega_p(u))$
$= \overline{\omega_p}(u)$
$\overline{\omega_k}$ is a FII of S.
Conversely.
Suppose that ϑ_p and $\overline{\omega_p}$ are FII of S. Let $u, x_1, x_2 \in S$.
$1 - \omega_p(x_1x_2) = \overline{\omega_p}(x_1x_2)$
$\geq \overline{\omega_p}(x_1) \wedge \overline{\omega_p}(x_2)$
$= (1 - \omega_p(x_1)) \wedge (1 - \omega_p(x_2))$
$= 1 - \omega_p(x_1) \vee \omega_p(x_2)$
$= 1 - \omega_p(x_1ux_2) = \overline{\omega_p}(x_1ux_2)$
$\geq \overline{\omega_p}(u) = 1 - \omega_p(u)$
which implies $\omega_p(x_1x_2) \leq \omega_p(x_1) \vee \omega_p(x_2)$
and $\omega_p(x_1ux_2) \leq \omega_p(u)$
This completes the proof.

4. HOMOMORPHISM OF PYTHAGOREAN FUZZY IDEALS IN SEMIGROUPS

Let f be a mapping from a set A to a set B. IF $P_1 = (\vartheta_{p_1}, \omega_{p_1})$ and $P_2 = (\vartheta_{p_2}, \omega_{p_2})$ are PFSs in A and B respectively then the preimage of B under f, denoted by $f^{-1}(p_2)$ is a PFS in p_1 defined by $f^{-1}(p_2) = (f^{-1}(\vartheta_{p_2}), f^{-1}(\vartheta_{p_2}))$.

Theorem 4.1 Let $f: S \to T$ be a homomorphism of semigroups. If $P_2 = (\vartheta_{p_2}, \omega_{p_2})$ is a PFBI of T. Then the preimage $f^{-1}(p_2) = (f^{-1}(\vartheta_{p_2}), f^{-1}(\vartheta_{p_2}))$ of P_2 under f is a PFBI of S.

Proof.
$f^{-1}(\vartheta_{p_2})(x_1x_2) = \vartheta_{p_2}(f(x_1x_2))$
$= \vartheta_{p_2}(f(x_1), f(x_2))$
$\geq min\{\vartheta_{p_2}(f(x_1)), \vartheta_{p_2}(f(x_2))\}$
$= min\{f^{-1}(\vartheta_{p_2}(x_1)), f^{-1}(\vartheta_{p_2}(x_2))\}$
and
$f^{-1}(\omega_{p_2})(x_1x_2) = \omega_{p_2}(f(x_1x_2))$
$= \omega_{p_2}(f(x_1), f(x_2))$
$\leq max\{\omega_{p_2}(f(x_1)), \omega_{p_2}(f(x_2))\}$
$= max\{f^{-1}(\omega_{p_2}(x_1)), f^{-1}(\omega_{p_2}(x_2))\}$.
Hence $f^{-1}(p_2) = (f^{-1}(\vartheta_{p_2}), f^{-1}(\omega_{p_2}))$ is a Pythagorean fuzzy sub-semigroup of S for any $u, x_1, x_2 \in S$.
$f^{-1}(\vartheta_{p_2})(x_1ux_2) = \vartheta_{p_2}(f(x_1ux_2))$
$= \vartheta_{p_2}(f(x_1), f(u)), f(x_2))$
$\geq min\{\vartheta_{p_2}(f(x_1)), \vartheta_{p_2}(f(x_2))\}$
$= min\{f^{-1}(\vartheta_{p_2}(x_1)), f^{-1}(\vartheta_{p_2}(x_2))\}$
and
$f^{-1}(\omega_{p_2})(x_1ux_2) = \omega_{p_2}(f(x_1ux_2))$
$= \omega_{p_2}(f(x_1), f(u)), f(x_2))$
$\leq max\{\omega_{p_2}(f(x_1)), \omega_{p_2}(f(x_2))\}$
$= max\{f^{-1}(\omega_{p_2}(x_1)), f^{-1}(\omega_{p_2}(x_2))\}$.
Therefore $f^{-1}(p_2) = (f^{-1}(\vartheta_{p_2}), f^{-1}(\omega_{p_2}))$ is a PFBI of S.

Theorem 4.2 Let $f: A \rightarrow B$ be a homomorphism of semigroups. If $p_2 = (\vartheta_{p_2}, \omega_{p_2})$ is a PFII of B, then preimage $f^{-1}(p_2) = (f^{-1}(\vartheta_{p_2}), f^{-1}(\omega_{p_2}))$ of p_2 under f is a PFII of S.
Proof. Assume that $p_2 = \vartheta_{p_2}, \omega_{p_2}$ is a PFII of S and let $x_1, x_2 \in S$.
Then

$$\begin{aligned} f^{-1}(\vartheta_{p_2})(x_1x_2) &= \vartheta_{p_2}(f(x_1x_2)) \\ &= \vartheta_{p_2}(f(x_1)f(x_2)) \\ &\geq \vartheta_{p_2}(f(x_1)) \wedge \vartheta_{p_2}(f(x_2)) \\ &= f^{-1}(\vartheta_{p_1}(x_1)) \wedge f^{-1}(\vartheta_{p_2}(x_2)) \end{aligned}$$

$$\begin{aligned} f^{-1}(\omega_{p_2})(x_1x_2) &= \omega_{p_2}(f(x_1x_2)) \\ &= \omega_{p_2}(f(x_1)f(x_2)) \\ &\leq \omega_{p_2}(f(x_1)) \vee \omega_{p_2}(f(x_2)) \\ &= f^{-1}(\omega_{p_1}(x_1)) \vee f^{-1}(\omega_{p_2}(x_2)). \end{aligned}$$

Hence $f^{-1}(p_2) = (f^{-1}(\vartheta_{p_2}), f^{-1}(\omega_{p_2}))$ is a PF sub-semigroup of S for any $u. x_1, x_2 \in S$,
we have

$$\begin{aligned} f^{-1}(\vartheta_{p_2})(x_1ux_2) &= \vartheta_{p_2}(f(x_1ux_2)) \\ &= \vartheta_{p_2}(f(x_1), f(u), f(x_2)) \\ &\geq \vartheta_{p_2}(f(u)) \\ &= f^{-1}(\vartheta_{p_2}(u)) \end{aligned}$$

$$\begin{aligned} f^{-1}(\omega_{p_2})(x_1ux_2) &= \omega_{p_2}(f(x_1ux_2)) \\ &= \omega_{p_2}(f(x_1), f(u), f(x_2)) \\ &\leq \omega_{p_2}(f(u)) \\ &= f^{-1}(\omega_{p_2}(u)). \end{aligned}$$

Therefore $f^{-1}(p_2) = (f^{-1}(\vartheta_{p_2}), f^{-1}(\omega_{p_2}))$ is a PFII of S.

5. CONCLUSION

In this paper, Pythagorean fuzzy sub-semigroup, Pythagorean fuzzy left(resp. right) ideal, Pythagorean fuzzy ideal, Pythagorean fuzzy bi-ideal, Pythagorean fuzzy interior ideal and Homomorphism of Pythagorean fuzzy ideal in semigroups are studied and investigated some properties with some necessary examples.

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