M-polynomial of the total-block graph of some chain graphs

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Abstract : For any graph G, its M-polynomial is defined as $M(G;x,y) = \sum_{i\leq j} M_{ij}(G)x^i y^j$ where $M_{ij}, i,j \geq 1$ is the number of edges uv of G such that $\{d_G(u), d_G(v)\} = \{i,j\}$. Physico-chemical properties of chemical graphs can be determined by topological indices. The degree based topological indices can be easily driven from M-polynomial. In this paper, we obtain the expressions for M-polynomial of the total-block graphs of some chain graphs. Also, we derive some degree based topological indices from the obtained polynomials.

2010 Mathematics Subject Classification: 05C07,05C31.

Keywords: graph transformations, total-block graph, topological indices, M-polynomial.

1. **Introduction :** It is well known that graphs provide a natural representation of the structure of covalently bonded molecules. The most common way to represent a molecular topology is in which each atom is replaced by a vertex and each chemical bond by an edge between the respective two vertices. If G is a graph, and if it can be transformed in some way into another graph G', so that the correspondence between G and G' is one-to-one, then the transformation $G \to G'$ preserves the entire information of G onto G'. In the chemical literature, there have been few earlier attempts to shift from ordinary molecular graphs to their transforms [16,17,18,31]. It is interesting to find some properties of graphs which are invariant. Topological indices and polynomials are foremost among them. Over the last decade there are numerous research papers devoted to topological indices and polynomials. Several topological indices have been defined in the literature. For details of topological indices one can refer to [19, 25]. For different topological indices and their applications one can refer to [3,4,6,11,12,13]. There are many graph polynomials [8, 31]. The M-polynomial [9] is one among other algebraic polynomials which was introduced in 2015 and useful in determining many degree-based topological indices. Recently, the study of M-polynomial is reported in [5,7,26,27,28].

2. **Notation and definitions**: Let G be a simple graph with vertex set $V(G)$ and edge set $E(G)$. The degree of a vertex $v \in V(G)$ is the number of edges incident with v in G and is denoted by $d_v(G)$. The chain graph $C = C(G_1, G_2, \dots, G_d; v_1w_1, v_2w_2, \dots, v_dw_d)$ of $\{G_i\}_{i=1}^{d}$ with respect to the vertices $\{v_i, w_i\}_{i=1}^{d}$ is the graph obtained from the graphs $G_1, G_2, \dots, G_d$ by identifying the vertices w_i and v_{i+1} for all $i = 1,2,\dots,d-1$, as shown in Fig 1.

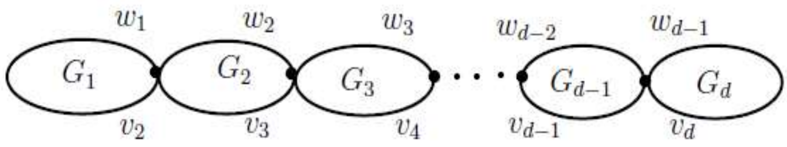

Fig. 1. The chain graph $C = C(G_1, G_2, ..., G_d; v_1w_1, v_2w_2, ..., v_dw_d)$

A path is a graph having alternating sequence of vertices and edges $v_0, e_1, v_1, e_2, \dots, v_{n-1}, e_n$, where all the vertices and edges are distinct and is denoted by $P_n, n \geq 2$. Hence, we can say that a path P_n is a chain graph where each G_i is an edge. A block of a graph G is a maximal nonseparable subgraph of G. A complete graph K_n is a $n-1$ regular graph with n vertices. A closed path is called a cycle and a cycle with n vertices is denoted by $C_n, n \geq 3$. The total-block graph $T_B(G)$ [22] of a graph G is a graph whose set of vertices is the union of the set of vertices and set of blocks of G and in which two vertices are adjacent if and only if the corresponding members of G are adjacent or incident. Many papers have been devoted to total-block graphs [1, 21, 22, 23]. The block-transformation graph $G^{\alpha\beta\gamma}$ was introduced in [2] where α, β, γ take values 0 or 1. By the definition of block-transformation graphs we can easily see that for any graph G, $T_B(G) = \mathrm{G}^{111}$. For undefined notations and terminology we refer [20]. Let $C^{'} = C^{'}(G_1, G_2, \dots, ; v_1w_1, v_2w_2, \dots, v_kw_k)$ be a chain graph where each $\{G_i\}_{i=1}^{k}$, k≥2, is a cycle $C_n, n \geq 3$. Let $C^{''} = C^{''}(G_1, G_2, \dots, G_k; v_1w_1, v_2w_2, \dots, v_kw_k)$ be a chain graph where each $\{G_i\}_{i=1}^{k}$, k≥2, is a complete graph $K_n, n \geq 4$.
In Fig 2, graphs P_5, $C^{'}, C^{''}$and their total-block graphs $T_B(P_5)$, $T_B(C^{'})$, $T_B(C^{''})$ for $n = 5$ and $k = 4$ have been shown. Our aim in this paper will be to find the M-polynomial and some of the degree based topological indices of $T_B(P_n)$, $T_B(C^{'}), T_B(C^{''})\ for\ k \geq 2$.

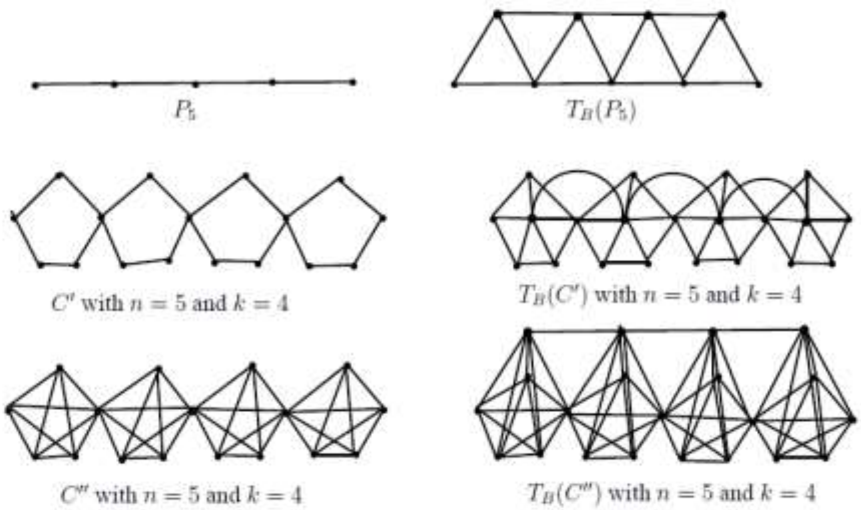

Fig. 2.

Many topological indices have been defined in the literature. Among them some standard topological indices are first Zagreb index [14], second Zagreb index [15], modified second Zagreb index [9], Randic′ index [29] harmonic index [10], symmetric division index [9] and inverse sum index [9]. The general form of these degree-based topological indices of a graph is given by $TI(G) = \sum_{e=uv\in E(G)} f(d_G(u), d_G(v))$ where $f = f(x, y)$ is a function appropriately chosen for the computation. Table 1 gives the standard topological indices defined by $f(x, y)$. All these topological indices can be obtained from a single expression called polynomials. For any graph G, its M-polynomial is defined as $M(G; x, y) = \sum_{i\leq j} M_{ij}\,(G)x^i y^j$ where $M_{ij}, i, j \geq 1$ is the number of edges uv of G such that $\{d_G(u), d_G(v)\} = \{i, j\}$. M-polynomial [9] is useful in determining many degree-based topological indices listed in Table 1. This motivates us to study M-polynomial of some graph operations on some chain graphs.

Table 1. [9] Operators to derive degree-based topological indices from M-polynomial.

Notation	Topological Index	$f(x,y)$	Derivation from $M(G;x,y)$
$M_1(G)$	First Zagreb	$x+y$	$(D_x + D_y)(M(G;x,y))\|_{x=y=1}$
$M_2(G$	Second Zagreb	xy	$(D_x D_y)(M(G;x,y))\|_{x=y=1}$
$M_2^m(G)$	Second modified Zagreb	$\frac{1}{xy}$	$(S_x S_y)(M(G;x,y))\|_{x=y=1}$
$S_D(G)$	Symmetric division	$\frac{x^2+y^2}{xy}$	$(D_x S_y + D_y S_x)(M(G;x,y))\|_{x=y=1}$
$H(G)$	Harmonic	$\frac{2}{x+y}$	$2S_x J M(G;x,y)\|_{x=1}$
$I_n(G)$	Inverse sum	$\frac{xy}{x+y}$	$S_x J D_x D_y(M(G;x,y))\|_{x=1}$

where $D_x = x\frac{\partial f(x,y)}{\partial x}$, $D_y = y\frac{\partial f(x,y)}{\partial y}$, $S_x = \int_0^x \frac{f(t,y)}{t}dt$, $S_y = \int_0^y \frac{f(x,t)}{t}dt$ and $J(f(x,y)) = f(x,x)$ are the operators.

3. M-polynomial of the total-block graph of some chain graphs

In this section, we find the M-polynomial of three graphs $T_B(P_n), T_B(C^{'}),\ T_B(C^{''})\ for\ k \geq 2$ and we also derive some topological indices (mentioned in Table 1) of these graphs from their respective M-polynomials.

Theorem 3A[22] If G is a connected graph with p vertices and q edges and if b_i is the number of blocks to which v_i belongs in G, then the total-block graph $T_B(G)$ has $\sum_{i=1}^{p} b_i + 1\ vertices\ and\ \sum_{i=1}^{p}\binom{b_i+1}{2}$ edges.

Proposition 3A[1] For a vertex v of a graph G, $d_{T_B(G)}(v) = d_G(v) + r$ where r is the number of blocks containing v.

Proposition 3B[1] If b is a block of a graph G and it has n vertices, then $d_{T_B(G)}(b) = \sum_{i=1}^{n} r_i$ where r_i is the number of blocks containing v_i and v_i is in b.

Lemma 3.1 For graph P_3, the M-polynomial of $T_B(P_3) is$

$M(T_B(P_3);x,y) = 2x^2y^3 + 2x^2y^4 + 2x^3y^4 + x^3y^3$ and $M_1(T_B(P_3)) = 42, M_2(T_B(P_3)) = 61, M_2^m(T_B(P_3)) = \frac{31}{36}, S_D(T_B(P_3)) = \frac{31}{2}, H(T_B(P_3)) = \frac{249}{105}$, and $I_n(T_B(P_3)) = \frac{2099}{210}$.

Proof. From the definition of the total-block graph of a graph G we see that in $T_B(P_3)$ we have $|E_{\{2,3\}}| = 2, |E_{\{2,4\}}| = 2, |E_{\{3,4\}}| = 2$ *and* $|E_{\{3,3\}}| = 1$, and from the derivation of $M(T_B(P_3);x,y)$ given in Table 1, we get $M_1(T_B(P_3)) = 42, M_2(T_B(P_3)) = 61, M_2^m(T_B(P_3)) = \frac{31}{36}, S_D(T_B(P_3)) = \frac{31}{2}, H(T_B(P_3)) = \frac{249}{105}$, and $I_n(T_B(P_3)) = \frac{2099}{210}$, hence the proof.

Theorem 3.1 For any path $P_n, n \geq 4$, the M-polynomial of $T_B(P_n)$ *is*

$$M(T_B(P_n);x,y) = 2x^2y^3 + 2x^2y^4 + 4x^3y^4 + (4n-13)x^4y^4.$$

Proof. We know that a path $P_n, n \geq 4$ has n vertices and $n - 1$ edges, also from Theorem 3A, Proposition 3A and Proposition 3B, we see that $T_B(P_n)$ has $2n - 1$ vertices and $4n - 1$ edges with 2 vertices of degree 2, 2 vertices of degree 3 and $2n - 5$ vertices with degree 4 also $|E_{\{2,3\}}| = 2, |E_{\{2,4\}}| = 2, |E_{\{3,4\}}| = 4$ *and* $|E_{\{4,4\}}| = 4n - 13$, hence the proof.

Corollary 3.1 For any path $P_n, n \geq 4$,

1. $M_1(T_B(P_n)) = 32n - 54$.
2. $M_2(T_B(P_n)) = 64n - 132$.
3. $M_2^m(T_B(P_n)) = \frac{11}{12} + \frac{4n-13}{16}$.
4. $S_D(T_B(P_n)) = \frac{53}{3} + 2(4n - 13)$.
5. $H(T_B(P_n)) = \frac{274}{105} + \frac{4n-13}{4}$.
6. $I_n(T_B(P_n)) = \frac{1252}{105} + 2(4n - 13)$.

Proof. Since the M-polynomial of $T_B(P_n)$ is $M(T_B(P_n);x,y) = 2x^2y^3 + 2x^2y^4 + 4x^3y^4 + (4n-13)x^4y^4$ and from the expressions given in Table 1 we have,

$$D_x = x\frac{\partial f(x,y)}{\partial x} = 4x^2y^3 + 4x^2y^4 + 12x^3y^4 + 4(4n-13)x^4y^4$$
$$D_y = y\frac{\partial f(x,y)}{\partial y} = 6x^2y^3 + 8x^2y^4 + 16x^3y^4 + 4(4n-13)x^4y^4$$
$$S_x = \int_0^x \frac{f(t,y)}{t}dt = x^2y^3 + x^2y^4 + \frac{4}{3}x^3y^4 + \frac{(4n-13)}{4}x^4y^4$$
$$S_y = \int_0^y \frac{f(x,t)}{t}dt = \frac{2}{3}x^2y^3 + \frac{1}{2}x^2y^4 + x^3y^4 + \frac{(4n-13)}{4}x^4y^4$$

$D_x|_{x=y=1} = 16n - 32.$
$D_y|_{x=y=1} = 16n - 22$
Therefore, $M_1(T_B(P_n)) = (D_x + D_y)(M(T_B(P_n);x,y))|_{x=y=1}$=$32n - 54$.

Now, $D_xD_y = 12x^2y^3 + 16x^2y^4 + 48x^3y^4 + 16(4n-13)x^4y^4$.

Therefore, $M_2(T_B(P_n)) = (D_xD_y)(M(T_B(P_n);x,y))\big|_{x=y=1} = 64n - 132$.

$$S_xS_y = \frac{1}{3}x^2y^3 + \frac{1}{4}x^2y^4 + \frac{1}{3}x^3y^4 + \frac{(4n-13)}{16}x^4y^4$$

Therefore, $M_2^m(T_B(P_n)) = (S_xS_y)(M(T_B(P_n);x,y))\big|_{x=y=1} = \frac{11}{12} + \frac{4n-13}{16}$

$$D_xS_y = \frac{4}{3}x^2y^3 + x^2y^4 + 3x^3y^4 + (4n-13)x^4y^4$$

$$D_yS_x = 3x^2y^3 + 4x^2y^4 + \frac{16}{3}x^3y^4 + (4n-13)x^4y^4$$

Therefore, $S_D(T_B(P_n)) = (D_xS_y + D_yS_x)(M(T_B(P_n);x,y))\big|_{x=y=1} = \frac{53}{3} + 2(4n-13)$

$$S_xJM(P_n;x,y) = \frac{2}{5}x^5 + \frac{2}{6}x^6 + \frac{4}{7}x^7 + \frac{4n-13}{8}x^8$$

Therefore,

$H(T_B(P_n)) = 2S_xJM(P_n;x,y)|_{x=1} = \dfrac{274}{105} + \dfrac{4n-13}{4}$

$I_n(T_B(P_n)) = S_xJD_xD_y(M(P_n;x,y))|_{x=1} = \frac{1252}{105} + 2(4n-13)$.

Lemma 3.2 For any chain graph $C' = C'(G_1, G_2; v_1w_1, v_2w_2)$ where each $\{G_i\}_{i=1}^{2}$, is a cycle $C_n, n \geq 3$, $M(T_B(C');x,y) = 8x^3y^3 + 4x^3y^n + 10x^3y^{n+1} + 2x^ny^{n+1} + x^{n+1}y^{n+1}$ and $M_1(T_B(C')) = 2(n+1)(n+10)$, $M_2(T_B(C')) = 3n^2 + 46n + 103$, $M_2^m(T_B(C')) = \frac{8}{9} + \frac{4}{3n} + \frac{10}{3(n+1)} + \frac{2}{n(n+1)}$, $S_D(T_B(C')) = 18 + \frac{4(n^2+9)}{3n} + \frac{10(n^2+2n+10)}{3(n+1)} + \frac{2(2n^2+2n+1)}{n(n+1)}$, $H(T_B(C')) = \frac{8}{3} + \frac{8}{n+3} + \frac{20}{n+4} + \frac{4}{2n+1} + \frac{1}{n+1}$ and $I_n(T_B(C')) = \frac{n+25}{2} + \frac{12n}{n+3} + \frac{30(n+1)}{n+4} + \frac{2n(n+1)}{2n+1}$.

Proof. From the definition of the total-block graph of a graph G we see that in $T_B(C')$ we have $|E_{\{3,3\}}| = 8, |E_{\{3,n\}}| = 4, |E_{\{3,n+1\}}| = 10, |E_{\{n,n+1\}}| = 2, |E_{\{n+1,n+1\}}| = 1$, and from the derivation of $M(T_B(C');x,y)$ given in Table 1, we get $M_1(T_B(C')) = 2(n+1)(n+10)$, $M_2(T_B(C')) = 3n^2 + 46n + 103$, $M_2^m(T_B(C')) = \frac{8}{9} + \frac{4}{3n} + \frac{10}{3(n+1)} + \frac{2}{n(n+1)}$, $S_D(T_B(C')) = 18 + \frac{4(n^2+9)}{3n} + \frac{10(n^2+2n+10)}{3(n+1)} + \frac{2(2n^2+2n+1)}{n(n+1)}$, $H(T_B(C')) = \frac{8}{3} + \frac{8}{n+3} + \frac{20}{n+4} + \frac{4}{2n+1} + \frac{1}{n+1}$ and $I_n(T_B(C')) = \frac{n+25}{2} + \frac{12n}{n+3} + \frac{30(n+1)}{n+4} + \frac{2n(n+1)}{2n+1}$, hence the proof.

Theorem 3.2 For any chain graph $C' = C'(G_1, G_2, \ldots, G_k; v_1w_1, v_2w_2, \ldots, v_pw_p)$ where each $\{G_i\}_{i=1}^{p}$, p≥ 3, is a cycle $C_n, n \geq 3$, $M(T_B(C');x,y) = 2x^{n+1}y^{n+2} + (p-3)x^{n+2}y^{n+2} + 4(p-1)x^3y^6 + 2(n-1)x^3y^{n+1} + (n-2)(p-2)x^3y^{n+2} + (np-4p+4)x^3y^3 + 2x^6y^{n+1} + 2(p-2)x^6y^{n+2}$.

Proof. We know that a cycle $C_n, n \geq 3$ has n vertices and n edges. From Theorem 3A, Proposition 3A and Proposition 3B, we see that T_BC' has $np - 2p + 2$ vertices of degree 3, $(p-1)$ vertices of degree 6, 2 vertices of degree $n+1$ and $p-2$ vertices of degree $(n+2)$

also $|E_{\{3,3\}}| = np - 4p + 4, |E_{\{3,6\}}| = 4(p-1), |E_{\{3,n+1\}}| = 2(n-1), |E_{\{3,n+2\}}| = (n-2)(p-2), |E_{\{6,n+1\}}| = 2, |E_{\{6,n+2\}}| = 2(p-2)\ |, E_{\{n+1,n+2\}}| = 2\ and\ |E_{\{n+2,n+2\}}| = p-3,$ hence the proof.

Corollary 3.2 For any chain graph $C' = C'(G_1, G_2, \dots, G_k; v_1w_1, v_2w_2, \dots, v_kw_k)$ where each $\{G_i\}_{i=1}^{k}$, $k \geq 3$, is a cycle $C_n, n \geq 3$,

1. $M_1(T_B(C')) = n^2p + 13np - 4n + 22p - 24.$
2. $M_2(T_B(C')) = 4n^2p - n^2 - 18n + 25np + 52p - 62.$
3. $M_2^m(T_B(C')) = \frac{np-2n-p+2}{3(n+2)} + \frac{2n-1}{3(n+1)} + \frac{np-2p+2}{9} + \frac{2}{(n+1)(n+2)} + \frac{p-3}{(n+2)^2}.$
4. $S_D\big(T_B(C')\big) = 2np + 4p - 8 + \frac{3np-4n+6p-10}{n+2} + \frac{2(4n+5)}{n+1} + \frac{n^2p+np-n-2p+3}{3}.$
5. $H(T_B(C')) = 2\left[\frac{2np+3p-2n-1}{(2n+3)(2n+4)} + \frac{3np-4p+4}{18} + \frac{2n^2+14n-6}{(n+4)(n+7)} + \frac{(p-2)(n^2+8n-6)}{(n+5)(n+8)}\right]$
6. $I_n\big(T_B(C')\big) = 2np + 3p - \frac{3}{2}n - 5 + (n+1)\left(\frac{2(n+2)}{2n+3} + \frac{6(n-1)}{n+4} + \frac{12}{n+7}\right) + \frac{3(n+2)(p-2)(n^2+10n+4)}{(n+5)(n+8)}.$

Proof. We have $M(T_B(C'); x, y) = 2x^{n+1}y^{n+2} + (p-3)x^{n+2}y^{n+2} + 4(p-1)x^3y^6 + 2(n-1)x^3y^{n+1} + (n-2)(p-2)x^3y^{n+2} + (np-4p+4)x^3y^3 + 2x^6y^{n+1} + 2(p-2)x^6y^{n+2}$

$$D_x = 2(n+1)x^{n+1}y^{n+2} + (n+2)(p-3)x^{n+2}y^{n+2} + 12(p-1)x^3y^6 + 6(n-1)x^3y^{n+1} + 3(n-2)(p-2)x^3y^{n+2} + 3(np-4p+4)x^3y^3 + 12x^6y^{n+1} + 2(p-2)x^6y^{n+2}$$

$$D_y = 2(n+2)x^{n+1}y^{n+2} + (p-3)(n+2)x^{n+2}y^{n+2} + 24(p-1)x^3y^6 + 2(n+1)(n-1)x^3y^{n+1} + (n^2-4)(p-2)x^3y^{n+2} + 3(np-4p+4)x^3y^3 + 2(n+1)x^6y^{n+1} + 2(p-2)(n+2)x^6y^{n+2}$$

$$S_x = \frac{2}{n+1}x^{n+1}y^{n+2} + \frac{(p-3)}{n+2}x^{n+2}y^{n+2} + \frac{4(p-1)}{3}x^3y^6 + \frac{2(n-1)}{3}x^3y^{n+1} + \frac{(n-2)(p-2)}{3}x^3y^{n+2} + \frac{(np-4p+4)}{3}x^3y^3 + \frac{1}{3}x^6y^{n+1} + \frac{(p-2)}{3}x^6y^{n+2}$$

$$S_y = \frac{2}{n+2}x^{n+1}y^{n+2} + \frac{(p-3)}{n+2}x^{n+2}y^{n+2} + \frac{2(p-1)}{3}x^3y^6 + \frac{2(n-1)}{n+1}x^3y^{n+1} + \frac{(n-2)(p-2)}{n+2}x^3y^{n+2} + \frac{(np-4p+4)}{3}x^3y^3 + \frac{2}{n+1}x^6y^{n+1} + \frac{2(p-2)}{n+2}x^6y^{n+2}$$

$$D_x|_{x=y=1} = 7np - n + 8p - 10$$

$$D_y\Big|_{x=y=1} = 6np - 3n + 14p - 14 + n^2p$$

Therefore, $M_1\big(T_B(C')\big) = \big(D_x + D_y\big)\big(M(C';x,y)\big)\Big|_{x=y=1}$=$n^2p + 13p - 4n + 22p - 24$.

$$D_xD_y = 2(n+2)(n+1)x^{n+1}y^{n+2} + (p-3)(n+2)^2x^{n+2}y^{n+2} + 72(p-1)x^3y^6 + 6(n^2-1)x^3y^{n+1} + 3(n^2-4)(p-2)x^3y^{n+2} + 9(np-4p+4)x^3y^3 + 12(n+1)x^6y^{n+1} + 12(p-2)(n+2)x^6y^{n+2}$$

Therefore, $M_2(T_B(C')) = \big(D_xD_y\big)\big(M(C';x,y)\big)\Big|_{x=y=1}$=$4n^2p + 25np + 52p - 18n - n^2 - 62$

$$S_xS_y = \frac{2}{(n+2)(n+1)}x^{n+1}y^{n+2} + \frac{(p-3)}{(n+2)^2}x^{n+2}y^{n+2} + \frac{2(p-1)}{9}x^3y^6 + \frac{2(n-1)}{3(n+1)}x^3y^{n+1} + \frac{(n-2)(p-2)}{3(n+2)}x^3y^{n+2} + \frac{(np-4p+4)}{9}x^3y^3 + \frac{1}{3(n+1)}x^6y^{n+1} + \frac{(p-2)}{3(n+2)}x^6y^{n+2}$$

Therefore,
$M_2^m\big(T_B(C')\big) = \big(S_xS_y\big)\big(M(C';x,y)\big)\Big|_{x=y=1}$=$\frac{np-2n-p+2}{3(n+2)} + \frac{2n-1}{3(n+1)} + \frac{np-2p+2}{9} + \frac{2}{(n+1)(n+2)} + \frac{p-3}{(n+2)^2}$

$$D_xS_y = \frac{2(n+1)}{n+2}x^{n+1}y^{n+2} + (p-3)x^{n+2}y^{n+2} + 2(p-1)x^3y^6 + \frac{6(n-1)}{n+1}x^3y^{n+1} + \frac{3(n-2)(p-2)}{n+2}x^3y^{n+2} + (np-4p+4)x^3y^3 + \frac{12}{n+1}x^6y^{n+1} + \frac{12(p-2)}{n+2}x^6y^{n+2}$$

$$D_yS_x = \frac{2(n+2)}{n+1}x^{n+1}y^{n+2} + (p-3)x^{n+2}y^{n+2} + 8(p-1)x^3y^6 + \frac{2(n^2-1)}{3}x^3y^{n+1} + \frac{(n^2-4)(p-2)}{3}x^3y^{n+2} + (np-4p+4)x^3y^3 + \frac{n+1}{3}x^6y^{n+1} + \frac{(p-2)(n+2)}{3}x^6y^{n+2}$$

Therefore, $S_D\big(T_B(C')\big) = \big(D_xS_y + D_yS_x\big)\big(M(C';x,y)\big)\big|_{x=y=1} = 2np + 4p - 8 + \frac{3np-4n+6p-10}{n+2} + \frac{2(4n+5)}{n+1} + \frac{n^2p+np-n-2p+3}{3}$

$$H(T_B(C')) = 2S_xJM(C';x,y)|_{x=1} = 2\left[\frac{2np+3p-2n-1}{(2n+3)(2n+4)} + \frac{3np-4p+4}{18} + \frac{2n^2+14n-6}{(n+4)(n+7)} + \frac{(p-2)(n^2+8n-6)}{(n+5)(n+8)}\right]$$

$$I_n\big(T_B(C')\big) = S_xJD_xD_y(M(C';x,y))|_{x=1}2np + 3p - \frac{3}{2}n - 5 + (n+1)\left(\frac{2(n+2)}{2n+3} + \frac{6(n-1)}{n+4} + \frac{12}{n+7}\right) + \frac{3(n+2)(p-2)(n^2+10n+4)}{(n+5)(n+8)}$$

Lemma 3.3 For any chain graph $C'' = C''(G_1, G_2; v_1w_1, v_2w_2)$ where each $\{G_i\}_{i=1}^{2}$, is a complete graph $K_n, n \geq 4$, $M\big(T_B(C''); x, y\big) = (n-1)(n-2)x^ny^n + 2(n-1)x^ny^{n+1} + 2(n-1)x^ny^{2n} + x^{n+1}y^{n+1} + 2x^{n+1}y^{2n}$ and $M_1\big(T_B(C'')\big) = 2(n^3+2n^2+2n+1)$, $M_2\left(T_B(C'')\right) = n^4+3n^3+3n^2+4n+1$, $M_2^m\big(T_B(C'')\big) = \frac{(n-1)^2}{n^2} + \frac{2n-1}{n(n+1)} + \frac{1}{(n+1)^2}$, $S_D\big(T_B(C'')\big) = 2n^2 - n + 1 + \frac{4n^3+5n^2-1}{n(n+1)}$, $H\big(T_B(C'')\big) = \frac{(n-1)(3n-2)}{3n} + \frac{4(n-1)}{2n+1} + \frac{1}{n+1} + \frac{4}{3n+1}$ and $I_n(T_B(C'')) = \frac{3n^3-n^2+n+3}{6} + \frac{2n(n^2-1)}{2n+1} + \frac{4n(n+1)}{3n+1}$.

Proof. From the definition of the total-block graph of a graph G we see that in $T_B(C'')$ we have $|E_{\{n,n\}}| = (n-1)(n-2)$, $|E_{\{n,n+1\}}| = 2(n-1)$, $|E_{\{n,2n\}}| = 2(n-1)$, $|E_{\{n+1,n+1\}}| = 1$, $|E_{\{n+1,2n\}}| = 2$, and from the derivation of $M(T_B(C''); x, y)$ given in Table 1, we get $M_1\big(T_B(C'')\big) = 2(n^3+2n^2+2n+1)$, $M_2\left(T_B(C'')\right) = n^4+3n^3+3n^2+4n+1$, $M_2^m\big(T_B(C'')\big) = \frac{(n-1)^2}{n^2} + \frac{2n-1}{n(n+1)} + \frac{1}{(n+1)^2}$, $S_D\big(T_B(C'')\big) = 2n^2 - n + 1 + \frac{4n^3+5n^2-1}{n(n+1)}$, $H\big(T_B(C'')\big) = \frac{(n-1)(3n-2)}{3n} + \frac{4(n-1)}{2n+1} + \frac{1}{n+1} + \frac{4}{3n+1}$ and $I_n(T_B(C'')) = \frac{3n^3-n^2+n+3}{6} + \frac{2n(n^2-1)}{2n+1} + \frac{4n(n+1)}{3n+1}$, hence the proof.

Theorem 3.3 For any chain graph $C'' = C''(G_1, G_2, \dots, G_k: v_1w_1, v_2w_2, \dots, v_kw_k)$ where each $\{G_i\}_{i=1}^{k}$, $k \geq 3$, is a complete graph $K_n, n \geq 4$, the M-polynomial of $T_B(C'')$ is

$M(C''; x, y) = (n^2 + 2np - 7n - 7p + 16)x^ny^n + 2(n-1)x^ny^{n+1} + (n-2)(p-2)x^ny^{n+2} + 2x^{n+1}y^{n+2} + (p-3)x^{n+2}y^{n+2} + 2(np - n - 2p + 3)x^ny^{2n} + 2x^{n+1}y^{2n} + 2(p-2)x^{n+2}y^{2n} + (p-2)x^{2n}y^{2n}$

Proof. We know that a complete graph $K_n, n \geq 3$ has n vertices and $\frac{n(n-1)}{2}$ edges, From Theorem 3A, Proposition 3A and Proposition 3B, it is we see that $T_B(C'')$ has $(np - 2p + 2)$ vertices of degree n, 2 vertices of degree $(n+1)$, $(p-2)$ vertices of degree $(n+2)$ and $(p-1)$ vertices of degree $2n$ also $|E_{\{n,n\}}| = n^2 + 2np - 7n - 7p + 16$, $|E_{n,n+1}| = 2(n-1)$, $|E_{\{n,n+2\}}| = (n-2)(p-2)$, $|E_{\{n+1,n+2\}}| = 2$, $|E_{\{n+2,n+2\}}| = p - 3$, $|E_{\{n,2n\}}| =$

$2(np-n-2p+3), |E_{\{n+1,2n\}}| = 2\ and\ |E_{\{n+2,2n\}}| = 2(p-2), |E_{\{2n,2n\}}| = p-2.$ Hence the proof.

Corollary 3.3 For any chain graph $C'' = C''(G_1, G_2, \dots, G_k; v_1w_1, v_2w_2, \dots, v_kw_k)$ where each $\{G_i\}_{i=1}^{k}$, $k \geq 3$, is a complete graph $K_n, n \geq 4$,

1. $M_1(T_B(C'')) = 2(n^3 - 10n^2 + 6n^2p - 8np + 18n + 2p - 3).$
2. $M_2(T_B(C'')) = n^4 - 7n^3p - 11n^3 - 6n^2p + 15n^2 + 8np + 4p - 12n - 8.$
3. $M_2^m(T_B(C'')) = \frac{4n^2p+12np-32n-35p+74}{4n^2} + \frac{2n-1}{n(n+1)} + \frac{np-2n-p+2}{n(n+2)} + \frac{np+p-n+1}{(n+1)(n+2)^2}$
4. $S_D(T_B(C'')) = 2n^2 + 9np - 19n - 20p + 37 + \frac{n^2p+np-n-2p+3}{n} + \frac{2(n^2+2n+2)}{n+1} + \frac{n^2p+2np-2n-2n^2+2}{n+2}$
5. $H(T_B(C'')) = 2\left[\frac{6n^2+20np-50n-55p+114}{12n} + \frac{2(3np+p-3n)}{(3n+1)(3n+2)} + \frac{2(2n-1)(n+2)}{(2n+1)(2n+3)} + \frac{n^2p-2n^2-3p+np-3n+5}{2(n+1)(n+2)}\right]$
6. $I_n(T_B(C'')) = \frac{n(3n^2+14np-29n-31p+60)}{6} + \frac{2(n+1)(2n^3+3n^2+2n+2)}{(2n+1)(2n+3)} + \frac{n^3p-2n^3+n^2p-3n^2-np-n+2p-6}{2(n+1)} + \frac{4n(3n^2p-3n^2+7np-9n+2p-2)}{(3n+1)(3n+2)}.$

Proof.Wehave

$M(C''; x, y) = (n^2 + 2np - 7n - 7p + 16)x^ny^n + 2(n-1)x^ny^{n+1} + (n-2)(p-2)x^ny^{n+2} + 2x^{n+1}y^{n+2} + (p-3)x^{n+2}y^{n+2} + 2(np-n-2p+3)x^ny^{2n} + 2x^{n+1}y^{2n} + 2(p-2)x^{n+2}y^{2n} + (p-2)x^{2n}y^{2n}$

$D_x = n(n^2 + 2np - 7n - 7p + 16)x^ny^n + 2n(n-1)x^ny^{n+1} + n(n-2)(p-2)x^ny^{n+2} + 2(n+1)x^{n+1}y^{n+2} + (n+2)(p-3)x^{n+2}y^{n+2} + 2n(np-n-2p+3)x^ny^{2n} + 2(n+1)x^{n+1}y^{2n} + 2(n+2)(p-2)x^{n+2}y^{2n} + 2n(p-2)x^{2n}y^{2n}$

$D_y = n(n^2 + 2np - 7n - 7p + 16)x^ny^n + 2(n^2-1)x^ny^{n+1} + (n^2-4)(p-2)x^ny^{n+2} + 2(n+2)x^{n+1}y^{n+2} + (n+2)(p-3)x^{n+2}y^{n+2} + 4n(np-n-2p+3)x^ny^{2n} + 4nx^{n+1}y^{2n} + 4n(p-2)x^{n+2}y^{2n} + 2n(p-2)x^{2n}y^{2n}$

Therefore,

$M_1\left(T_B(C'')\right) = \left(D_x + D_y\right)\left(M(C''; x, y)\right)\big|_{x=y=1} = 2(n^3 - 10n^2 + 6n^2p - 8np + 18n + 2p - 3).$

$D_xD_y = n(n^2 + 2np - 7n - 7p + 16)x^ny^n + 2n(n^2-1)x^ny^{n+1} + n(n^2-4)(p-2)x^ny^{n+2} + 2(n+1)(n+2)x^{n+1}y^{n+2} + (n+2)^2(p-3)x^{n+2}y^{n+2} + 4n(np-n-2p+3)x^ny^{2n} + 4n()x^{n+1}y^{2n} + 4n(p-2)x^{n+2}y^{2n} + 2n(p-2)x^{2n}y^{2n}$

Therefore, $M_2(T_B(C'')) = \left(D_xD_y\right)\left(M(C''; x, y)\right)\big|_{x=y=1} = n^4 - 7n^3p - 11n^3 - 6n^2p + 15n^2 + 8np + 4p - 12n - 8$

$S_x = \frac{(n^2+2np-7n-7p+16)}{n} x^n y^n + \frac{2(n-1)}{n} x^n y^{n+1} + \frac{(n-2)(p-2)}{n} x^n y^{n+2} + \frac{2}{n+1} x^{n+1} y^{n+2} + \frac{(p-3)}{n+2} x^{n+2} y^{n+2} + \frac{2(np-n-2p+3)}{n} x^n y^{2n} + \frac{2}{n+1} x^{n+1} y^{2n} + \frac{2(p-2)}{n+2} x^{n+2} y^{2n} + \frac{p-2}{2n} x^{2n} y^{2n}$

$S_y = \frac{(n^2+2np-7n-7p+16)}{n} x^n y^n + \frac{2(n-1)}{n+1} x^n y^{n+1} + \frac{(n-2)(p-2)}{n+2} x^n y^{n+2} + \frac{2}{n+2} x^{n+1} y^{n+2} + \frac{(p-3)}{n+2} x^{n+2} y^{n+2} + \frac{(np-n-2p+3)}{n} x^n y^{2n} + \frac{1}{n} x^{n+1} y^{2n} + \frac{(p-2)}{n} x^{n+2} y^{2n} + \frac{p-2}{2n} x^{2n} y^{2n}$

Therefore,

$$M_2^m(T_B(C'')) = (S_x S_y)(M(C''; x, y))\big|_{x=y=1} = \frac{4n^2p + 12np - 32n - 35p + 74}{4n^2} + \frac{2n-1}{n(n+1)} + \frac{np - 2n - p + 2}{n(n+2)} + \frac{np + p - n + 1}{(n+1)(n+2)^2}$$

$D_x S_y = (n^2 + 2np - 7n - 7p + 16) x^n y^n + \frac{2n(n-1)}{n+1} x^n y^{n+1} + \frac{n(n-2)(p-2)}{n+2} x^n y^{n+2} + \frac{2(n+1)}{n+2} x^{n+1} y^{n+2} + (p-3) x^{n+2} y^{n+2} + (np - n - 2p + 3) x^n y^{2n} + \frac{n+1}{n} x^{n+1} y^{2n} + \frac{(n+2)(p-2)}{n} x^{n+2} y^{2n} + (p-2) x^{2n} y^{2n}$

$D_y S_x = (n^2 + 2np - 7n - 7p + 16) x^n y^n + \frac{2(n^2-1)}{n} x^n y^{n+1} + \frac{(n^2-4)(p-2)}{n} x^n y^{n+2} + \frac{2(n+2)}{n+1} x^{n+1} y^{n+2} + (p-3) x^{n+2} y^{n+2} + 4(np - n - 2p + 3) x^n y^{2n} + \frac{4n}{n+1} x^{n+1} y^{2n} + \frac{4n(p-2)}{n+2} x^{n+2} y^{2n} + (p-2) x^{2n} y^{2n}$

Therefore,

$$S_D(T_B(C'')) = (D_x S_y + D_y S_x)(M(C''; x, y))\big|_{x=y=1} = 2n^2 + 9np - 19n - 20p + 37 + \frac{n^2p + np - n - 2p + 3}{n} + \frac{2(n^2 + 2n + 2)}{n+1} + \frac{n^2p + 2np - 2n - 2n^2 + 2}{n+2}$$

$$H(T_B(C'')) = 2 S_x J M(C''; x, y)\big|_{x=1} = 2\left[\frac{6n^2 + 20np - 50n - 55p + 114}{12n} + \frac{2(3np + p - 3n)}{(3n+1)(3n+2)} + \frac{2(2n-1)(n+2)}{(2n+1)(2n+3)} + \frac{n^2p - 2n^2 - 3p + np - 3n + 5}{2(n+1)(n+2)}\right]$$

$$I_n(T_B(C'')) = S_x J D_x D_y(M(C''; x, y))\big|_{x=1} = \frac{n(3n^2 + 14np - 29n - 31p + 60)}{6} + \frac{2(n+1)(2n^3 + 3n^2 + 2n + 2)}{(2n+1)(2n+3)} + \frac{n^3p - 2n^3 + n^2p - 3n^2 - np - n + 2p - 6}{2(n+1)} + \frac{4n(3n^2p - 3n^2 + 7np - 9n + 2p - 2)}{(3n+1)(3n+2)}$$

4. Conclusion

Many of the chemical graphs and connection networks are chain graphs. Hence it is important to study the topological properties of these graphs even under transformations. In this paper we have derived the M-polynomial and six major topological indices of the total-block graphs $T_B(P_n), T_B(C''), T_B(C'')$.

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