The Toxin Effect on an Eco-epidemiological Model

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Abstract- This paper examines an ecotoxicant-epidemiological model involving infectious disease in prey of kind SI. The susceptible prey and predator released some toxic material to each other. This model is a modified version from the classic Lotka-Volterra predator prey model as a struggle. The conditions for the existence and local stability of equilibrium points are obtained. The global dynamics is studied methodically and numerically for one set of initial points and for diverse sets of parameters values.

Keywords: Eco-epidemiological model, Prey-predator model, SI epidemic disease, Toxin.

I.INTRODUCTION

Predation is according to the definition of ecologists, a biological interaction between two organisms, one of which of which is the predator, the subcultures on an organisms or a number of other organisms known as prey. The predator may or may not kill its prey before the embryos are on it, but the act of predation causes, from point of view of the previous scientist the death of prey always.

Given that predators occupy the first rank in the food chain that supports the environmental balance, therefore the means by which the prey maintains its life can be varied. It is a kind of defence mechanism, as the physical features of some animals make these animals unwanted meals for predators.

Moreover, the poisons a chemical used as a weapon, whether it is injected directly into the victim's body through tusks of up to an inch length, for example, or through sharp payments such as the needle tip.

In most cases, snakes and snakes sit at the top of any list that encompasses the most poisonous animals in the word, but not all of these organisms are able to excrete poison, but that certain species of them went through stages of development that made it able to inject toxins through grooves or tubes in their teeth.

There are two main and distinct areas that must be studied: Epidemiology and Ecology, as the dynamics of ecosystems, are strongly influenced by some epidemiological diseases [1-4]. One of the branches of mathematical biology that deals with environmental and epidemiological issues in that one is for the epidemiological environment. Most models of infectious diseases and methods of transmission arise from the classic work of (Kermack et al. (1927)),[5-7].

However, industrial wastes in seas and rivers have a major impact on the life of living organisms as they produce very dangerous toxic substances. Due to the properties of some accumulated non-degradable chemicals, Where these non-dissolving substances turn into toxic chemicals after they are released into the aquatic environment. This will affect the organisms that live in the water, including fish, [8-11].

Mathematical modelling of population models has been developed to study the problems posed by toxins in the environment, Keong et al. in [12] studied a prey-predator model where, it was suggested that both species secrete one toxin over another, in addition to the effect of harvesting on both species.

In this paper, a mathematical model of three first-order non-linear differential equations has been proposed to study the effect of the toxic substance released by the prey on the predator and vice versa in the presence of an incurable disease in the prey community, and I assume that the disease does not transfer to the predator society and the attack function has been chosen from a Lotka-Volterra type. The conditions for the existence and local stability of equilibrium points are obtained. The global dynamics is studied analytically and numerically for one set of primary points and diverse sets of parameters values.

II MATHEMATICAL MODEL

Consider the following ecotoxicant-epidemiological model:-

$$\begin{aligned} \frac{dS}{dT} &= rS\left(1-\frac{S+I}{K}\right)-\alpha SI-b_1S^2X-c_1\text{SX}, \\ \frac{dI}{dT} &= \alpha SI-c_2IX-d_1I, \\ \frac{dX}{dT} &= e_1c_1\text{SX}+e_2c_2IX-b_2SX^2-d_2\text{X}. \end{aligned} \tag{2.1}$$

Where the variables and the parameters of the above system are illustrated in the following table:

Table 1:- Variables and the parameters of system (2.1)

Parameter	Representation of the parameter
$S(T)$	The density of susceptible prey at time T
$I(T)$	The density of infected prey at time T
$X(T)$	The density of predator at time T
$r>0$	The intrinsic growth rate of susceptible prey
$K>0$	Carrying capacity
$\alpha>0$	Infection rate
$c_1>0$ *and* $c_2>0$	Maximum attack rates for susceptible and infected, respectively.
$0<e_1<1$ *and* $0<e_2<1$	The uptake rates of food from the susceptible and infected prey, respectively.
$d_1>0$	The mortality rate of infected prey
$d_2>0$	The death rate of the predator in the absence of its feeding

$b_1 > 0$ *and* $b_2 > 0$	Toxicant rates for susceptible prey and predator respectively

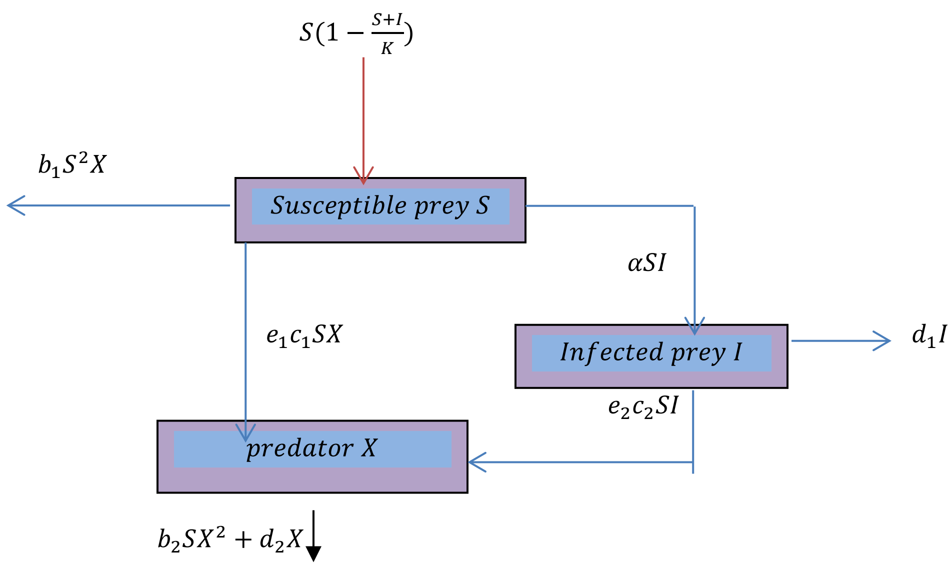

Fig.1: Block illustration for ecotoxicant-epidemiological model assumed by system (2.1)

In order to reduce the number of parameters of system (2.1), the next dimensionless variables and constants are reached:

$$t = r\,T\,, s = \frac{S}{k}\ , i = \frac{I}{k}\ , x = \frac{X}{k}\ . l_1 = \frac{\alpha k}{r}\ , l_2 = \frac{b_1K^2}{r}\ , l_3 = \frac{c_1K}{r}\ , l_4 = \frac{c_2K}{r}\ , l_5 = \frac{d_1}{r},$$
$$l_6 = \frac{e_1c_1K}{r}\ , l_7 = \frac{e_2c_2K}{r}\ , l_8 = \frac{b_2K^2}{r}\ , l_9 = \frac{d_2}{r}.$$

Then the non dimension system is:

$$\begin{aligned}
\frac{ds}{dt} &= s\big(1-(s+i)\big) - l_1si - l_2s^2x - l_3sx = f_1(s,i,x)\\
\frac{di}{dt} &= l_1si - l_4ix - l_5i\ = f_2(s,i,x) \qquad\qquad (2.2)\\
\frac{dx}{dt} &= l_6sx + l_7ix - l_8sx^2 - l_9x = f_3(s,i,x)
\end{aligned}$$

with the next initial condition $s(0) \geq 0, i(0) \geq 0$ and $x(0) \geq 0$.

It is easy to check that the solution of system (2.2) exist and unique.

Theorem (2.1): The solutions of system (2.2) that start in R^3_+ are uniformly bounded.

Proof.

Let $(s(t), i(t), x(t))$ be any solution of the system (2.2) with non-negative initial condition $(s(0), i(0), x(0))$. Agreeing with the first equation of system (2.2) we need:

$$\frac{ds}{dt} \leq s(1-s).$$

So, we get:

$\lim\limits_{t\to\infty}\sup s(t) \le 1$. Take the function

$G(t) = s(t) + i(t) + x(t)$.

Therefore,

$\frac{dG}{dt} < 2s - (l_3 - l_6)sx - (l_4 - l_7)ix - (s + l_5\, i + l_9 x)$.

Now, hence from the natural facts $l_6 < l_3$, and $l_7 < l_4$, thus

$$\frac{dG}{dt} \le 2 - LG, \qquad \text{where } \mathrm{L} = \min\{1, l_5, l_9\}.$$

Now, by the comparison theorem [13], we get:

$$G(t) \le \frac{2}{L} + \left(G(0) - \frac{2}{L}\right) e^{-Lt} \quad .$$

Thus $0 \le G(t) \le \frac{2}{L}$ as $t \to \infty$, and the proof is complete ■

III THE EQUILIBRIUM POINTS:

System (2.2) has five equilibrium points which are given below.

1) The trivial equilibrium point $E_0 = (0,0,0)$ and always exist.

2) The axial equilibrium point $E_1 = (1,0,0)$

3) The equilibrium point $E_2 = (\hat{s}, \hat{\imath}, 0)$, where $\hat{s} = \frac{l_5}{l_1}$ $, and\ \hat{\imath} = \frac{l_1 - l_5}{l_1(1 + l_1)}$ exist provided that:

$$l_5 < l_1 . \tag{3.2e}$$

4) The equilibrium point $E_3 = (\bar{\bar{s}}, 0, \bar{\bar{x}})$ exists by solving the next equations :

$$1 - s - l_2 sx - l_3 x = 0 \tag{3.3a}$$

$$l_6 s - l_8 sx - l_9 = 0 \tag{3.3b}$$

From eq. $(3.3b)$ we get,

$$s = \frac{l_9}{l_6 - l_8 x}. \tag{3.3c}$$

By substitute equations (3.3c) in (3.3a), we get,

$$l_3 l_8 x^2 - (l_8 + l_3 l_6)x + l_6 - l_9(1 + l_2) = 0. \tag{3.3d}$$

Now, by discarte rule eq. $(3.3d)$ has unique positive roots provided that:

$$l_6 < l_9(1 + l_2) \tag{3.3e}$$

So , the point $E_4 = (\bar{\bar{s}}, 0, \bar{\bar{x}})$, where $\bar{\bar{s}} = s(\bar{\bar{x}})$ exist if with condition (3.3e) the next condition holds:

$$\bar{\bar{x}} < \frac{l_6}{l_8}. \tag{3.3f}$$

5) The equilibrium point $E_5 = (s^*, i^*, x^*)$ exists by solving the following set of equations:

$$1 - (s + i) - l_1 i - l_2 sx - l_3 x = 0 \qquad (3.4a)$$

$$l_1 s - l_4 x - l_5 = 0 \qquad (3.4b)$$

$$l_6 s + l_7 i - l_8 sx - l_9 = 0 \qquad (3.4c)$$

From equation $(3.4b)$ we have,

$$s = \frac{l_5 + l_4 x}{l_1}. \qquad (3.4e)$$

By replacing equation (3.4e) in equation (3.4c) results:

$$i = \frac{l_1 l_9 + (l_5 + l_4 x)(l_8 x - l_6)}{l_1 l_7}. \qquad (3.4f)$$

By replacing equations (3.4e) and (3.4f) in equation (3.4a) results:

$$M_1 x^2 + M_2 x + M_3 = 0, \qquad (3.4g)$$

where:

$M_1 = -[(1 + l_1) l_4 l_8 + l_2 l_4 l_7]$,
$M_2 = -[l_4 l_7 - (1 + l_1)(l_4 l_6 + l_5 l_8) + l_1 l_3 l_7]$

$M_3 = l_7(l_1 - l_5) - (1 + l_1)(l_1 l_9 - l_5 l_6) - l_2 l_5 l_7$.

So, eq. $(3.4h)$ has a unique positive root, namely x^* if:

$$(1 + l_1)(l_4 l_6 + l_5 l_8) > l_7(l_4 + l_1 l_3) \qquad (3.4i)$$
$$l_5 l_6 > l_1 l_9 \qquad (3.4j)$$
$$l_1 > l_5 \qquad (3.4k)$$
$$l_7(l_1 - l_5) - (1 + l_1)(l_1 l_9 - l_5 l_6) > l_2 l_5 l_7 \qquad (3.4l)$$

So, $E_5 = (s^*, i^*, x^*)$ where $s^* = s(x^*), i^* = i(x^*)$ which are positive if the following condition holds:

$$x^* > \frac{l_6}{l_8}. \qquad (3.4l)$$

IV LOCAL STABILITY ANALYSIS.

In this section, the stability of system (2.2) discussed:-

The Jacobian matrix $J(s, i, x)$ of the dimensionless system can be written:

$$J = \begin{bmatrix} J_{11} & J_{12} & J_{13} \\ J_{21} & J_{22} & J_{23} \\ J_{31} & J_{32} & J_{33} \end{bmatrix}, \qquad (4.1)$$

where

$J_{11} = 1 - 2s - (1 + l_1)i - 2l_2 sx - l_3 x$, $J_{12} = -(1 + l_1)s$,

$J_{13} = -s(l_2 s + l_3)$, $J_{21} = l_1 i$, $J_{22} = l_1 s - l_4 x - l_5$, $J_{23} = -l_4 i$

$J_{31} = l_6 x - l_8 x^2$, $J_{32} = l_7 x$, $J_{33} = l_6 s + l_7 i - 2 l_8 s x - l_9$.

4. 1 Local stability of E_0

At E_0 the Jacobian matrix is:

$$J_0 = J(E_0) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -l_5 & 0 \\ 0 & 0 & -l_9 \end{bmatrix}. \quad (4.1a)$$

Then the eigenvalues of $J(E_0)$ are $\lambda_{0s} = 1, \lambda_{0i} = -l_5$ and $\lambda_{0x} = -l_9$

Thus, the equilibrium point E_0 is saddle (unstable).

4.2 Local stability of E_1

At E_1 the Jacobian matrix become

$$J(E_1) = \begin{bmatrix} -1 & -(1 + l_1) & -(l_1 + l_3) \\ 0 & l_1 - l_5 & 0 \\ 0 & 0 & l_6 - l_9 \end{bmatrix}. \quad (4.2a)$$

Then the eigenvalues of $J(E_1)$ are $\lambda_{1s} = -1, \lambda_{1i} = l_1 - l_5$ and $\lambda_{1x} = l_6 - l_9$

Thus, the point E_1 stable, iff :

$l_1 < l_5$, (4.2b)

$l_6 < l_9$. (4.2c)

Otherwise, E_1 is unstable.

4.3 Local stability of E_2

At E_2 the Jacobian matrix is:

$$J(E_2) = [k_{ij}]_{3\times 3} \ . \quad (4.3a)$$

Here

$k_{11} = 1 - 2\hat{s} - (1 + l_1)\hat{\imath}$, $k_{12} = -(1 + l_1)\hat{s}$, $k_{13} = -\hat{s}(l_2\hat{s} + l_3)$,

$k_{21} = l_1 \hat{\imath}$, $k_{22} = 0, k_{23} = -l_4\hat{\imath}$, $k_{31} = 0, k_{32} = 0, k_{33} = l_6\hat{s} + l_7\hat{\imath} - l_9$.

Then the characteristic equation of $J(E_2)$ is given by:

$[\lambda^2 - \text{tr(A)}\,\lambda + \text{Det(A)}]\ [l_6\hat{s} + l_7\hat{\imath} - l_9 - \lambda] = 0,$

where:

$tr(A) = \lambda_{2s} + \lambda_{2i} = (1 - 2\hat{s} - (1 + l_1)\hat{\imath})$

$Det(A) = \lambda_{2s}.\lambda_{2i} = l_1\hat{\imath}(1 + l_1)\hat{s} > 0$

So, either

$[\lambda^2 - \text{tr(A)}\lambda + \text{Det(A)}] = 0, where$

$$A = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & 0 \end{bmatrix},$$

which gives the two eigenvalues λ_{2s} and λ_{2i} are negative provided that

$$2\hat{s} + (1 + l_1)\hat{\imath} > 1. \quad (4.3d)$$

Or

$l_6\hat{s} + l_7\hat{\imath} - l_9 - \lambda = 0$, which gives

$\lambda_{2x} = l_6\hat{s} + l_7\hat{\imath} - l_9$, which is negative provided that

$$l_6\hat{s} + l_7\hat{\imath} < l_9 \quad (4.3c)$$

Therefore, E_2 is stable under conditions (4.3d) and (4.3c) , it is unstable otherwise.

4.4 Local stability of E_3

The Jacobian matrix of system (2.2) at the free disease equilibrium point $E_3 = (\bar{\bar{s}}, 0, \bar{\bar{x}})$, can be written as:

$$J(E_3) = [r_{ij}]_{3\times 3}, \quad (4.4a)$$

Here

$r_{11} = 1 - 2\bar{\bar{s}} - 2l_2\bar{\bar{s}}\bar{\bar{x}} - l_3\bar{\bar{x}}\,, r_{12} = -(1 + l_1)\bar{\bar{s}}_1, r_{13} = -\bar{\bar{s}}(l_2\bar{\bar{s}} + l_3),$

$r_{21} = 0, r_{22} = l_1\bar{\bar{s}} - l_4\bar{\bar{x}} - l_5\,, r_{23} = 0, r_{31} = (l_6 - l_8\bar{\bar{x}})\bar{\bar{x}}, r_{32} = l_7\bar{\bar{x}}$,

$r_{33} = l_6\bar{\bar{s}} - 2l_8\bar{\bar{s}}\bar{\bar{x}} - l_9.$

Then the characteristic equation of $J(E_3)$ is given by:

$$[\ \lambda^2 + \bar{\bar{M}}_1\lambda + \bar{\bar{M}}_2\ ]\,(l_1\bar{\bar{s}} - l_4\bar{\bar{x}} - l_5 - \lambda\,) = 0, \quad (4.4b)$$

where:

$$\lambda_{3s} + \lambda_{3x} = \bar{\bar{\mathrm{M}}}_1 = [(1 - 2\bar{\bar{s}} - 2l_2\bar{\bar{s}}\bar{\bar{x}} - l_3\bar{\bar{x}}) + (\bar{\bar{s}}(l_6 - 2l_8\overline{\overline{x)}} - l_9)]$$

$$\lambda_{3s}.\lambda_{3x} = \bar{\bar{M}}_2 = (1 - 2\bar{\bar{s}}(1 + l_2\bar{\bar{x}}) - l_3\bar{\bar{x}})(\bar{\bar{s}}(l_6 - 2l_8\bar{\bar{x}} - l_9) + \bar{\bar{x}}\bar{\bar{s}}(l_2\bar{\bar{s}} + l_3)(l_6 - l_8\bar{\bar{x}}\,)$$

So, either

$l_1\bar{\bar{s}} - l_4\bar{\bar{x}} - l_5 - \lambda = 0, \qquad that\ is\ ,$

$$\lambda_{3i} = l_1\bar{\bar{s}} - l_4\bar{\bar{x}} - l_5 < 0, \quad (4.4c)$$

provided that :

$$l_1\bar{\bar{s}} < l_4\bar{\bar{x}} + l_5. \quad (4.4d)$$

Or

$\left[\ \lambda^2 + \bar{\bar{M}}_1\lambda + \bar{\bar{M}}_2\ \right] = 0$, which gives the two other eigenvalues $\lambda_{3s}\ and\ \lambda_{3x}$ which are negative if in addition to the existing condition (3.3*f*) the following conditions hold:

$$1 < 2\bar{\bar{s}}(1 + l_2\bar{\bar{x}}) + l_3\bar{\bar{x}} \qquad (4.4e)$$

$$\bar{\bar{x}} > \frac{l_6}{2l_8}. \qquad (4.4f)$$

Therefore, E_3 is stable under condition (3.3f) and conditions (4.4d- 4.4*f*) are hold., it is unstable otherwise.

4.5 Local stability of E_4

At E_4 the Jacobian matrix is :

$$J(E_4) = [j_{ij}]_{3\times3}. \qquad (4.5a)$$

Here

$j_{11} = 1 - 2s^* - (1 + l_1)i^* - 2l_2s^*x^* - l_3x^*$, $j_{12} = -(1 + l_1)s^*$,

$j_{13} = -s^*(l_2s^* + l_3)$, $j_{21} = l_1i^*$, $j_{22} = 0, j_{23} = -l_4i^*$

$j_{31} = x^*(l_6 - l_8x^*)$, $j_{32} = l_7x^*$, $j_{33} = l_6s^* + l_7i^* - 2l_8s^*x^* - l_9$.

Then the characteristic equation of $J(E_4)$ is given by:

$$\lambda^3 + F_1\lambda^2 + F_2\lambda + F_3 = 0. \qquad (4.5b)$$

Where:

$F_1 = -(j_{11} + j_{33})$,

$F_2 = j_{11}j_{33} - j_{23}j_{32} - j_{12}j_{21} + j_{13}j_{31}$,

$F_3 = j_{32}(j_{11}j_{23} - j_{13}j_{21}) + j_{21}(j_{21}j_{33} - j_{23}j_{31})$.

Now by Routh Hurwitz criterion the roots of eq. $(4.5b)$, have negative real parts iff

$F_1 > 0$, $F_3 > 0$, and $\Delta = (F_1F_2 - F_3)F_3 > 0$.

$Now, F_i > 0, i = 1,3$, provided that

$$2s^* + (1 + l_1)i^* + 2l_2s^*x^* + l_3x^* > 1, \qquad (4.5c)$$

$$x^* < \frac{l_6}{l_8}, \qquad (4.5d)$$

$j_{23}j_{31} < j_{21}j_{33}$.

Further, it is easy to check that:

$\Delta = S_1 - S_2$, $where$

$$S_1 = -(j_{11} + j_{33})(j_{11}j_{33} - j_{23}j_{32} - j_{12}j_{21} + j_{13}j_{31}) - j_{21}(j_{21}j_{33} - j_{23}j_{31})$$

$$S_2 = -(j_{11} + j_{33})j_{13}j_{31} + j_{32}(j_{11}j_{23} - j_{13}j_{21})$$

Hence $\Delta > 0$, if in addition to conditions $(4.5c) and (4.5i)$, the following condition

$$S_1 > S_2. \tag{4.5e}$$

So, E_5 is locally stable, it is unstable otherwise.

V **GLOBAL STABILITY ANALYSIS**

In this section the global stability analysis for the dimensionless system (2.2) for the local stable points is studied:-

Theorem (5.1):
The point $E_1 = (1,0,0)$ is globally asymptotically stable provided that with the Basin of attraction of $Int.R^3_+$ that satisfies the next conditions:

$$s > 1, \tag{5.1a}$$
$$(s-1)^2 > l_1 i + l_3 x. \tag{5.1b}$$

Proof: Consider the following function
$L_1(s,i,x,y) = (s - 1 - \ln s) + i + x.$
It is easy to see that $L_1(s,i,x) \epsilon\, C^1(R^3_+, R)$, and $L_1(E_1) = 0$, and $L_1(s,i,x) > 0$;
$\forall (s,i,x) \neq E_1$, by differentiating L_1 with respect to time t and use the equations in the system , we get:-

$$\frac{dL_1}{dt} = -(s-1)^2 - (s-1)i - l_5 i - l_2 sx(s-1) - (l_3 - l_6)sx - (l_4 - l_7)ix + l_1 i + l_3 x.$$

Now, according to the natural facts and $condition$ $(5.1a)$ we get:

$$\frac{dL_1}{dt} < -(s-1)^2 + l_1 i + l_3 x.$$

So, $\frac{dL_1}{dt} < 0$ under condition (5.1b) . Hence E_1 is globally asymptotically stable.

Theorem (5.2) :
The point $E_2 = (\hat{s}, \hat{\imath}, 0)$ of system (2.2) is globally asymptotically stable with the Basin of attraction of $Int.R^3_+$ that satisfies the next conditions:

$$s > \hat{s}, \tag{5.2a}$$
$$i > \hat{\imath}, \tag{5.2b}$$
$$(s-\hat{s})^2 + l_2 sx(s-\hat{s}) > l_3 \hat{s} x + l_1 (s-\hat{s})(i-\hat{\imath}) + l_4 \hat{\imath}. \tag{5.2c}$$

Proof: Consider the following function

$$L_2(s,i,x) = \left(s - \hat{s} - \hat{s}\ln\frac{s}{\hat{s}}\right) + \left(i - \hat{\imath} - \hat{\imath}\ln\frac{i}{\hat{\imath}}\right) + x.$$

It is easy to see that $L_2(s,i,x) \epsilon\, C^1(R^3_+, R)$, and $L_2(E_2) = 0$, and $L_2(s,i,x) > 0$;
$\forall (s,i,x) \neq E_2$, by differentiating L_2 with respect to time t and use the equations in the system , we get:-

$$\frac{dL_2}{dt} < -(s-\hat{s})^2 - l_2 sx(s-\hat{s}) - (l_3 - l_6)sx - (l_4 - l_7)ix + l_3 \hat{s} x + l_1 (s-\hat{s})(i-\hat{\imath}) + l_4 \hat{\imath}.$$

Then $\frac{dL_2}{dt} < 0$ under the natural facts $and\ conds.(5.2a - 5.2c)$, and then E_2 is globally asymptotically stable.

Theorem (5.3) :

The predator-free equilibrium point $E_3 = (\bar{\bar{s}}, 0, \bar{\bar{x}})$ of system (2.2) is globally asymptotically stable with the Basin of attraction of $Int. R_+^3$ that satisfies the next conditions:

$$s > \bar{\bar{s}}, \tag{5.3a}$$
$$x > \bar{\bar{x}}, \tag{5.3b}$$
$$(1 + l_2)(s - \bar{\bar{s}})^2 + l_6\bar{\bar{s}}x > \bar{\bar{s}}(l_1 i + l_3 x) + l_3 s\bar{\bar{x}}. \tag{5.3c}$$

Proof: Consider the following function

$$L_3(s,i,x) = \left(s - \bar{\bar{s}} - \bar{\bar{s}}\, ln \frac{s}{\bar{\bar{s}}} \right) + i + \left(x - \bar{\bar{x}} - \bar{\bar{x}} ln \frac{i}{\bar{\bar{x}}} \right).$$

It is easy to see that $L_3(s,i,x) \epsilon\, C^1(R_+^3, R)$, and $L_3(E_2) = 0$, and $L_3(s,i,x) > 0$; $\forall (s,i,x) \neq E_3$. , by differentiating L_3 with respect to time t and use the equations in the system , we get:-

$\frac{dL_3}{dt} = -(1 + l_2)\,(s - \bar{\bar{s}})^2 - (l_2\bar{\bar{s}} + l_8 x)(s - \bar{\bar{s}})(x - \bar{\bar{x}}) - l_8 s(x - \bar{\bar{x}})^2 - (s - \bar{\bar{s}})i\ - (l_3 - l_6)sx - (l_4 - l_7)ix - (l_3 - l_6)\bar{\bar{s}}\bar{\bar{x}} - l_5 i - l_6(s\bar{\bar{x}} + \bar{\bar{s}}x) - l_7 i\bar{\bar{x}} + \bar{\bar{s}}(l_1 i + l_3 x) + l_3 s\bar{\bar{x}}.$

So, according to the natural facts and conditions (5.3a) and (5.3b) we get:

$\frac{dL_2}{dt} < -(1 + l_2)(s - \bar{\bar{s}})^2 - l_6\bar{\bar{s}}x + \bar{\bar{s}}(l_1 i + l_3 x) + l_3 s\bar{\bar{x}}$

Then $\frac{dL_3}{dt} < 0$ under condition (5.3c), and so, E_3 is globally asymptotically stable.

Theorem (5.4):

The positive equilibrium point $E_4 = (s^*, i^*, x^*)$ of system (2.2) is globally asymptotically stable with the Basin of attraction of $Int. R_+^3$ that satisfies the next conditions:

$$s > s^*, \tag{5.4a}$$
$$i > i^*, \tag{5.4b}$$
$$x < x^*, \tag{5.4c}$$
$$K > L, \tag{5.4d}$$

where

K=$(1 + l_1)(s - s^*)^2 + (s - s^*)(i - i^*) + l_3\, s^*\,(s - s^*)(x - x^*) + l_8 s^*(x^*)^2$

L=$l_1(si^* + s^*i) + l_3(sx^* + s^*x) + l_4(xi^* + x^*i) + l_8 xx^*(s + s^*)$.

Proof: Consider the following function

$$L_4(s,i,x) = \left(s - s^* - s^*\, ln \frac{s}{s^*} \right) + \left(i - i^* - i^*\, ln \frac{i}{i^*} \right) + \left(x - x^* - x^*\, ln \frac{x}{x^*} \right).$$

It is easy to see that $L_4(s,i,x) \epsilon\, C^1(R_+^3, R)$, and $L_4(E_4) = 0$, and $L_4(s,i,x) > 0\,; \forall (s,i,x) \neq E_4$, by differentiating L_4 with respect to time t and use the equations in the system , we get:-

$$\begin{aligned}\frac{dL_4}{dt} &< -(1 + l_1)(s - s^*)^2 - (s - s^*)(i - i^*) - l_3\, s^*\,(s - s^*)(x - x^*) - l_8 s^*(x^*)^2 - (l_3 - l_6)sx \\ &\quad - (l_4 - l_7)ix - (l_3 - l_6)s^*x^* - (l_4 - l_7)i^*x^* + l_1(si^* + s^*i) + l_3(sx^* + s^*x) \\ &\quad + l_4(xi^* + x^*i) + l_8 xx^*(s + s^*). \\ &= -K + L.\end{aligned}$$

Then $\frac{dL_4}{dt}$ is negative definite under conditions $(5.4a) - (5.4c)$, with the natural facts mentioned in theorem (2.1). Hence E_4 is a globally asymptotically stable ■

VI NUMERICAL SIMULATION :

In this section, The behaviour of the system (2) is studied numerically to confirm the above analytic result. The phase portrait and the time series of the solutions of the system (6.1) are drawn for the following set of parameter

$$\left.\begin{array}{c} l_1 = 0.2\,, l_2 = 0.5\,, l_3 = 0.4\,, l_4 = 0.4\,, l_5 = 0.001\,, l_6 = 0.3\,, l_7 = 0.3 \\ l_8 = 1, l_9 = 0.001\,, \end{array}\right\} \quad (6.1)$$

and with the initial point (2.5,1.5,1.5).

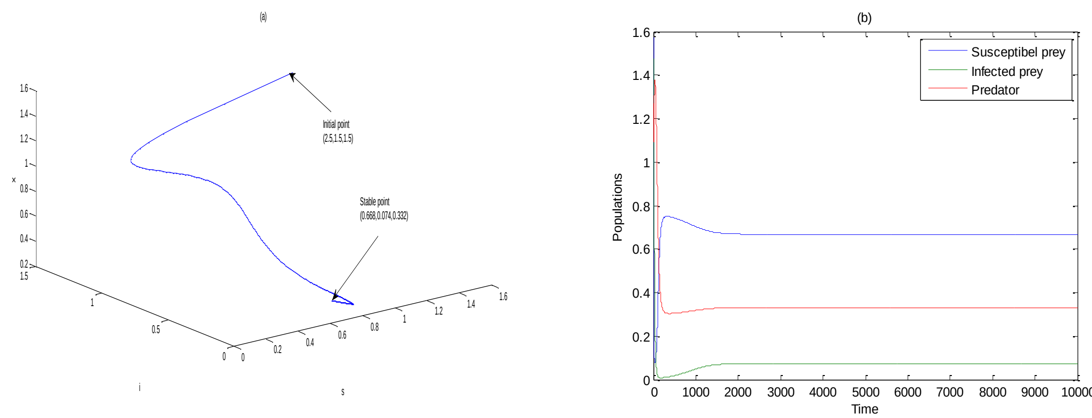

Fig. 2 :- (a) The phase portrait of system (2.2) using data (6.1) started from initial point (2.5,1.5,1.5) which approaches to $E_4 = (0.668,0.074,0.332)$ (b) The time series of the attractor in fig. (6.1a).

Fig. (2) shows that the conditions which have been mentioned in Theorem (5,4) are satisfied, and the positive fixed point exists and it is specified as $E_4 = (0.668,0.074,0.332)$.

Now, For the set that given in Eq. (6.1) and various one factor each time, the proposed model is solved numerically, and the results are summarized in the following table:

Table 2: Numerical behaviour of the system (2.2) for the data assumed in (6.1) with changing one parameter at each time.

Range of parameter	Numerical behaviour of the system (2)	The bifurcation point
$0.1 < l_1 \leq 0.159$ $0.159 < l_1 < 1.5$	The solution converges to E_3. The solution converges to E_4.	$l_1 = 0.159$
$0.1 < l_2 \leq 0.372$ $0.372 < l_2 \leq 2$	The solution converges to E_4. The solution converges to E_3.	$l_2 = 0.372$
$0.4 < l_3 \leq 0.678$ $0.678 < l_3 \leq 0.8$	The solution converges to E_4. The solution converges to E_3.	$l_3 = 0.678$
$0.4 < l_4 \leq 0.5024$ $0.5024 < l_4 < 0.8$	The solution converges to E_4. The solution converges to E_3.	$l_4 = 0.5024$
$0.001 < l_5 \leq 0.031$	The solution converges to E_4.	

$0.031 < l_5 \leq 1$	The solution converges to E_3.	$l_5 = 0.031$
$0.1 \leq l_6 \leq 0.3$	The solution converges to E_4.	
$0.1 \leq l_7 \leq 0.3$	The solution converges to E_4.	
$0.12 < l_8 \leq 0.8399$ $0.8399 < l_8 \leq 2$	The solution converges to E_3 The solution converges to E_4.	$l_8 = 0.8399$
$0.001 \leq l_9 \leq 0.250$	The solution converges to E_4	

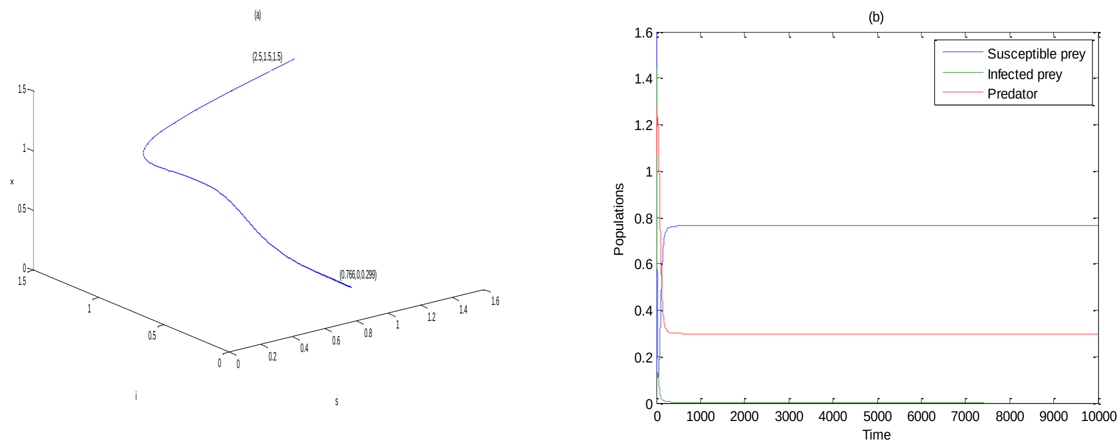

Fig. (3):- (a) The phase portrait of system (2.2) using data (6.1) started from the initial point (2.5,1.5,1.5) which approaches to $E_3 = (0.766,0,0.299)$ (b) The time series of the attractor in fig. (6.2a). for typical value $l_1 = 0.1$.

Moreover, if the parameters $l_1, l_5, l_6, and\ l_9$ are varying to $l_1 = 0.001, l_5 = 0.01, l_6 = 0.1$ and $l_9 = 0.2$ in Eq. (6.1), then the conditions which have been mentioned in Theorem (5.1) are satisfied, and the axial equilibrium point exists, and it is given as $E_1 = (1,0,0)$. See Fig (3).

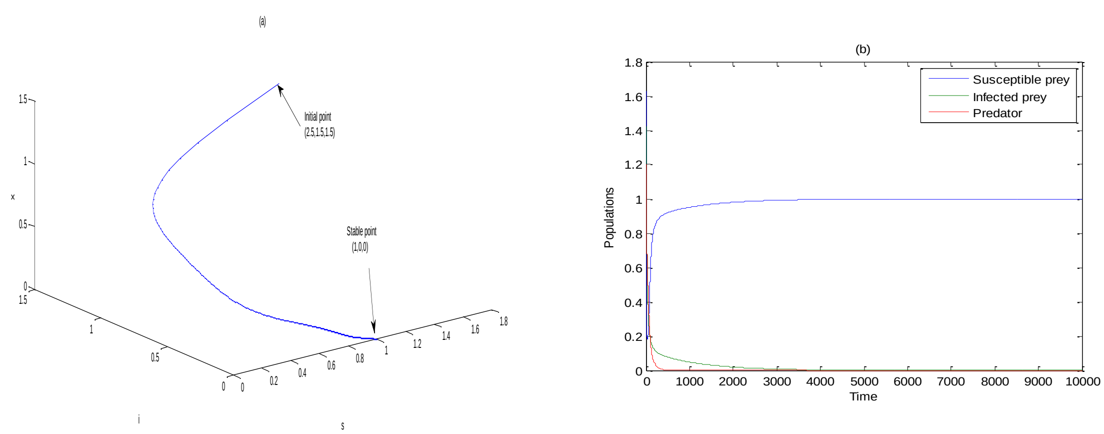

Fig. (4):- (a) The phase portrait of system (2.2) using data (6.1) started from the initial point (2.5,1.5,1.5) which approaches to $E_1 = (1,0,0)$. (b) The time series of the attractor in fig. (4a).

Finally, for the following parameters

$$\left.\begin{matrix} l_1 = 0.01\,, l_2 = 0.1\,, l_3 = 0.3\,, l_4 = 0.3\,, l_5 = 0.001\,, l_6 = 0.1\,, l_7 = 0.1 \\ l_8 = 1, l_9 = 0.5\,, \end{matrix}\right\} \quad (6.2)$$

the conditions which have been mentioned in Theorem (5.2) are satisfied, and the free predator equilibrium point exists, and it is given as $E_2 = (\hat{s}, \hat{\imath}, 0)$. See Fig (5).

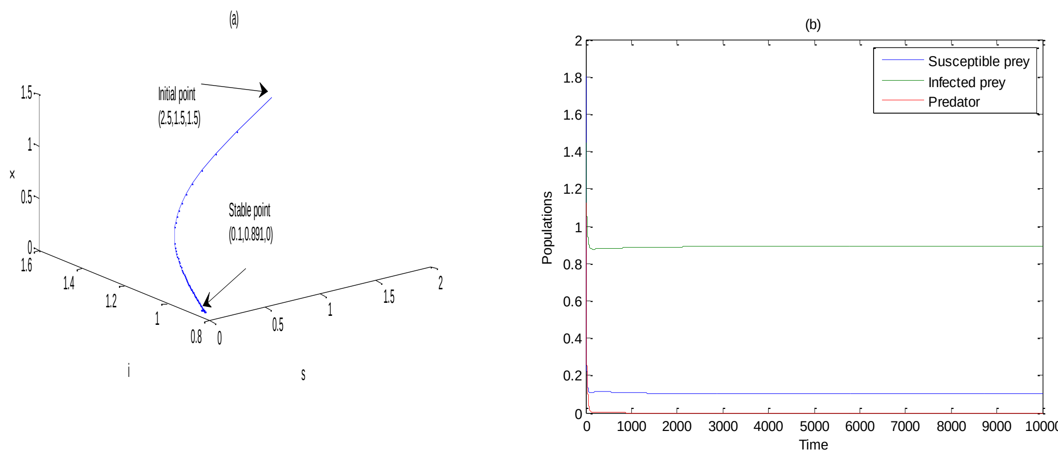

Fig. (5):- (a) The phase portrait of system (2.2) using data (6.2) started from initial point (2.5,1.5,1.5) which approaches to $E_2 = (0.1,0.891,0)$. (b) The time series of the attractor in fig.(5a).

VII CONCLUSION

A mathematical model of three first-order non-linear differential equations has been proposed to study the effect of the toxic substance released by the prey on the predator and vice versa in the presence of an incurable disease in the prey community. We assume that the disease does not transfer to the predator society, and the attack function has been chosen from a Lotka-Volterra type. The conditions for the existence, local and global stability of steady points are obtained analytically. For the set that given in eq. (6.1) and various one factor each time, the proposed model has been solved numerically, and the following results have been summarized as follows:

1- There is no periodic solution of system (2.2) in Int. R_+^3

2- The trajectory of the system (2.2) approaches asymptotically to the globally stable positive point $E_4 = (0.668,0.074,0.332)$.

3- The existence of the positive point relies solely on the $l_1, l_2, l_3\ l_4, l_5$ and l_8. In particular, if $0.1 < l_1 \leq 0.159$ system (2.2) settle down to the disease-free equilibrium point

4- It is detected that the behaviour of the system (2.2) does not change if l_6, l_7 and l_9 is varied.

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